Chapter 36
Generalized weighted arithmetic means

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Dedicated to the memory of S.M. Ulam
on the 100th anniversary of his birth

Abstract Means which are the sum of single variable functions are considered. It is shown among other that if such a mean is weighted quasi-arithmetic, or subtranslative or subadditive then it must be a weighted quasi-arithmetic mean. Conditions under which the functions of the form $f(x) = ax + b$ are affine or convex with respect to such a mean are presented. Invariance of a weighted arithmetic mean with respect to the relevant mean-type mappings is considered. Some open problems are presented.

Key words: mean, generalized weighted mean, mean-convex function mean-affine functions, subadditivity, subtransitivity, functional equation, functional inequalities.

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36.1 Introduction

A function $M : I^2 \to \mathbb{R}$ is called a *mean* in an interval $I \subseteq \mathbb{R}$, if

$$\min(x,y) \le M(x,y) \le \max(x,y) \ , \qquad x,y \in I \ .$$

If for all $x,y \in I$, $x \neq y$, these inequalities are sharp, M is called *strict*; and *symmetric,* if $M(x,y) = M(y,x)$ for all $x,y \in I$. If M is a mean in I, then $M(J^2) = J$, for every subinterval $J \subseteq I$. Moreover M is *reflexive*, i.e.

$$M(x,x) = x \ , \qquad x \in I \ .$$

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Every reflexive function $M : I^2 \to \mathbb{R}$ which is increasing with respect to each variable is a mean in I. If $I = (0,\infty)$ and

$$M(tx,ty) = tM(x,y) , \qquad t,x,y > 0 ,$$

then M is positively homogeneous.

For a strictly monotonic function $\gamma : I \to \mathbb{R}$ and $p \in (0,1)$, the function $M : I^2 \to I$, $M = \mathscr{A}_p^{[\gamma]}$,

$$\mathscr{A}_p^{[\gamma]}(x,y) := \gamma^{-1}\left(p\gamma(x) + (1-p)\gamma(y)\right) , \qquad x,y \in I ,$$

is a mean, and it is called *quasi-arithmetic weighted mean.* The function γ is called a *generator* of the mean and p its *weight*. The symmetric mean

$$\mathscr{A}^{[\gamma]} := \mathscr{A}_{1/2}^{[\gamma]}$$

is called *quasi-arithmetic* mean.

Taking here $\gamma(x) = ax + b$, $a \neq 0$, we get

$$\mathscr{A}_p^{[\gamma]}(x,y) = \mathscr{A}_p(x,y) := px + (1-p)y , \qquad x,y \in I .$$

If $p = 1/2$ we write $\mathscr{A} = \mathscr{A}_{1/2}$.

Note that the weighted arithmethic mean is sum of two functions of single variable.

In this paper we examine the means which are of the form $M(x,y) = \varphi(x) + \psi(y)$ for some functions $\varphi, \psi : I \to \mathbb{R}$. In section 36.2 we observe that M is a mean iff $\psi(x) = x - \varphi(x)$ for all $x \in I$ and both φ and ψ are increasing – in particular φ and ψ must be nonexpansive, i.e. Lipschitz continuous with a Lipschitz constant 1. We denote this mean by $\mathscr{W}^{[\varphi]}$. For obvious reason, the means of this type can be treated as a generalization of the classical weighted arithmetic means. We show that if $\mathscr{W}^{[\varphi]} \leq \mathscr{W}^{[\gamma]}$ then $\mathscr{W}^{[\varphi]} = \mathscr{W}^{[\gamma]}$, and $\mathscr{W}^{[\varphi]}$ is symmetric iff $\mathscr{W}^{[\varphi]} = \mathscr{A}$.

In section 36.3 we prove that a mean $\mathscr{W}^{[\varphi]}$ defined on $[0,\infty)^2$ is weighted quasi-arithmetic iff it is weighted arithmetic. The positive homogeneity of $\mathscr{W}^{[\varphi]}$ is also considered.

In sections 36.4 and 36.5 we consider conditions under which the functions of the form $f(x) = ax + b$ are affine or convex with respect to $\mathscr{W}^{[\varphi]}$. It turns out that if the function $f(x) = ax$ is $\mathscr{W}^{[\varphi]}$-convex, then it is $\mathscr{W}^{[\varphi]}$-affine and, under the differentiability of φ at 0 and the condition $a \neq 1$, the mean $\mathscr{W}^{[\varphi]}$ must be a weighted arithmetic. Some related results concerning general convexity are also given.

Recall that the notion of a convex function with respect to two given means was introduced by G. Aumann [3] and, for the first time was considered by J. Aczél [1] (cf. also J. Matkowski and J. Rätz [9]).

In section 36.6 we show that $\mathscr{W}^{[\varphi]}$ is subtranslative iff it is weighted arithmetic. In section 36.7 we deal with subadditivity of $\mathscr{W}^{[\varphi]}$. In section 36.8 we consider the invariance of a weighted arithmetic mean with respect to the mean-type mapping of the form $\left(\mathscr{W}^{[\varphi]},\mathscr{W}^{[\psi]}\right)$ and we apply the obtained result in solving a functional equation. In section 36.9 we propose a generalization of the weighted quasi-arithmetic means. In section 36.10 the finite dimensional counterparts of the means $\mathscr{W}^{[\varphi]}$ are discussed.

At the end we present some open problems related to Daróczy-Páles identity.

36.2 A generalization of weighted arithmetic mean

We begin with

Theorem 36.1. *Let I be an interval and $\varphi,\psi : I \to \mathbb{R}$. The function $M : I^2 \to \mathbb{R}$,*

$$M(x,y) := \varphi(x) + \psi(y), \qquad x,y \in I, \tag{36.1}$$

is a (strict) mean if, and only if, the functions φ and ψ are (strictly) increasing and

$$\varphi(x) + \psi(x) = x\,, \qquad x \in I\,.$$

Proof. Suppose that M given by (1) is a mean. Setting $y = x$ in (1) we get $\varphi(x) + \psi(x) = x$, whence $\psi(x) = x - \varphi(x)$ for $x \in I$. Hence, taking $x,y \in I$, $x \le y$, from the definition of the mean we have

$$x = \min(x,y) \le \varphi(x) + y - \varphi(y) \le \max(x,y) = y\,,$$

i.e. $\varphi(x) \le \varphi(y)$ and $x - \varphi(x) \le y - \varphi(y)$,which proves that φ and ψ are increasing. Conversely, suppose that φ and ψ are increasing and such that $\varphi(x) + \psi(x) = x$ for $x \in I$. Take arbitrary $x,y \in I$, assume that $x \le y$. Then we have $\varphi(x) \le \varphi(y)$ and $x - \varphi(x) \le y - \varphi(y)$, whence $x \le \varphi(x) + y - \varphi(y) \le y$, which means that $\min(x,y) \le M(x,y) \le \max(x,y)$. In the case when M is a strict mean we argue analogously. □

Corollary 36.1. *Let I be an interval and $\varphi : I \to \mathbb{R}$. Then*

(i) the functions φ and $\mathrm{id}_I - \varphi$ are nondecreasing iff the function $\mathscr{W}^{[\varphi]} : I^2 \to I$ defined by

$$\mathscr{W}^{[\varphi]}(x,y) := \varphi(x) + y - \varphi(y)\,, \qquad x,y \in I\,, \tag{36.2}$$

is a mean;

(ii) the functions φ and $\mathrm{id}_I - \varphi$ are strictly increasing iff $\mathscr{W}^{[\varphi]}$ is a strict mean.

