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POWER MEANS GENERATED BY SOME MEAN-VALUE THEOREMS

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ABSTRACT. According to a new mean-value theorem, under the conditions of a function f ensuring the existence and uniqueness of Lagrange's mean, there exists a unique mean M such that

$$\frac{f(x)-f(y)}{x-y} = M\left(f'(x), f'(y)\right).$$

The main result says that, in this equality, M is a power mean if, and only if, M is either geometric, arithmetic or harmonic. A Cauchy relevant type result is also presented.

INTRODUCTION

In a recent paper [4] the following counterpart of the Lagrange mean-value theorem has been proved. *If a real function f defined on an interval $I \subset \mathbb{R}$ is differentiable, and f' is one-to-one, then there exists a unique mean function $M : f'(I) \times f'(I) \to f'(I)$, such that*

$$\frac{f(x)-f(y)}{x-y} = M\left(f'(x), f'(y)\right), \qquad x, y \in I,\ x \neq y.$$

In this connection the following problem arises. Given a mean M, determine all differentiable real functions f such that

$$\frac{f(x)-f(y)}{x-y} = M\left(f'(x), f'(y)\right), \qquad x, y \in I,\ x \neq y. \tag{1}$$

In the case when M is the geometric mean this equation has appeared in [3] and was useful in solving an open problem related to convex functions (cf. Remark 5).

In the first section we consider equation (1) in the case when $M = M^{[\varphi]}$, where

$$M^{[\varphi]}(u, v) = \varphi^{-1}\left(\frac{\varphi(u)+\varphi(v)}{2}\right), \qquad u, v \in J,$$

and $\varphi : f'(I) \to \mathbb{R}$ is a continuous and strictly monotonic function; so M is a quasi-arithmetic mean of a generator φ. Assuming three times continuous differentiability of f, and twice continuous differentiability of φ, we give some necessary conditions for equality (1) (Theorem 1). Applying this result, in the next section we give a

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complete solution of the problem in the case when $M^{[\varphi]}$ is positively homogeneous, that is, when either $\varphi(t) = At^p + B$ or $\varphi(t) = A\log t + B$ for some real p, A, B such that $A \neq 0 \neq p$. Then $M = M^{[p]} : (0,\infty)^2 \to (0,\infty)$ is a power mean, that is,

$$M^{[p]}(u,v) := \begin{cases} \left(\frac{u^p+v^p}{2}\right)^{1/p} & \text{if} \quad p \neq 0, \\ \sqrt{uv} & \text{if} \quad p = 0. \end{cases}$$

The main result (Theorem 2) says that equality (1) with $M = M^{[p]}$ holds if, and only if, the mean M is either arithmetic ($M^{[1]}(u,v) = \frac{u+v}{2}$), geometric ($M^{[0]}(u,v) = \sqrt{uv}$) or harmonic ($M^{[-1]}(u,v) = \frac{2uv}{u+v}$).

Assume that the functions $f, g : I \to \mathbb{R}$ satisfy the conditions ensuring the existence and uniqueness of the classical Cauchy mean-value. Then (cf. [4]) there exists a unique mean $M : J^2 \to J$, with $J := \frac{f}{g'}'(I)$, such that

$$\frac{f(x)-f(y)}{g(x)-g(y)} = M\left(\frac{f'(x)}{g'(x)}, \frac{f'(y)}{g'(y)}\right), \qquad x, y \in I,\ x \neq y.$$

Applying Theorem 2, we determine all power means M and the functions f, g satisfying this equation.

1. The case when M is quasi-arithmetic

In this section we prove:

Theorem 1. *Let $I, J \subset \mathbb{R}$ be intervals. Suppose that*
$f : I \to \mathbb{R}$ is three times continuously differentiable, $f''(x) \neq 0$ for $x \in I$;
$\varphi : J \to \mathbb{R}$ is twice continuously differentiable, and $\varphi'(u) \neq 0$ for $u \in J$.
If

$$\frac{f(x)-f(y)}{x-y} = M^{[\varphi]}\left(f'(x), f'(y)\right), \qquad x, y \in I,\ x \neq y, \tag{2}$$

then there exists $C \in \mathbb{R}$, $C \neq 0$, such that

$$\varphi'\left(f'(x)\right) = \frac{C}{f''(x)^{2/3}}, \qquad x \in I, \tag{3}$$

and

(4)
$$f''(x)\left(f'(x) - \frac{f(x)-f(y)}{x-y}\right)^3 = f''(y)\left(\frac{f(x)-f(y)}{x-y} - f'(y)\right)^3, \qquad x, y \in I, x \neq y.$$

Proof. Without any loss of generality we can assume that $\varphi'(x) > 0$ in J. Suppose that (2) holds true. Then, from the definition of the quasi-arithmetic mean,

$$2\varphi\left(\frac{f(x)-f(y)}{x-y}\right) = \varphi\left(f'(x)\right) + \varphi\left(f'(y)\right), \qquad x, y \in I,\ x \neq y.$$

