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Aequationes Mathematicae

Quotient mean, its invariance with respect to a quasi-arithmetic mean-type mapping, and some applications

Janusz Matkowski

Abstract. Under some conditions on the functions f and g defined in a real interval I the function

$$Q^{[f,g]}(x,y) := \left(\frac{f}{g}\right)^{-1}\left(\frac{f(x)}{g(y)}\right)$$

is a strict mean in I. We examine the invariance of this mean with respect to the weighted quasi-arithmetic mean type mapping generated by f and g.

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1. Introduction

Let the real functions f and g be continuous, positive, and of different type of strict monotonicity in a real interval I. Then the function $Q^{[f,g]}: I^2 \to \mathbb{R}$ defined by

$$Q^{[f,g]}(x,y) := \left(\frac{f}{g}\right)^{-1}\left(\frac{f(x)}{g(y)}\right), \qquad x,y \in I,$$

is a strict mean in I, and we call it a *quotient mean*. In this note we examine its invariance with respect to the weighted quasi-arithmetic mean type mapping $(A_p^{[f]}, A_r^{[g]})$, where $A_p^{[f]}$ denotes the weighted quasi-arithmetic mean of the generator f and weight p. Theorem 1 says that $Q^{[f,g]}$ is $(A_p^{[f]}, A_r^{[g]})$-invariant, that is

$$Q^{[f,g]} \circ \left(A_p^{[f]}, A_r^{[g]}\right) = Q^{[f,g]}$$

iff the product fg is a constant function and $p + r = 1$.

As an application we effectively determine the limit of the sequence of the mean-type mapping $\left(A_p^{[f]}, A_r^{[g]}\right)$ (Theorem 2).

For $I = (0, \infty), f(x) = x, g(x) = \frac{1}{x}$ and $p = r = \frac{1}{2}$ we have $Q^{[f,g]} = G, A_p^{[f]} = A, A_r^{[g]} = H$, where G, A and H denote the geometric, arithmetic and harmonic means, respectively. Since the product fg is a constant function, Theorem 1 implies

$$G \circ (A, H) = G,$$

the known invariance of the geometric mean with respect to the mean-type mapping $(A, H) : (0, \infty)^2 \to (0, \infty)$. Applying this fact (cf. Theorem 2), we conclude that

$$\lim_{n \to \infty} (A, H)^n = (G, G),$$

where $(A, H)^n$ is the n-th iterate of the mapping (A, H).

Let us note that this invariance relation $G \circ (A, H) = G$ is equivalent to the classical Pythagorean harmony proportion

$$\frac{A}{G} = \frac{G}{H}.$$

An application in solving a functional equation is given (Corollary 3).

2. Result on invariance

Recall that a function $M : I^2 \to \mathbb{R}$ is called a *mean* in an interval $I \subseteq \mathbb{R}$, if

$$\min(x, y) \le M(x, y) \le \max(x, y), \qquad x, y \in I.$$

If for all $x, y \in I, x \neq y$, these inequalities are strict, M is called *strict*; and *symmetric*, if $M(x, y) = M(y, x)$ for all $x, y \in I$.

If M is a mean in I, then $M(J^2) = J$ for every subinterval $J \subseteq I$. Moreover M is *reflexive*, i.e.

$$M(x, x) = x, \qquad x \in I.$$

It is easy to see that every reflexive function $M : I^2 \to \mathbb{R}$ that is (strictly) increasing with respect to each variable is a (strict) mean in I.

For any continuous and strictly monotonic function $f : I \to \mathbb{R}$ and $p \in (0, 1)$, the function $M = A_p^{[f]} : I^2 \to I$,

$$A_p^{[f]}(x, y) := f^{-1}\left(pf(x) + (1 - p)f(y)\right), \qquad x, y \in I,$$

is a mean. $A_p^{[f]}$ is called a *weighted quasi-arithmetic mean*, f is called its *generator*, and p its weight.

The means of this type form one of the most important class of mean values. For a survey of other classes of means cf., for instance, Bullen–Mitrinović–Vasić [2], Bullen [3].

We begin with the following easy to verify

Remark 1. Let $I \subset \mathbb{R}$ be an interval. If $f, g : I \to (0, \infty)$ are continuous, strictly monotonic, and such that $\frac{f}{g}$ is one-to-one, then the function $Q^{[f,g]} : I^2 \to \mathbb{R}$ defined by

$$Q^{[f,g]}(x, y) := \left(\frac{f}{g}\right)^{-1} \left(\frac{f(x)}{g(y)}\right), \qquad x, y \in I,$$

is a strict mean in I if, and only if, either f is increasing and g is decreasing, or f is decreasing and g is increasing.

In the sequel, if $f, g : I \to (0, \infty)$ are continuous and of different types of strict monotonicity, the function $Q^{[f,g]}$ is called a *quotient mean* and the functions f and g its *generators*.