Remark 36.1. Let I be an interval and $\varphi : I \to \mathbb{R}$. Then

(i) $\mathscr{W}^{[\varphi]}$ is a mean iff φ is nondecreasing and non-expansive, i.e.

$$x \le y \Longrightarrow 0 \le \varphi(y) - \varphi(x) \le y - x\,;$$

(ii) $\mathscr{W}^{[\varphi]}$ is a strict mean iff φ is strictly increasing and contractive, i.e.

$$x < y \Longrightarrow 0 < \varphi(y) - \varphi(x) < y - x .$$

Note the following easy to verify

Property 36.1. The following conditions are equivalent

(i) $\mathscr{W}^{[\varphi]}$ is symmetric;
(ii) there is a $c \in \mathbb{R}$ such that

$$\varphi(x) = \frac{x}{2} + c , \qquad x \in I ;$$

(iii) $\mathscr{W}^{[\varphi]} = \mathscr{A}$, where $\mathscr{A}(x,y) = \frac{x+y}{2}$ for $x, y \in I$.

Property 36.2. Let I be an interval and assume that $\varphi, \gamma : I \to \mathbb{R}$, $\mathrm{id}_I - \varphi$ and $\mathrm{id}_I - \gamma$ are nondecreasing. Then the following conditions are equivalent:

(i) $\mathscr{W}^{[\varphi]} \le \mathscr{W}^{[\gamma]}$;
(ii) $\mathscr{W}^{[\varphi]} = \mathscr{W}^{[\gamma]}$;
(iii) there is a $c \in \mathbb{R}$ such that $\gamma = \varphi + c$.

Proof. By definitions of the means $\mathscr{W}^{[\varphi]}$ and $\mathscr{W}^{[\gamma]}$, the inequality $\mathscr{W}^{[\varphi]} \le \mathscr{W}^{[\gamma]}$ is equivalent to the inequality $\varphi(x) - \gamma(x) \le \varphi(y) - \gamma(y)$ for all $x, y \in I$, which implies that $\varphi - \gamma$ is a constant function. □

36.3 When $\mathscr{W}^{[\varphi]}$ is weighted quasi arithmetic or homogeneous

The main result of this section reads as follows

Theorem 36.2. *Let $I \subset \mathbb{R}$ be an interval and suppose that $\varphi : I \to \mathbb{R}$ and $\mathrm{id}_I - \varphi$ are nondecreasing. Then $\mathscr{W}^{[\varphi]}$ defined by (2) is a weighted quasi-arithmetic mean if, and only if, there are $p \in (0,1)$ and $c \in \mathbb{R}$, $c \neq 0$, such that $\varphi(x) = px + c$ for $x \in I$, that is iff,*

$$\mathscr{W}^{[\varphi]}(x,y) = px + (1-p)y , \qquad x, y \in I .$$

Proof. If $\mathscr{W}^{[\varphi]}$ is a weighted quasi-arithmetic mean, then there is a continuous strictly monotonic function $\gamma : I \to \mathbb{R}$ and $p \in (0,1)$ such that

$$\mathscr{W}^{[\varphi]}(x,y) = \varphi(x) + \psi(x) = \gamma^{-1}\left(p\gamma(x) + (1-p)\gamma(y)\right) , \qquad x, y \in I , \qquad (36.3)$$

where

$$\psi(x) = x - \varphi(x) , \qquad x \in I .$$

Put $J := \gamma(I)$ and take arbitrary $u, v \in J$. Setting $x := \gamma^{-1}(u)$, $y := \gamma^{-1}(b)$ in (36.3) gives

$$\gamma^{-1}(pu+(1-p)v) = \varphi\circ\gamma^{-1}(u)+\psi\circ\gamma^{-1}(v)\,, \qquad u,v\in J\,.$$

Hence, setting

$$f := \gamma^{-1}\,, \qquad g := \varphi\circ\gamma^{-1}\,, \qquad h := \psi\circ\gamma^{-1}\,,$$

we obtain the functional equation

$$f(pu+(1-p)v) = g(u)+h(v)\,, \qquad u,v\in J\,.$$

Interchanging u and v we hence get

$$f((1-p)u+pv) = g(v)+h(u)\,, \qquad u,v\in J\,.$$

Adding the respective sides of these two equations we obtain

$$f(pu+(1-p)v)+f((1-p)u+pv) = [g(u)+h(u)]+[g(v)+h(v)]$$

for all $u,v\in J$. Taking here $v=u$ gives

$$g(u)+h(u) = f(u)\,, \qquad u,v\in J\,,$$

whence

$$f(pu+(1-p)v)+f((1-p)u+pv) = f(u)+f(v)\,, \qquad u,v\in J\,,$$

that is f is p-Wright affine. In view of Ng's theorem [11], there is an additive function $\alpha:\mathbb{R}\to\mathbb{R}$ and $b\in\mathbb{R}$ such that $f(u)=\alpha(u)+b$ for all $u\in J$. Since $f=\gamma^{-1}$ is continuous, we have $\alpha(u)=au$ for all $u\in\mathbb{R}$. Hence $\gamma^{-1}(u)=au+b$ for all $u\in J$ and, obviously, $a\neq 0$. From (36.2) we obtain our result. □

Let us also note the following easy to prove

Theorem 36.3. *Let $\varphi: I\to\mathbb{R}$ and $\mathrm{id}_{(0,\infty)}-\varphi$ be nondecreasing. The mean $\mathscr{W}^{[\varphi]}$ is homogeneous iff there is $p\in[0,1]$ such that*

$$\mathscr{W}^{[\varphi]}(x,y) = px+(1-p)y\,, \qquad x,y>0\,.$$

36.4 $\mathscr{W}^{[\varphi]}$-affine functions

Recall the following

Definition 36.1. (cf. [9]) Let $I,J\subset\mathbb{R}$, $J\subset I$ be intervals, and let $M: I^2\to I$ be a mean. A function $f: J\to I$ is said to be affine with respect M, briefly, M-affine, if

$$f(M(x,y)) = M(f(x),f(y))\,, \qquad x,y\in J\,.$$

Remark 36.2. It easy to see that, under the assumptions of the above definition, the following three statements are valid:

(i) if f is one-to-one, onto and M-affine, then f^{-1} is M-affine;
(ii) if $J = I$ and $f, g : I \to I$ are M-affine then so is $f \circ g$;
(iii) the function $f = \mathrm{id}_{(0,\infty)}$ and the constant function $f = c$ for $c \in I$ are M-affine.

