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A FUNCTIONAL EQUATION CHARACTERIZING HOMOGRAPHIC FUNCTIONS

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Abstract. Some functional equations related to homographic functions and their characterization are presented.

1. Introduction

If $I \subset \mathbb{R}$ is an interval and $f\colon I \to \mathbb{R}$ is a homographic function

$$f(x) = \frac{ax+b}{cx+d}, \quad x \in I,$$

$(ad - bc \neq 0)$ then, it is easy to verify that

$$(*) \qquad \left(\frac{f(x)-f(y)}{x-y}\right)^2 = f'(x)f'(y), \quad x, y \in I,\ x \neq y,$$

cf. [1] where this equation has appeared in a problem related to convex functions.

Replacing here $\sqrt{f'}$ by an arbitrary function g we get the functional equation

$$\frac{f(x)-f(y)}{x-y} = g(x)g(y), \quad x, y \in I,\ x \neq y,$$

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with two unknown functions f and g. We show that, without any regularity assumptions, this equation characterizes the homographic function and their derivative (Corollary 1).

In Section 1, we consider the functional equation

$$\frac{f(x)-f(y)}{x-y}=pg(x)g(y), \quad x,y\in X,\ x\neq y,$$

assuming that X is an arbitrary subset of $\mathbb{K}\in\{\mathbb{R},\mathbb{C}\}$ such that $\operatorname{card} X\geq 3$ and $f,g\colon X\to\mathbb{K}$ are the unknown functions. Theorem 1 gives the general solution for $p=1$, and Corollary 1 for $p\neq 0$.

In Section 2, Theorem 2 describes the general solution of the functional equation with four unknown functions

$$\frac{f(x)-F(y)}{x-y}=g(x)G(y), \quad x,y\in X,\ x\neq y.$$

A remark on a general solution of the functional equation

$$\frac{f(x)-F(y)}{h(x)-h(y)}=g(x)G(y), \quad x,y\in X,\ x\neq y,$$

with five unknown functions f,F,g,G,h defined on an arbitrary set such X such that $\operatorname{card} X\geq 3$ ends up the paper.

Another, completely different, approach in a characterization of the homographic functions, more closer to the invariance of double ratio of four points, is implicitly given in [1].

2. Functional equation with two functions

The main result of this section reads as follows:

THEOREM 1. *Let $X\subset\mathbb{K}$ be a set such that* $\operatorname{card} X\geq 3$. *The functions $f,g\colon X\to\mathbb{K}$ satisfy the functional equation*

$$\frac{f(x)-f(y)}{x-y}=g(x)g(y), \quad x,y\in X,\ x\neq y, \tag{1}$$

if, and only if, either

f *is an arbitrary constant and* g *is the zero function;*

or

$$f(x)=\frac{ax+b}{cx+d},\quad g(x)=\frac{1}{cx+d},\quad x\in X,$$

for some $a,b,c,d\in\mathbb{K}$ *such that* $ad-bc=1$.

Proof. Assume that $f,g\colon X\to\mathbb{R}$ satisfy equation (1).

If there is $y\in X$ such that $g(y)=0$, then, by (1), $f(x)=f(y)$ for all $x\in X$, that f is constant. It follows that $g(x)=0$ for all $x\in X$. Obviously, if f is constant and $g=0$ then equation (1) is satisfied.

Now we can assume that $g(x)\neq 0$ for all $x\in X$. From (1) we have

$$f(x)-f(y)=g(x)g(y)(x-y),\quad x,y\in X,\ x\neq y.$$

Since $f(x)-f(y)=[f(x)-f(z)]+[f(z)-f(y)]$, we hence get

$$g(x)g(y)\,(x-y)=g(x)g(z)\,(x-z)+g(z)g(y)\,(z-y)$$

for all $x,y,z\in X$, $x\neq y\neq z\neq x$, or equivalently, dividing both sides by $g(x)g(y)g(z)$,

$$\frac{x-y}{g(z)}=\frac{x-z}{g(x)}+\frac{z-y}{g(y)},\quad x,y,z\in X,\ x\neq y\neq z\neq x,$$

whence, for all $x,y,z\in X$, $x\neq y\neq z\neq x$,

$$\frac{1}{g(z)}=\frac{1}{x-y}\left[\left(\frac{1}{g(x)}-\frac{1}{g(y)}\right)z+\left(\frac{x}{g(x)}-\frac{y}{g(y)}\right)\right].$$

Since the right side does not depend on z and, by assumption, $\operatorname{card}X\geq 3$, it follows that there are $c,d\in\mathbb{K}$ such that

$$\frac{1}{g(z)}=cz+d,\quad z\in X,$$

whence

$$g(x)=\frac{1}{cx+d},\quad x\in X. \tag{2}$$

Setting this function into equation (1) we get

$$f(x)=\frac{1}{cy+d}\frac{x-y}{cx+d}+f(y),\quad x,y\in X,\ x\neq y,$$

whence, as the right side does not depend on y, we conclude that

$$f(x)=\frac{ax+by}{cx+d}, \quad x\in X, \tag{3}$$

for some $a,b\in\mathbb{K}$.