Clearly, in the definition of $Q^{[f,g]}$, without any loss of generality, we can assume that f is increasing and g is decreasing.

Remark 2. The mean $Q^{[f,g]}$ is symmetric iff the product fg is a constant function and, consequently iff,

$$Q^{[f,g]}(x, y) = f^{-1}\left(\sqrt{f(x)f(y)}\right) = g^{-1}\left(\sqrt{g(x)g(y)}\right), \qquad x, y \in I.$$

Let $I \subset \mathbb{R}$ be an interval and $M, N, K : I^2 \to I$ be means. A mean $K : I^2 \to I$ is called *invariant with respect to the mean-type mapping* $(M, N) : I^2 \to I^2$ (briefly, K is (M, N)-*invariant*), if

$$K(M(x, y), N(x, y)) = K(x, y), \quad x, y \in I,$$

(cf. [5], also [6–8]). The mean K is also referred to as the Gauss composition of means M and N (cf. Daróczy and Páles [4]). The invariant mean is useful when we are looking for the limit of the sequence of iterates of the mean-type mapping $(M, N) : I^2 \to I^2$.

In this context, the following result is a motivation for introducing quotient means.

Theorem 1. *Let $I \subset \mathbb{R}$ be an interval and $p, r \in (0, 1)$ be fixed. Suppose that the functions $f, g : I \to (0, \infty)$ are continuous functions, f strictly increasing, g strictly decreasing, and such that $\frac{f}{g}$ is one-to-one. Then the following conditions are equivalent*

(i) *the quotient mean $Q^{[f,g]}$ is invariant with respect to the mean-type mapping $(A_p^{[f]}, A_r^{[g]})$, that is*

$$Q^{[f,g]} \circ \left(A_p^{[f]}, A_r^{[g]}\right) = Q^{[f,g]}; \tag{2.1}$$

(ii) *the product fg is a constant function and*

$$p + r = 1.$$

(iii) *for all $x, y \in I$,*

$$Q^{[f,g]}(x,y) = f^{-1}\left(\sqrt{f(x)f(y)}\right), \quad A_r^{[g]}(x,y) = f^{-1}\left(\frac{f(x)f(y)}{pf(x)+(1-p)f(y)}\right).$$

Proof. Assume that condition (i) holds true, that is $Q^{[f,g]}$ is invariant with respect to the mean-type mapping $(A_p^{[f]}, A_r^{[g]})$. From (2.1), by the definitions of $Q^{[f,g]}, A_p^{[f]}$ and $A_r^{[g]}$, we get

$$\frac{pf(x)+(1-p)f(y)}{rg(x)+(1-r)g(y)} = \frac{f(x)}{g(y)}, \qquad x, y \in I,$$

which reduces to the equality

$$g(y)[(p+r-1)f(x)+(1-p)f(y)] = rf(x)g(x), \qquad x, y \in I. \qquad (2.2)$$

Since the right-hand side does not depend on y, taking here an arbitrary fixed x, we get

$$g(y) = \frac{c}{b+f(y)}, \qquad y \in I, \qquad (2.3)$$

for some $c > 0$ and $b \in \mathbb{R}$. Setting this function into (2.2) we get

$$\frac{c}{b+f(y)}\left[(p+r-1)f(x)+(1-p)f(y)\right] = rf(x)\frac{c}{b+f(x)}, \quad x, y \in I,$$

which reduces to the equality

$$[f(x)-f(y)]\left[(1-p-r)f(x)-b(p-1)\right] = 0 \quad x, y \in I.$$

The assumptions on f imply that $b = 0$ and $p + r = 1$, whence, by (2.3),

$$f(x)g(x) = c, \qquad x \in I.$$

Thus condition (i) implies condition (ii).

Assume condition (ii). Thus $g = c/f$ for a nonzero constant c and $r = 1-p$. Hence, by the definitions of $Q^{[f,g]}$ and $A_r^{[g]}$ we get, for all $x, y \in I$,

$$Q^{[f,g]}(x,y) = f^{-1}\left(\sqrt{f(x)f(y)}\right), \quad A_r^{[g]}(x,y) = f^{-1}\left(\frac{f(x)f(y)}{pf(x)+(1-p)f(y)}\right),$$

that is condition (iii) holds true.