Theorem 36.4. *Let $I = \mathbb{R}$ or $I = [0,\infty)$. Suppose that $\varphi : I \to \mathbb{R}$ and $\mathrm{id}_{(0,\infty)} - \varphi$ are strictly increasing.*

(i) If $I = \mathbb{R}$ and $f : \mathbb{R} \to \mathbb{R}$ is $\mathscr{W}^{[\varphi]}$-affine, then there exist an additive function $\alpha : \mathbb{R} \to \mathbb{R}$ and $b \in \mathbb{R}$ such that

$$f(x) = \alpha(x) + b\,, \qquad x \in \mathbb{R}\,.$$

If moreover f is bounded from above (from below) or measurable in a neighborhood of a point then $f(x) = ax + b$ for some $a, b \in \mathbb{R}$ and for all $x \in \mathbb{R}$.
(ii) If $I = [0,\infty)$ and $f : [0,\infty) \to [0,\infty)$ is $\mathscr{W}^{[\varphi]}$-affine, then there are $a, b \in \mathbb{R}$ such that

$$f(x) = \alpha(x) + b\,, \qquad x \geq 0\,.$$

Proof. 1) We can assume without any loss of generality that $\varphi(0) = 0$. Suppose that $f : \mathbb{R} \to \mathbb{R}$ is $\mathscr{W}^{[\varphi]}$-affine. From Definition 36.1 we have

$$f\left(\varphi(x) + y - \varphi(y)\right) = \varphi(f(x)) + f(y) - \varphi(f(y))\,, \qquad x, y \in \mathbb{R}\,.$$

Taking here $y = 0$ we get

$$f\left(\varphi(x)\right) = \varphi(f(x)) + f(0) - \varphi(f(0))\,, \qquad x \in \mathbb{R}\,.$$

and taking $x = 0$ we get

$$f\left(y - \varphi(y)\right) = \varphi(f(0)) + f(0) - \varphi(f(y))\,, \qquad y \in \mathbb{R}\,.$$

Hence, setting $b := f(0)$, and adding the respective sides of these two equations, we obtain

$$f\left(\varphi(x) + y - \varphi(y)\right) = f(\varphi(x)) - b + f(y - \varphi(y))\,, \qquad x, y \in \mathbb{R}\,.$$

that is,

$$f\left(\varphi(x) + \psi(y)\right) - b = \left[f(\varphi(x)) - b\right] + \left[f(\psi(y)) - b\right]\,, \qquad x, y \in \mathbb{R}\,,$$

where $\psi(y) = y - \varphi(y)$ for $y \in \mathbb{R}$. As $\mathscr{W}^{[\varphi]}$ is a strict mean (cf. Corollary 36.1(*ii*)), we have $\varphi(\mathbb{R}) = (r_1, r_2)$, $\psi(\mathbb{R}) = (s_1, s_2)$ where $-\infty \leq r_1 < r_2 \leq \infty$. It follows that the function $\alpha : \mathbb{R} \to \mathbb{R}$ defined by

$$\alpha(u) := f(u) - c\,, \qquad u \in \mathbb{R}\,,$$

satisfies the functional equation

$$\alpha(u+v) = \alpha(u) + \alpha(v) , \qquad u \in (r_1, r_2), v \in (s_1, s_2) .$$

Since at least one of the numbers r_1, r_2, s_1, s_2 is not finite, it follows that

$$\alpha(u+v) = \alpha(u) + \alpha(v) , \qquad u, v \in \mathbb{R} ,$$

which shows that α is additive. The remaining part of the theorem is well known (cf. Aczél [2], Kuczma [5]) and the proof of part (i) is completed.

In case (ii) the argument showing the additivity of α is analogous. The nonnegativity of f implies that f is of the desired form. □

Theorem 36.5. *Let $I \subset \mathbb{R}$ be a closed interval and $\varphi : I \to \mathbb{R}$ a nondecreasing function such that* $\mathrm{id}_{(0,\infty)} - \varphi$ *is nondecreasing. Suppose that $f : I \to I$, $f(x) = ax + b$ $(x \in I)$ for some $a, b \in \mathbb{R}$. Then*

(i) f is $\mathscr{W}^{[\varphi]}$-affine iff the function

$$I \ni x \longrightarrow \varphi(ax+b) - a\varphi(x) \text{ is constant ;}$$

(ii) if f is $\mathscr{W}^{[\varphi]}$-affine, $a \neq 1$ and φ is differentiable at the point $x_0 = \frac{b}{1-a}$, then

$$f(x) = px + c , \qquad x \in I ,$$

for some $p \in [0,1]$, $c \in \mathbb{R}$; moreover

$$\mathscr{W}^{[\varphi]} = px + (1-p)y , \qquad x, y \in I ;$$

(iii) assuming that f is $\mathscr{W}^{[\varphi]}$-affine and $a = 1$, if either

$$b > 0 \quad \text{and} \quad \lim_{x\to\infty} \frac{\varphi(x)}{x} \quad \text{exists} ,$$

or

$$b < 0 \quad \text{and} \lim_{x\to-\infty} \frac{\varphi(x)}{x} \quad \text{exists} ,$$

then the conclusion of (ii) remains valid.

Proof. The $\mathscr{W}^{[\varphi]}$-affinity of f, that is the equality

$$a\left(\varphi(x) + y - \varphi(y)\right) = \varphi\left(ax+b\right) + ay + b - \varphi(ay+b) , \qquad x, y \in I ,$$

simplifies to

$$\varphi\left(ax+b\right) - a\varphi(x) = \varphi(ay+b) - a\varphi(y) , \qquad x, y \in I ,$$

that is equivalent to the fact that, for some $C \in \mathbb{R}$,

$$\varphi\left(ax+b\right) - a\varphi(x) = C , \qquad x \in I .$$

To prove (ii) note that this equation can be written in the form

$$\left[\varphi(ax+b) - \frac{C}{1-a}\right] = a\left[\varphi(x) - \frac{C}{1-a}\right], \qquad x \in I .$$

Setting $\phi(x) := \varphi(x) - \frac{C}{1-a}$ we get

$$\phi(ax+b) = a\gamma(x), \qquad x \in I .$$

Taking into account that f maps I into itself and considering the four possible cases $I = \mathbb{R}$, I is unbounded from below, I is unbounded from above, and I is bounded, it is easy to see that, without any loss of generality, we can assume that $|a| < 1$. It follows that $x_0 := \frac{b}{1-a}$, the only fixed point of f (which of course belongs to I) is contractive. Moreover, setting $x = x_0$ in the above equation gives $\phi(x_0) = 0$. Setting

$$\gamma(x) := \frac{\phi(x)}{x - x_0}, \qquad x \in I,\ x \neq x_0 ,$$

in the above functional equation gives

$$\gamma(x) = \gamma(ax+b), \qquad x \in I,\ x \neq x_0 ,$$

whence, by induction,

$$\gamma(x) = \gamma\left(a^n x + b\frac{1-a^n}{1-a}\right), \qquad x \in I,\ x \neq x_0,\ n \in \mathbb{N} .$$

The differentiability of φ at x_0 implies the existence of $p := \lim_{x \to x_0} \gamma(x)$. Therefore, letting $n \to \infty$ in the above formula, we obtain $\gamma(x) = p$ for $x \in I \setminus \{x_0\}$, and the proof of (ii) is completed.