Substituting the functions (2) and (3) into (1) we obtain

$$\frac{f(x)-f(y)}{x-y}=\frac{ad-bc}{(cx+d)\,(cy+d)}=(ad-bc)\,g(x)g(y), \quad x,y\in X,\ x\neq y,$$

which implies that $ad-bc=1$. This completes the proof. □

Corollary 1. *Let* $X\subset\mathbb{K}$ *be a set such that* $\operatorname{card}X\geq 3$ *and let* $p\in\mathbb{K}\backslash\{0\}$ *be fixed. The functions* $f,g\colon X\to\mathbb{K}$ *satisfy the functional equation*

$$\frac{f(x)-f(y)}{x-y}=pg(x)g(y), \quad x,y\in X,\ x\neq y, \tag{4}$$

if, and only if, either

f *is an arbitrary constant and* g *is the zero function;*

or

$$f(x)=\frac{ax+b}{cx+d}, \quad g(x)=\frac{1}{cx+d}, \quad x\in X,$$

for some $a,b,c,d\in\mathbb{K}$ *such that*

$$ad-bc=p.$$

Proof. It is enough to apply Theorem 1 with f replaced by f/p. □

Corollary 2. *Let* $I\subset\mathbb{R}$ *be a an interval. A differentiable function* $f\colon I\to\mathbb{R}$ *satisfies equation (*):*

$$\left(\frac{f(x)-f(y)}{x-y}\right)^2=f'(x)f'(y), \quad x,y\in J,\ x\neq y,$$

if, and only if, either f *constant or*

$$f(x)=\frac{ax+b}{cx+d}, \quad x\in I,$$

for some $a, b, c, d \in \mathbb{K}$ *such that*

$$ad - bc \neq 0.$$

PROOF. Obviously, f is constant iff $f'(x) = 0$ for some $x \in I$. Therefore it is enough consider the case when f' is of the constant sign in I. Without any loss of generality, we can assume that f' is positive in I. Then, clearly,

$$\frac{f(x) - f(y)}{x - y} > 0, \quad x, y \in J, \ x \neq y.$$

Hence, setting $g := \sqrt{f'}$ we get the functional equation (1). Applying Theorem 1 we obtain

$$f(x) = \frac{ax + b}{cx + d}, \quad x \in I.$$

Now it easy to verify that f satisfies equation (*). □

3. Functional equation with four unknown functions

Applying Theorem we shall prove the following

THEOREM 2. *Let* $X \subset \mathbb{K}$ *be a set such that* $\operatorname{card} X > 3$. *The functions* $f, F, g, G\colon X \to \mathbb{K}$ *satisfy the functional equation*

$$\frac{f(x) - F(y)}{x - y} = g(x)G(y), \quad x, y \in X, \ x \neq y, \tag{5}$$

if, and only if, one of the following cases occurs:

(i) *for some* $a, b, c, d, m \in \mathbb{C}$ *such that* $ad - bc \neq 0$,

$$f(x) = F(x) = \frac{ax + b}{cx + d}, \quad g(x) = \frac{G(x)}{ac - bd} = \frac{1}{cx + d}, \quad x \in X;$$

(ii) *the functions* f *and* F *are constant,* $f = F$, $g(x) = 0$ *for all* $x \in X$ *and* G *is arbitrary;*

(iii) *the functions* f *and* F *are constant,* $f = F$, $G(x) = 0$ *for all* $x \in X$ *and* g *is arbitrary.*