Now, simple calculations show that condition (iii) implies (i). This completes the proof. □

Remark 3. Taking $I = (0, \infty), f(x) = x, g(x) = \frac{1}{x}$ and $p \in (0,1)$, we get,

$$Q^{[f,g]}(x,y) = G(x,y) = \sqrt{xy},$$

$$A_p^{[f]}(x,y) = px + (1-p)y, \quad A_{1-p}^{[g]}(x,y) = \frac{xy}{px+(1-p)y},$$

for all $x, y > 0$. Since $f(x)g(x) = 1$ for all $x > 0$, we have

$$Q^{[f,g]}\left(A_p^{[f]}(x,y), A_{1-p}^{[g]}(x,y)\right) = \sqrt{(px + (1-p)y)\left(\frac{xy}{px + (1-p)y}\right)} = \sqrt{xy}$$
$$= Q^{[f,g]}(x,y),$$

for all $x, y > 0$.

For $p = \frac{1}{2}$ we hence get $G \circ (A, H) = G$.

3. An application

From [8] (cf. also [5,7]) we quote the following

Theorem 2. *Let $I \subset \mathbb{R}$ be an interval and let $M, N : I^2 \to I$ be continuous means. If for every point $(x, y) \in I^2, x \neq y$, we have*

$$0 < \max(M(x,y), N(x,y)) - \min(M(x,y), N(x,y)) < \max(x,y) - \min(x,y), \tag{3.1}$$

then

(1) *there exists a unique continuous (M, N)-invariant mean $K : I^2 \to I$, that is*

$$K(M(x,y), N(x,y)) = K(x,y), \qquad x, y \in I;$$

(2) *the sequence $(M, N)^n, n \in \mathbb{N}$, of the iterates of the mean-type mapping $(M, N) : I^2 \to I^2$ converges to the mean-type mapping (K, K), that is*

$$\lim_{n\to\infty} (M, N)^n(x,y) = (K(x,y), K(x,y)), \qquad (x,y) \in I^2.$$

Applying Theorem 1 and 2 we obtain the following

Corollary 1. *Let $I \subset \mathbb{R}$ be an interval and $p \in (0,1)$ be fixed. Suppose that $f : I \to (0, \infty)$ is continuous and f strictly increasing, and $g = c/f$ for a constant $c > 0$. Then the sequence $\left((A_p^{[f]}, A_{1-p}^{[g]})^n\right)_{n\in\mathbb{N}}$ of the iterates of the mean-type mapping $(A^{[f]}, A^{[g]})$ converges pointwise and*

$$\lim_{n\to\infty} \left(A_p^{[f]}, A_{1-p}^{[g]}\right)^n = \left(Q^{[f,g]}, Q^{[f,g]}\right),$$

where

$$Q^{[f,g]}(x,y) = f^{-1}\left(\sqrt{f(x)f(y)}\right), \quad x, y \in I.$$

Proof. Since the means $M := A^{[f]}$ and $N := A^{[g]}$ are strict, assumption (3.1) of Theorem 2 is satisfied. □

By Remark 3 we hence obtain

Corollary 2. *Let $f(x) = x, g(x) = \frac{1}{x}$ for $x > 0$, and $p \in (0, 1)$. Then*

$$A_p^{[f]}(x, y) = px + (1 - p)y, \quad A_{1-p}^{[g]}(x, y) = \frac{xy}{px + (1 - p)y}, \quad x, y > 0,$$

and, for all $x, y > 0$,

$$\lim_{n \to \infty} \left(A_p^{[f]}(x, y), A_{1-p}^{[g]}(x, y)\right)^n = (G(x, y), G(x, y)).$$

Corollary 3. *Let $p \in (0, 1)$ be fixed. Suppose that a function $F : (0, \infty)^2 \to \mathbb{R}$ is continuous on the diagonal $\{(x, x) : x > 0\}$. Then F satisfies the functional equation*

$$F\left(px + (1 - p)y, \frac{xy}{px + (1 - p)y}\right) = F(x, y), \qquad x, y > 0, \qquad (3.2)$$

if, and only if, there is a continuous function $\varphi : (0, \infty) \to \mathbb{R}$ such that

$$F(x, y) = \varphi(\sqrt{xy}), \qquad x, y > 0.$$

Proof. Assume that F satisfies Eq. (3.2). From (3.2), by induction, we have

$$F(x, y) = F\left(\left(px + (1 - p)y, \frac{xy}{px + (1 - p)y}\right)^n\right), \quad x, y > 0; n \in \mathbb{R}.$$

From Corollary 2, letting here $n \to \infty$, by the continuity of F on the diagonal, we get

$$F(x, y) = F(\sqrt{xy}, \sqrt{xy}), \qquad x, y > 0,$$

whence, setting

$$\varphi(t) := F(t.t), \qquad t > 0,$$

we obtain $F(x, y) = \varphi\left(\sqrt{xy}\right)$ for all $x, y > 0$.

The converse implication is easy to verify. □

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Janusz Matkowski
Faculty of Mathematics, Computer Science and Econometrics
University of Zielona Góra
Podgórna 50
65246 Zielona Gora
Poland
e-mail: J.Matkowski@wmie.uz.zgora.pl

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