We omit similar reasoning in the case (iii). □

Remark 36.3. According to Corollary 36.1, the function φ in the above result is Lipschitzian and, consequently, it is absolutely continuous. Nevertheless, in the second part of this theorem, the assumption of differentiability of φ at the fixed point of f is essential. In fact we have the following

Proposition 36.1. *Let $a \in (0,1)$ be fixed. Suppose that $\varphi_0 : [a,1] \to \mathbb{R}$ is increasing, nonexpansive and such that $\varphi_0(a) = a\varphi_0(1)$. Then there exists a unique increasing nonexpansive function $\varphi : (0,\infty) \to \mathbb{R}$ such that $\varphi\big|_{[a,1]} = \varphi_0$, and the function $f : (0,\infty) \to (0,\infty)$, $f(x) = ax$, is $\mathscr{W}^{[\varphi]}$-affine.*

Proof. Since

$$\bigcup_{k \in \mathbb{Z}} \left(a^{k+1}, a^k\right] = (0,\infty), \qquad \left(a^{k+1}, a^k\right] \cap \left(a^{m+1}, a^m\right] = \emptyset,\ k, m \in \mathbb{Z},\ k \neq m ,$$

(where $\mathbb{Z}$ stands for the set of integers), the function $\varphi : (0,\infty) \to \mathbb{R}$,

$$\varphi(x) := a^k \varphi_0\left(a^{-k}x\right), \qquad x \in \left(a^{k+1}, a^k\right],\ k \in \mathbb{Z},$$

is correctly defined. Note that

$$\varphi(a^k) := a^k \varphi_0(1), \qquad k \in \mathbb{Z},$$

and, for every $k \in \mathbb{Z}$, the function φ is left-continuous at the point a^k. Since $\varphi_0(a) = a\varphi_0(1)$, we have

$$\varphi(a^k+) = \lim_{x \to a^k+} \varphi_0\left(a^{a-k+1}x\right) = a^{k-1}\varphi_0(a) = a^k\varphi_0(1) = \varphi\left(a^k\right),$$

so the function φ is continuous at each point a^k and, consequently, φ is continuous. The increasing monotonicity of φ_0 implies that, for every $k \in \mathbb{Z}$, the function φ is increasing on the interval $\left(a^{k+1}, a^k\right]$. The continuity of φ implies that φ is increasing on $(0,\infty)$. Since φ_0 is nonexpansive, for a fixed $k \in \mathbb{Z}$, and arbitrary $x, y \in \left(a^{k+1}, a^k\right]$, $x < y$, we have

$$\begin{aligned} 0 \le \varphi(y) - \varphi(x) &= a^k\left[\varphi_0\left(a^{-k}y\right) - \varphi_0\left(a^{-k}x\right)\right] \\ &\le a^k\left[\left(a^{-k}y\right) - \left(a^{-k}x\right)\right] = y - x, \end{aligned}$$

which proves that φ is nonexpansive on every interval $\left(a^{k+1}, a^k\right]$. The continuity of φ at every point a^k implies hat φ is nonexpansive on $(0,\infty)$. Thus we have shown that the function φ satisfies condition (i) of Remark 36.1 and, consequently, $\mathscr{W}^{[\varphi]}$ is a mean.

Now take arbitrary $x > 0$. There is a unique $k \in \mathbb{Z}$ such that $x \in \left(a^{k+1}, a^k\right]$. Then $ax \in \left(a^{k+2}, a^{k+1}\right]$ and, by the definition of φ we have

$$\varphi(ax) = a^{k+1}\varphi_0(a^{-k+1}(ax)) = a(a^k\varphi_0(a^{-k}x)) = a\varphi(x)$$

which proves that that mean $\varphi(ax) - a\varphi(x)$ is constant on $(0,\infty)$ and, in view of Theorem 36.5(i), the function $f(x) = ax$ for $x > 0$, is $\mathscr{W}^{[\varphi]}$-affine. □

Remark 36.4. Proposition 36.1 gives a construction of all $\varphi : (0,\infty) \to \mathbb{R}$ such that the function $f : (0,\infty) \to (0,\infty)$, $f(x) = f(1)x$, is $\mathscr{W}^{[\varphi]}$-affine. Moreover it shows that the converse of Theorem 36.5 is false.

Let us note the following

Lemma 36.1. *Let $M : (0,\infty)^2 \to (0,\infty)$ be a continuous mean. If there are $a, b > 0$, $a \ne 1 \ne b$, such that $\log(b-a)$ is irrational, and the functions $f, g : (0,\infty) \to (0,\infty)$, $f(x) = ax$, $g(x) = bx$ are M-affine, then M is positively homogeneous.*

Proof. By assumption we have $M(ax, ay) = aM(x,y)$, $M(bx, by) = bM(x,y)$ for all $x, y > 0$, whence, by induction,

$$M(a^k x, a^k y) = a^k M(x,y), \quad M(b^m x, b^m y) = b^m M(x,y), \quad k, m \in \mathbb{Z},\ x, y > 0,$$

that is

$$M(tx,ty) = tM(x,y)\ , \quad t \in D,\ x,y > 0\ ,$$

where $D := \{a^k b^m : k,m \in \mathbb{Z}\}$. Since $\log(b-a)$ is irrational, by Kronecker theorem, D is a dense in $(0,\infty)$. The continuity of M implies that $M(tx,ty) = tM(x,y)$ for $t,x,y > 0$. □

As an immediate consequence of this lemma and Theorem 36.4 we obtain

Theorem 36.6. *Let $I = (0,\infty)$ and suppose that $\varphi : I \to \mathbb{R}$ and $\mathrm{id}_I - \varphi$ are nondecreasing. If there are $a,b > 0$, $a \neq 1 \neq b$, such that $\log(b-a)$ is irrational, and the functions $f,g : (0,\infty) \to (0,\infty)$, $f(x) = ax$, $g(x) = bx$ are $\mathscr{W}^{[\varphi]}$-affine, then $\mathscr{W}^{[\varphi]}$ is a weighted arithmetic mean.*

Theorem 36.7. *Let $I = \mathbb{R}$ and suppose that $\varphi : I \to \mathbb{R}$ and $\mathrm{id}_I - \varphi$ are nondecreasing. If there are $a,b > 0$, $a \neq 1 \neq b$, such that a/b is irrational, and the functions $f,g : \mathbb{R} \to \mathbb{R}$, $f(x) = x+a$, $g(x) = x+b$ are $\mathscr{W}^{[\varphi]}$-affine, then $\mathscr{W}^{[\varphi]}$ is a weighted arithmetic mean.*

In connection with the above two result note that the conditions under which the simultaneous system of functional equations

$$\varphi(Ax+\alpha) = a\varphi(x)\ , \qquad \varphi(Bx+\beta) = b\varphi(x)$$

has a nontrivial continuous solution is given in [8].