Proof. Assume that $g(x)G(x) \neq 0$ for all $x \in X$. Interchanging x and y in (5) and we get

$$\frac{f(y) - F(x)}{y - x} = g(y)G(x), \quad x, y \in X,\ x \neq y.$$

Dividing the respective sides of these two equations we obtain

$$\frac{f(x) - F(y)}{F(x) - f(y)} = \frac{m(x)}{m(y)}, \quad x, y \in X,\ x \neq y, \tag{6}$$

where

$$m(x) := \frac{g(x)}{G(x)}, \quad x \in X.$$

Setting an arbitrary chosen $y = y_0$ in (6) we get

$$F(x) = \frac{\alpha f(x) + \beta}{m(x)} + \gamma, \quad x \in X \setminus \{y_0\}, \tag{7}$$

for some $\alpha, \beta, \gamma \in \mathbb{R}$. From (5) and (6) we obtain

$$\frac{f(x) - \frac{\alpha f(y)+\beta}{m(y)} - \gamma}{\frac{\alpha f(x)+\beta}{m(x)} + \gamma - f(y)} = \frac{m(x)}{m(y)}, \quad x, y \in X \setminus \{y_0\},\ x \neq y,$$

whence, after simplification,

$$f(x)\left[m(y) - \alpha\right] = \left[\alpha + \gamma - f(y)\right] m(x) + \gamma m(y) + 2\beta \tag{8}$$

for all $x, y \in X \setminus \{y_0\}$, $x \neq y$.

Consider the case when

$$m(x) = \alpha, \quad x \in X \setminus \{y_0\}.$$

Of course $\alpha \neq 0$. In this case (8) implies that $\gamma = -\frac{\beta}{\alpha}$ and, from (7),

$$F(x) = f(x), \quad x \in X \setminus \{y_0\}.$$

Thus from (5) we get the functional equation

$$\frac{f(x) - f(y)}{x - y} = g(x)G(y), \quad x, y \in X \setminus \{y_0\}, \tag{9}$$

with four unknown functions. The symmetry of the left-hand side implies that

$$\frac{g(x)}{G(x)} = \frac{g(y)}{G(y)} = a, \quad x, y \in X \setminus \{y_0\}, \ x \neq y,$$

whence

$$G(y) = \frac{1}{a} g(y), \quad y \in X \setminus \{y_0\},$$

and from (9) we get

$$\frac{f(x) - f(y)}{x - y} = \frac{1}{\alpha} g(x) g(y), \quad x, y \in X \setminus \{y_0\}, \ x \neq y.$$

Repeating this reasoning with y_0 replaced by y_0', $y_0' \neq y_0$, we conclude that

$$\frac{f(x) - f(y)}{x - y} = \frac{1}{\alpha} g(x) g(y), \quad x, y \in X \setminus \{y_0'\}, \ x \neq y.$$

Both equations imply that

$$\frac{f(x) - f(y)}{x - y} = \frac{1}{\alpha} g(x) g(y), \quad x, y \in X, \ x \neq y.$$

Applying Corollary 1 with $p := \frac{1}{\alpha}$ we obtain the "if" part of our result.

To finish the proof it is enough to show that the function m must coincide with the constant α. For an indirect argument assume that there is $y_1 \in X$ such that

$$m(y_1) \neq \alpha.$$

Then, from (8) we get

$$f(x) = Am(x) + B, \quad x \in X \setminus \{y_0\}, \tag{10}$$

for some $A, B \in \mathbb{R}$.

We shall show that in this case m and f are constant functions in $X \setminus \{y_0\}$. Assume first that $A = 0$. Then $f(x) = B$ for all $x \in X \setminus \{y_0\}$ and from (7)

$$F(x) = \frac{l}{m(x)} + \gamma, \quad x \in X \setminus \{y_0\}, \tag{11}$$

where $l := \alpha B + \beta$. If l were zero then we would have $F(x) = \gamma$ for all all $x \in X \setminus \{y_0\}$, and from (7),

$$\frac{B-\gamma}{\gamma - B} = \frac{m(x)}{m(y)}, \quad x, y \in X \setminus \{y_0\}, \ x \neq y,$$

that is $m(y) = -m(x)$ for all $x, y \in X \setminus \{y_0\}$, $x \neq y$. This is impossible as the set $X \setminus \{y_0\}$ has at least three points and m cannot disappear at any point. Thus $l \neq 0$.Put $C := B - \gamma$. Setting the function (11) into (6) we get

$$\frac{C - \frac{l}{m(y)}}{\frac{l}{m(x)} - C} = \frac{m(x)}{m(y)}, \quad x, y \in X, \ x \neq y,$$

whence

$$C\left[m(x) + m(y)\right] = 2l, \quad x, y \in X \setminus \{y_0\}, \ x \neq y.$$

Since $l \neq 0$, it follows that m is a constant function in $X \setminus \{y_0\}$.