36.5 $\mathscr{W}^{[\varphi]}$-convex functions

Definition 36.2. (Aczél [1], Aumann [3], also [9]) Let $I,J \subset \mathbb{R}$, $I \subset J$, be intervals and $M : I^2 \to I$ be a mean. A function $f : J \to I$ is said to be

1^0 convex with respect to M, briefly M-convex, if

$$f(M(x,y)) \le M(f(x),f(y))\ , \qquad x,y \in J\ ,$$

2^0 M-concave, if

$$f(M(x,y)) \ge M(f(x),f(y))\ , \qquad x,y \in J\ .$$

Proposition 36.2. *Let $I,J \subset \mathbb{R}$, $I \subset J$, be intervals and $M,N : I^2 \to I$ some means.*

1^0 *If a decreasing function $f : J \to I$ is M-convex and $M \le N$, then f is N-convex.*
2^0 *If a decreasing function $f : J \to I$ is M-convex and $M \le N$, then f is N-convex.*

Proof. Applying in turn: the inequality $M \le N$ and the decreasing monotonicity of f, the M-convexity of f and again the inequality $M \le N$, we get, for all $x,y \in J$,

$$f(N(x,y)) \le f(M(x,y)) \le M(f(x),f(y)) \le N(f(x),f(y)) \ ,$$

which proves 1^0. The proof of 2^0 is similar. □

From Property 36.1(ii) we obtain

Corollary 36.2. *Let $I,J \subset \mathbb{R}$, $I \subset J$, be intervals. Suppose that $\varphi : I \to \mathbb{R}$, $\mathrm{id}_I - \varphi$ are nondecreasing and there is $p \in (0,1)$ such that either*

$$\mathscr{W}^{[\varphi]}(x,y) \le px + (1-p)y \ , \qquad x,y \in I \ ,$$

or

$$\mathscr{W}^{[\varphi]}(x,y) \ge px + (1-p)y \ , \qquad x,y \in I \ .$$

If $f : J \to I$ is $\mathscr{W}^{[\varphi]}$-convex ($\mathscr{W}^{[\varphi]}$-concave), then f is Jensen-convex (resp. Jensen concave).

Proof. If $\mathscr{W}^{[\varphi]}(x,y) \le px + (1-p)y$ for all $x,y \in I$, then, in view of Property 36.2(ii) with $\gamma(x) = px$ ($x \in I$), we have $\mathscr{W}^{[\varphi]}(x,y) = px + (1-p)y$ for all $x,y \in I$. It follows that

$$f\,(px + (1-p)y) \le pf(x) + (1-p)f(y) \ , \qquad x,y \in I \ .$$

By Kuhn's theorem [6], f satisfies the inequality with p replaced by $1/2$, i.e. f is Jensen convex (for a simple proof of Kuhn's result see Daróczy and Páles [4]. □

Theorem 36.8. *Let $I = (0,\infty)$ and suppose that $\varphi : I \to \mathbb{R}$, $\mathrm{id}_I - \varphi$ are nondecreasing.*

(i) The function $f : (0,\infty) \to (0,\infty)$, $f(x) = f(1)x$, is $\mathscr{W}^{[\varphi]}$-convex iff it is $\mathscr{W}^{[\varphi]}$-afine.
(ii) Let C_φ denote the set of all $a > 0$ such that the function $f(x) = ax$ ($x > 0$) is $\mathscr{W}^{[\varphi]}$-convex. Then

$$\text{either } \ C_\varphi = (0,\infty) \quad \text{or} \quad C_\varphi = \{d^k : k \in \mathbb{Z}\} \ \text{for some } d > 0 \ .$$

Proof. 1) Put $a := f(1)$. By the definition of $\mathscr{W}^{[\varphi]}$ the function $f(x) = ax$ ($x \in I$) is $\mathscr{W}^{[\varphi]}$-convex iff

$$a\,(\varphi(x) + y - \varphi(y)) \le \varphi(ax) + ay - \varphi(ay) \ , \qquad x,y \in I \ ,$$

that is, iff

$$a\varphi(x) - \varphi(ax) \le a\varphi(y) - \varphi(ay) \ , \qquad x,y \in I \ .$$

Obviously, this inequality is equivalent to the equality

$$a\varphi(x) - \varphi(ax) = a\varphi(y) - \varphi(ay) \ , \qquad x,y \in I \ ,$$

which holds iff

$$a\,(\varphi(x) + y - \varphi(y)) \le \varphi(ax) + ay - \varphi(ay) \ , \quad x,y \in I \ ,$$

that is iff f is $\mathscr{W}^{[\varphi]}$-afine. Now the first part follows from Theorem 36.5(i).

2) Suppose that $a,b \in C_{\varphi}$. Then, according to what has been already shown, the functions $f(x) = ax, g(x) = bx$ $(x > 0)$ are $\mathscr{W}^{[\varphi]}$-affine. In view of Remark 36.2, the function $f \circ g^{-1}$ is also $\mathscr{W}^{[\varphi]}$-affine and, consequently, $ab^{-1} \in C_{\varphi}$. It follows that C_{φ} is a subgroup of the multiplicative group $(0,\infty)$. It is obvious that every such a subgroup is either dense or discrete. If C_{φ} is discrete then, which is a well known fact, it must be a cyclic subgroup of $(0,\infty)$, i.e. there is a number $d > 0$ such that $C_{\varphi} = \{d^k : k \in \mathbb{Z}\}$. □

36.6 Translativity, subtranslativity and supertranslativity of $\mathscr{W}^{[\varphi]}$

Recall that a mean $M : (0,\infty)^2 \to (0,\infty)$ is *translative* (cf. Aczél [2, p. 234]) if

$$M(x+t, y+t) = M(x,y) + t \ , \qquad x,y,t > 0 \ .$$

We call a mean $M : (0,\infty)^2 \to (0,\infty)$ *subtranslative* if

$$M(x+t, y+t) \le M(x,y) + t \ , \qquad x,y,t > 0 \ ;$$

and *supertranslative* if the reversed inequality holds true.

Theorem 36.9. *Let $I = (0,\infty)$ and suppose that $\varphi : I \to \mathbb{R}$, $\mathrm{id}_I - \varphi$ are nondecreasing. The following conditions are equivalent:*

(i) $\mathscr{W}^{[\varphi]}$ *is* subtranslative;
(ii) $\mathscr{W}^{[\varphi]}$ *is* supertranslative;
(iii) $\mathscr{W}^{[\varphi]}$ *is* translative;
(iv) $\mathscr{W}^{[\varphi]}$ *is a weighted arithmetic mean.*

Proof. Suppose that $\mathscr{W}^{[\varphi]}$ is subtranslative. Then from the definition of $\mathscr{W}^{[\varphi]}$ we easily get

$$\varphi(x+t) + \varphi(y) \le \varphi(y+t) + \varphi(x) \ , \qquad x,y,t > 0.$$

Interchanging x and y we infer that

$$\varphi(x+t) + \varphi(y) = \varphi(y+t) + \varphi(x) \ , \qquad x,y,t > 0 \ ,$$

whence

$$\varphi(x+t) - \varphi(x) = \varphi(y+t) - \varphi(y) \ , \qquad x,y,t > 0 \ .$$

It follows that

$$\varphi(x+t) - \varphi(x) = \beta(t) \ , \qquad x,t > 0 \ ,$$

for some function $\beta : (0,\infty) \to \mathbb{R}$. Solving this Pexider equation we conclude that, for some real constant c, the function $\varphi - c$ is additive. The conditions on φ imply that

$$\varphi(x) = px - c \quad (x > 0)$$

for some $p \in [0,1]$. Now all the remaining statements are obvious. □

36.7 Subadditivity and superadditivity of $\mathscr{W}^{[\varphi]}$

The subadditivity and superadditivity of a mean plays important role in the theory of convexity with respect to the mean. Suppose, for instance, that $M : (0,\infty)^2 \to (0,\infty)$ is a superadditive mean and $I \subset (0,\infty)$ is an open interval. It is shown in [9] that if $f, g : I \to (0,\infty)$ are M-convex in I, then $f + g$ is M-convex.

Theorem 36.10. *Let $I = (0,\infty)$ and suppose that $\varphi : I \to \mathbb{R}$, $\mathrm{id}_I - \varphi$ are nondecreasing. The following conditions are equivalent*

(i) the mean $\mathscr{W}^{[\varphi]}$ is subadditive (or superadditive);
(ii) there are $p \in [0,1]$ and $c \in \mathbb{R}$ such that

$$\varphi(x) = px + c\,, \qquad x > 0\,;$$

(iii) there is $p \in [0,1]$ such that

$$\mathscr{W}^{[\varphi]}(x,y) = px + (1-p)y\,, \qquad x, y > 0\,.$$

Proof. Suppose that $\mathscr{W}^{[\varphi]}$ is subadditive, i.e.

$$\mathscr{W}^{[\varphi]}(x_1 + x_2, y_1 + y_2) \le \mathscr{W}^{[\varphi]}(x_1, y_1) + \mathscr{W}^{[\varphi]}(x_2, y_2)\,, \qquad x_1, x_2, y_1, y_2 > 0\,.$$

Hence, by the definition of $\mathscr{W}^{[\varphi]}$,

$$\varphi(x_1 + x_2) - \varphi(x_1) - \varphi(x_2) \le \varphi(y_1 + y_2) - \varphi(y_1) - \varphi(y_2)\,, \qquad x_1, x_2, y_1, y_2 > 0\,,$$

which, obviously implies that there is a constant $c \in \mathbb{R}$ such that

$$\varphi(x_1 + x_2) - \varphi(x_1) - \varphi(x_2) = -c\,, \qquad x_1, x_2, y_1, y_2 > 0\,.$$

Writing this equality in the form

$$\varphi(x_1 + x_2) - c = [\varphi(x_1) - c] + [\varphi(x_2) - c]\,, \quad x_1, x_2, y_1, y_2 > 0\,,$$

we conclude that $\varphi - c$ is additive. Since the remaining statements are obvious, the proof is complete. □

36.8 Invariance of the arithmetic mean with respect to the mean-type mappings of $\mathscr{W}^{[\varphi]}$-type

Let $M,N : I^2 \to I$ be means. A mean $K : I^2 \to I$ is called *invariant with respect to the mean-type mapping* $(M,N) : I^2 \to I^2$ (briefly, K is (M,N)*-invariant),* if

$$K(M(x,y),N(x,y)) = K(x,y) , \qquad x,y \in I .$$

The invariant mean is useful when we are looking for the limits of the sequence of iterates of the mean-type mapping $(M,N) : I^2 \to I^2$ (cf. [10]).

Let us note that the proportion

$$x : \frac{x+y}{2} = \frac{2xy}{x+y} : y ,$$

the base of the theory of harmony made by Pythagorean school, can be written in the form

$$\sqrt{\frac{x+y}{2} \cdot \frac{2xy}{x+y}} = \sqrt{xy} .$$

Setting

$$A(x,y) = \frac{x+y}{2} , \quad H(x,y) = \frac{2xy}{x+y} , \quad G(x,y) = \sqrt{xy}$$

for the arithmetic, harmonic and geometric mean, respectively, we hence get

$$G \circ (A,H) = G$$

which says that the geometric mean G is (A,H)-invariant. This fact allows to determine effectively the limit of the sequence of iterates $((A,H)^n)_{n\in\mathbb{N}}$ of the mean-type mapping $(A,H) : (0,\infty)^2 \to (0,\infty)^2$.

We prove the following

Theorem 36.11. *Let $I \subset \mathbb{R}$ be an interval, $p \in (0,1)$, and $M : I^2 \to I$ be a mean. Suppose that $\varphi : I \to \mathbb{R}$ and $\mathrm{id}_I - \varphi$ are nondecreasing. The weighted arithmetic mean $\mathscr{A}_p$ is invariant with respect to the mean-type mapping $\left(\mathscr{W}^{[\varphi]}, M\right)$, i.e.*

$$\mathscr{A}_p \circ \left(\mathscr{W}^{[\varphi]}, M\right) = \mathscr{A}_p ,$$

if, and only if,

$$M = \mathscr{W}^{[\psi]}$$

where, for some $c \in \mathbb{R}$,

$$\psi(x) = \frac{p}{1-p}\,(x - \varphi(x)) + c , \qquad x \in I .$$

Proof. If $\mathscr{A}_p$ is $\left(\mathscr{W}^{[\varphi]}, M\right)$ -invariant then, by the definition of the involved means, we have