Now we can assume that $A \neq 0$. Setting f given by (10) into (8), after simple calculations, we get

$$m(x)\left[2Am(y) - \alpha A - \alpha + \beta - \gamma\right] = \gamma m(y) - \beta m(y) + \alpha B + 2\beta, \ \ x, y \in X \setminus \{y_0\}.$$

This equality implies that m is a constant function. Indeed, if $(X \setminus \{y_0\}) \ni x \to m(x)$ were not constant then we would have

$$2Am(y) - \alpha A - \alpha + \beta - \gamma = 0, \quad y \in X \setminus \{y_0\},$$

and, as $A \neq 0$,

$$m(y) = \frac{\alpha A + \alpha - \beta + \gamma}{2A}, \quad y \in X \setminus \{y_0\},$$

that is a contradiction.

Thus the functions m and, by (10), f are constant in $X \setminus \{y_0\}$. In view of (7) also F is constant. Then $D := f - F$ is a constant and, from (5), $D \neq 0$. From (6) we get

$$m(y) = -m(x), \quad x, y \in X \setminus \{y_0\}, \ x \neq y,$$

that is impossible.

Now consider the case when $g(x)G(x) = 0$ for some $x \in X$.

Let $Z_G := \{x \in X\colon G(x) = 0\}$. Suppose that $Z_G \neq \emptyset$ and assume that $y_0 \in Z_G$. Then, by equation (5), $f(x) = F(y_0) =: \gamma$ for all $x \in X$, so f is a constant function. Similarly, if there is $x_0 \in Z_g$ then $F(x) = f(x_0)$ for all $x \in X$. Moreover $F = f$ on $Z_G \cup Z_g$. From (5) we have

$$\frac{\gamma - F(y)}{x - y} = g(x)G(y), \quad x, y \in X,\ x \neq y,$$

and, after interchanging a and y,

$$\frac{\gamma - F(y)}{y - x} = g(y)G(x), \quad x, y \in X,\ x \neq y$$

Putting $Y := X \setminus (Z_G \cup Z_g)$ and

$$m(x) := \frac{g(x)}{G(x)}, \quad x \in Y,$$

we hence get

$$\frac{\gamma - F(y)}{F(x) - \gamma} = \frac{m(x)}{m(y)}, \quad x, y \in Y,\ x \neq y,$$

that is

$$m(x)\left[F(x) - \gamma\right] = m(y)\left[\gamma - F(y)\right], \quad x, y \in Y,\ x \neq y,$$

whence, for a constant β,

$$F(x) = \frac{\beta}{m(x)} + \gamma \ \text{ for all } x \in Y, \quad \text{and} \quad F(x) = \frac{-\beta}{m(x)} + \gamma, \quad x \in Y.$$

It follows that $\beta = 0$ and, consequently, $F(x) = \gamma$ for all $x \in Y$. Thus we have shown that F and f are constant and equal on X. Hence, setting these functions into (5) we get $g(x)G(y) = 0$ for all $x, y \in X$, $x \neq y$. It follows that either g or G is the zero function. The proof is completed. □

4. Remark on a functional equation with five unknown functions

Applying Theorem 2 we get the following

REMARK 1. *Let X be an arbitrary set such that* $\operatorname{card} X > 3$. *The functions $f, F, g, G, h\colon X \to \mathbb{K}$ where h is one-to-one satisfy the functional equation*

$$\frac{f(x)-F(y)}{h(x)-h(y)}=g(x)G(y), \quad x,y\in X,\ x\neq y,$$

if, and only if, one of the following cases occurs:

(i) *for some* $a,b,c,d,m\in\mathbb{C}$ *such that* $ad-bc\neq 0$,

$$f(x)=F(x)=\frac{ah(x)+b}{ch(x)+d}, \quad g(x)=\frac{G(x)}{ac-bd}=\frac{1}{ch(x)+d}, \quad x\in X,$$

(ii) *the functions* f *and* F *are constant,* $f=F$, $g(x)=0$ *for all* $x\in X$ *and* G *is arbitrary;*

(iii) *the functions* f *and* F *are constant,* $f=F$, $G(x)=0$ *for all* $x\in X$ *and* g *is arbitrary.*

To get this remark it is enough to apply Theorem 2 to the functional equation

$$\frac{\left(f\circ h^{-1}\right)(x)-\left(F\circ h^{-1}\right)(y)}{x-y}=\left(g\circ h^{-1}\right)(x)\left(G\circ h^{-1}\right)(y),$$

$x,y\in h\,(X)$, $x\neq y$.

Reference

[1] Matkowski J., *Generalized convex functions and a solution of a problem of Zs. Páles*, Publ. Math. Debrecen **73** (2008), no. 3–4, 421–460.

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