$$p\left[\varphi(x)+y-\varphi(y)\right]+(1-p)M(x,y)=px+(1-p)y\,, \qquad x,y\in I\,,$$

whence, after simple calculations,

$$M(x,y)=\frac{p}{1-p}\left(x-\varphi(x)\right)+\left[y-\frac{p}{1-p}\left(y-\varphi(y)\right)\right]\,, \qquad x,y\in I\,,$$

and, consequently,

$$M(x,y)=\mathscr{W}^{[\psi]}(x,y)\,, \qquad x,y\in I\,,$$

where, by Property 36.2(*iii*), there is $c\in\mathbb{R}$ such that

$$\psi(x)=\frac{p}{1-p}\left(x-\varphi(x)\right)+c\,, \qquad x\in I\,.$$

The converse implication is easy to verify. □

Hence, applying the main result of [10] we obtain

Corollary 36.3. *Let $I\subset\mathbb{R}$ be an interval, $p\in(0,1)$. Suppose that $\varphi: I\to\mathbb{R}$ and $\mathrm{id}_I-\varphi$ are nondecreasing. If*

$$\psi(x)=\frac{p}{1-p}\left(x-\varphi(x)\right)+c\,, \qquad x\in I\,,$$

then $\psi: I\to\mathbb{R}$ and $\mathrm{id}_I-\psi$ are nondecreasing, and the sequence of iterates

$$\left(\left(\mathscr{W}^{[\varphi]},\mathscr{W}^{[\psi]}\right)^n\right)_{n\in\mathbb{N}}$$

of the mean-type mapping $\left(\mathscr{W}^{[\varphi]},\mathscr{W}^{[\psi]}\right)$ converges to the mean-type mapping $(\mathscr{A}_p,\mathscr{A}_p)$.

Corollary 36.4. *Let $I\subset\mathbb{R}$ be an interval. Suppose that $\varphi: I\to\mathbb{R}$ and $\mathrm{id}_I-\varphi$ are nondecreasing and $F: I^2\to\mathbb{R}$ is continuous on the diagonal $\{(x,x): x\in I\}$. Then the function F satisfies the functional equation*

$$F\left(\varphi(x)+y-\varphi(y),x-\varphi(x)+\varphi(y)\right)=F(x,y)\,, \qquad x,y\in I\,,$$

if, and only if, there is a continuous function $f: I\to\mathbb{R}$ such that

$$F(x,y)=f\left(\frac{x+y}{2}\right)\,, \qquad x,y\in I\,.$$

Proof. Putting $\psi(x)=x-\varphi(x)$ for $x\in I$, we can write the above functional equation in the form

$$F(x,y) = F\left(\mathscr{W}^{[\varphi]}(x,y), \mathscr{W}^{[\psi]}(x,y)\right), \qquad x,y \in I,$$

whence, by induction,

$$F(x,y) = F\left(\left(\mathscr{W}^{[\varphi]}(x,y), \mathscr{W}^{[\psi]}(x,y)\right)^n\right), \qquad x,y \in I, n \in \mathbb{N},$$

where

$$\left(\mathscr{W}^{[\varphi]}(x,y), \mathscr{W}^{[\psi]}(x,y)\right)^n = \left(\mathscr{W}^{[\varphi]}, \mathscr{W}^{[\psi]}\right)^n (x,y)$$

and $\left(\mathscr{W}^{[\varphi]}, \mathscr{W}^{[\psi]}\right)^n$ is the nth iterate of the mean-type mapping $\left(\mathscr{W}^{[\varphi]}, \mathscr{W}^{[\psi]}\right)$. Letting $n \to \infty$ and, applying the previous corollary for $p = 1/2$ and making use of the continuity of F, we obtain

$$F(x,y) = F\left(\frac{x+y}{2}, \frac{x+y}{2}\right) = f\left(\frac{x+y}{2}\right), \qquad x,y \in I,$$

where $f(x) := F(x,x)$ for $x \in I$. The converse implication is easy to verify. □

36.9 A generalization of the weighted quasi-arithmetic means

We begin this section with the following easy to verify

Remark 36.5. Let $J \subset \mathbb{R}$ be an interval and $\gamma : J \to \mathbb{R}$ be a continuous strictly monotonic function and let $I = \gamma(J)$. Suppose that $\varphi : I \to \mathbb{R}$ and and $\mathrm{id}_I - \varphi$ are nondecreasing. Then the function $M_{\gamma,\varphi} : J^2 \to J$,

$$M_{\gamma,\varphi}(x,y) := \gamma^{-1}\left(\varphi\left(\gamma(x)\right) + \gamma(y) - \phi(\varphi(y))\right), \qquad x,y \in J,$$

is a mean.

For $\varphi(x) = px$ with $p \in (0,1)$, the mean $M_{\gamma,\varphi}$ reduces to the weighted quasi-arithmetic mean.

We prove the following

Theorem 36.12. *Let $\gamma : [0,\infty) \to \mathbb{R}$ be differentiable strictly monotonic and suppose that a differentiable $\varphi : I \to \mathbb{R}$ and $\mathrm{id}_I - \varphi$ are nondecreasing in $I := \gamma([0,\infty))$. Then the mean $M_{\gamma,\varphi}$ is homogeneous if, and only if, there is $p \in [0,1]$ such that*

$$M_{\gamma,\varphi}(x,y) = px + (1-p)y, \qquad x,y \geq 0.$$

Proof. Suppose that $M_{\gamma,\varphi}$ is homogeneous, i.e.

$$M_{\gamma,\varphi}(tx,ty) = tM_{\gamma,\varphi}(x,y), \qquad x,y,t \geq 0.$$

My the definition of $M_{\gamma,\varphi}$ we can write this equality in the form

$$\gamma^{-1}\left(\varphi\left(\gamma(tx)\right)+\gamma(ty)-\phi\left(\varphi(ty)\right)\right)=t\gamma^{-1}\left(\varphi\left(\gamma(x)\right)+\gamma(y)-\phi\left(\varphi(y)\right)\right)$$

for $x,y,t\geq 0$. Differentiating both sides with respect to t and then setting $t=0$ and $p:=\varphi'[\gamma(0]$ gives $\gamma^{-1}\left(\varphi\left(\gamma(x)\right)+\gamma(y)-\phi\left(\varphi(y)\right)\right)=px+(1-p)y$ for $x,y\geq 0$.
□

36.10 Finite dimensional counterparts of $\mathscr{W}^{[\varphi]}$

In a similar way as Theorem 36.1 we can prove the following

Theorem 36.13. *Let $I\subset\mathbb{R}$ be an interval and $\varphi_1,\ldots,\varphi_k:I\to\mathbb{R}$. Then the function $M:I^k\to\mathbb{R}$ defined by*

$$M(x_1,\ldots,x_k):=\varphi_1(x_1)+\ldots+\varphi_k(x_k)$$

is a mean, i.e.

$$\min(x_1,\ldots,x_k)\leq M(x_1,\ldots,x_k)\leq \max M(x_1,\ldots,x_k)\,,\qquad x_1,\ldots,x_k\in I\,,$$

if, and only if, the functions $\varphi_1,\ldots,\varphi_k$ are nondecreasing and

$$\varphi_1(x)+\ldots+\varphi_k(x)=x\,,\qquad x\in I\,.$$

Thus, if the nondecreasing functions $\varphi_1,\ldots,\varphi_k:I\to\mathbb{R}$ are summing up to the identity, the function

$$\mathscr{W}^{[\varphi_1,\ldots,\varphi_{k-1}]}(x_1,\ldots,x_k)=\sum_{i=1}^{k-1}\varphi_i(x_i)+x_k-\sum_{i=1}^{k-1}\varphi_i(x_k)\,,\qquad x_1,\ldots,x_k\in I\,,$$

is a mean a k-variable mean in I.

Let us note the following easy to show

Theorem 36.14. *Let $I\subset\mathbb{R}$ be an interval. Suppose that the functions $\varphi_i:I\to\mathbb{R}$, $i=1,\ldots,k-1$, and*

$$\mathrm{id}\big|_I-\sum_{i=1}^{k-1}\varphi_i\,,$$

are nondecreasing. Then the following conditions are equivalent

(i) the mean $\mathscr{W}^{[\varphi_1,\ldots,\varphi_{k-1}]}$ is symmetric;
(ii) the mean $\mathscr{W}^{[\varphi_1,\ldots,\varphi_{k-1}]}$ is subadditive;
(iii) there are $p_i\geq 0$, $i=1,\ldots,k$, such that $\sum_{i=1}^{k}p_i=1$ and

$$\mathscr{W}^{[\varphi_1,\ldots,\varphi_{k-1}]}(x_1,\ldots,x_k)=\sum_{i=1}^{k}p_ix_i\,,\qquad x_1,\ldots,x_k\in I\,.$$

Remark 36.6. Let $I \subset \mathbb{R}$ be an interval. Suppose that the functions $\varphi_i, \psi_i : I \to \mathbb{R}$, $i = 1, ..., k-1$, and

$$\mathrm{id}\big|_I - \sum_{i=1}^{k-1} \varphi_i , \qquad \mathrm{id}\big|_I - \sum_{i=1}^{k-1} \psi_i$$

are nondecreasing. Then the following conditions are equivalent

(i) $\mathscr{W}^{[\varphi_1,...,\varphi_{k-1}]} \leq \mathscr{W}^{[\psi_1,...,\psi_{k-1}]}$;
(ii) $\mathscr{W}^{[\varphi_1,...,\varphi_{k-1}]} = \mathscr{W}^{[\psi_1,...,\psi_{k-1}]}$.

Proof. Assume that inequality (i) holds true. From the definition of $\mathscr{W}^{[\varphi_1,...,\varphi_{k-1}]}$ we have

$$\mathscr{W}^{[\varphi_1,...,\varphi_{k-1}]}(x_1,...,x_k) = \sum_{i=1}^{k-1} [\varphi_i(x_i) - \varphi_i(x_k)] + x_k , \qquad x_1,...,x_k \in I ,$$

and, similarly,

$$\mathscr{W}^{[\psi_1,...,\psi_{k-1}]}(x_1,...,x_k) = \sum_{i=1}^{k-1} [\psi_i(x_i) - \psi_i(x_k)] + x_k , \qquad x_1,...,x_k \in I .$$

Let us fix arbitrarily $j \in \{1,...,k-1\}$ and take $x, y \in I$. Hence, putting $x_i = x_k = y$ for all $i \in \{1,...,k-1\}$, $i \neq j$, in inequality (i) we get

$$\varphi_j(x) - \varphi_j(y) \leq \psi_j(x) - \psi_j(y) , \qquad x,y \in I ,$$

whence

$$\varphi_j(x) - \psi_j(x) \leq \varphi_j(y) - \psi_j(y) , \qquad x,y \in I .$$

Since $x, y \in I$ are arbitrary, it follows that

$$\varphi_j(x) - \psi_j(x) = \varphi_j(y) - \psi_j(y) , \qquad x,y \in I ,$$

for any $j \in \{1,...,k-1\}$ and $j \in \{1,...,k-1\}$. □

36.11 Final remark and some questions

For a fixed $p \in (0,1)$ denote the weighted arithmetic mean

$$\mathscr{A}_p(x,y) = px + (1-p)y , \qquad x,y \in \mathbb{R} .$$

Daróczy and Páles [4] observed the following identity

$$\mathscr{A}_p\left(\mathscr{A}_p\left(\frac{x+y}{2}, x\right), \mathscr{A}_p\left(y, \frac{x+y}{2}\right)\right) = \frac{x+y}{2} , \qquad x,y \in \mathbb{R} ,$$

which appears very useful in the theory of convex functions.

Note that if γ is continuous strictly increasing in an interval I and $p \in (0,1)$, then

$$\mathscr{A}_p^{[\gamma]}\left(\mathscr{A}_p^{[\gamma]}\left(\mathscr{A}^{[\gamma]},x\right),\mathscr{A}_p^{[\gamma]}\left(y,\mathscr{A}^{[\gamma]}\right)\right) = \mathscr{A}_p^{[\gamma]}(x,y)\,, \qquad x,y \in I\,,$$

which reduces to the previous identity for $\gamma = id|_I$.

In this connection the following problems arise.

Let $I \subset \mathbb{R}$ be an interval and let $\varphi : I \to \mathbb{R}$ and $\mathrm{id}_I - \varphi$ be strictly increasing.

Problem 36.1. Suppose that the following equation

$$\mathscr{W}^{[\varphi]}\left(\mathscr{W}^{[\varphi]}\left(\frac{x+y}{2},x\right),\mathscr{W}^{[\varphi]}\left(y,\frac{x+y}{2}\right)\right) = \frac{x+y}{2}\,, \qquad x,y \in I\,,$$

is satisfied. Does it imply that $\mathscr{W}^{[\varphi]}$ coincides with $\mathscr{A}_p$ for some $p \in (0,1)$?

Problem 36.2. Do there exist $\mathscr{W}^{[\varphi]}$, being not weighted quasi-arithmetic, and a quasi-arithmetic mean $\mathscr{A}^{[\gamma]}$ such that

$$\mathscr{W}^{[\varphi]}\left(\mathscr{W}^{[\varphi]}\left(\mathscr{A}^{[\gamma]}(x,y),x\right),\mathscr{W}^{[\varphi]}\left(y,\mathscr{A}^{[\gamma]}(x,y)\right)\right) = \mathscr{A}^{[\gamma]}(x,y)\,, \qquad x,y \in I\,?$$

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