Differentiating both sides, first with respect to x and then with respect to y, we get

$$2\varphi'\left(\frac{f(x)-f(y)}{x-y}\right)\frac{f'(x)(x-y) - f(x) + f(y)}{(x-y)^2} = \varphi'\left(f'(x)\right)f''(x), \tag{5}$$

$$2\varphi'\left(\frac{f(x)-f(y)}{x-y}\right)\frac{f'(y)(y-x) - f(y) + f(x)}{(x-y)^2} = \varphi'\left(f'(y)\right)f''(y) \tag{6}$$

for all $x, y \in I$, $x \neq y$. Subtracting the respective sides of these equalities and dividing the obtained differences by $x - y$ we have

$$2\varphi'\left(\frac{f(x)-f(y)}{x-y}\right)\frac{[f'(x)+f'(y)](x-y)-2[f(x)-f(y)]}{(x-y)^3} = \frac{\varphi'(f'(x))\,f''(x)-\varphi'(f'(y))\,f''(y)}{x-y} \tag{7}$$

for all $x, y \in I$, $x \neq y$. Applying L'Hospital's rule (or Taylor's theorem) we easily get

$$\lim_{y\to x}\frac{[f'(x)+f'(y)](x-y)-2[f(x)-f(y)]}{(x-y)^3} = \frac{f'''(x)}{6}, \qquad x \in I.$$

Hence, letting $y \to x$ in (7), we obtain

$$2\varphi'(f'(x))\,\frac{f'''(x)}{6} = \varphi''(f'(x))\,[f''(x)]^2 + \varphi'(f'(x))\,f'''(x),$$

whence

$$3\varphi''(f'(x))\,[f''(x)]^2 + 2\varphi'(f'(x))\,f'''(x) = 0, \qquad x \in I.$$

Assume first that $f''(x) \neq 0$ for all $x \in I$. Then, by the Darboux property of a derivative, f'' is of a constant sign in I. Dividing both sides by $f''(x)\varphi'(f'(x))$ we hence get

$$3\frac{\varphi''(f'(x))}{\varphi'(f'(x))}f''(x) + 2\frac{f'''(x)}{f''(x)} = 0, \qquad x \in I,$$

or, equivalently,

$$\left(3\log\varphi'(f'(x)) + 2\log f''(x)\right)' = 0, \qquad x \in I,$$

whence, after simple calculation,

$$\varphi'(f'(x)) = \frac{C}{f''(x)^{2/3}}, \qquad x \in I,$$

for some $C \in \mathbb{R}$, $C \neq 0$.

Hence, dividing the respective sides of (5) and (6) (by the assumption it can be done), we obtain

$$\frac{f'(x)(x-y)-f(x)+f(y)}{f'(y)(y-x)-f(y)+f(x)} = \left(\frac{f''(y)}{f''(x)}\right)^{1/3}, \qquad x, y \in I,\ x \neq y,$$

which implies (4). This completes the proof. □

Remark 1. The function $f(x) = \alpha x + \beta$, $x \in I$, satisfies equation (1) with an arbitrary mean $M : J^2 \to J$. Therefore in Theorem 1 we assume that f' is not a constant function.

Remark 2. From equation (1) we have

$$f(x) - f(y) = M(f'(x), f'(y))\,(x-y), \qquad x, y \in I.$$

Since $f(x) - f(y) = [f(x) - f(z)] + [f(z) - f(y)]$, for all $x, y, z \in I$, we get

$$M(f'(x), f'(y))\,(x-y) = M(f'(x), f'(z))\,(x-z) + M(f'(z), f'(y))\,(z-y).$$

Assuming that f' is one-to-one and putting $h := (f')^{-1}$, we hence obtain

$$M(u,v)\,[h(u)-h(v)] = M(u,w)\,[h(u)-h(w)] + M(w,v)\,[h(w)-h(v)]$$

for all $u, v, w \in J := f'(I)$.

If $f(x) = \frac{a}{2}x^2 + bx + c$ for some $a, b, c \in \mathbb{R}$, then $h(u) = \frac{u-b}{a}$. It is easy to verify that h and $M(u, v) = \frac{u+v}{2}$ satisfy this equality.

2. The case when M is a power mean

In this part we assume that M in equation (1) is a power mean. The main result reads as follows.

Theorem 2. *Let $I \subset \mathbb{R}$ be an interval. Suppose that $f : I \to \mathbb{R}$ is differentiable, $f'(x) > 0$ for $x \in I$, and f' is not constant in I. Then*

$$\frac{f(x) - f(y)}{x - y} = M^{[p]}\left(f'(x), f'(y)\right), \qquad x, y \in I,\ x \neq y, \tag{8}$$

for some $p \in \mathbb{R}$ if, and only if, one of the following cases occurs:

- **(i):** $p = 0$ *(that is, $M^{[p]}$ is the geometric mean) and for some $a, b, c \in \mathbb{R}$, $ac - b \neq 0$,*
$$f(x) = \frac{ax + b}{x + c}, \qquad x \in I;$$
- **(ii):** $p = 1$ *(that is, $M^{[p]}$ is the arithmetic mean) and for some $a, b, c \in \mathbb{R}$, $a \neq 0$,*
$$f(x) = \frac{a}{2}x^2 + bx + c, \qquad x \in I;$$
- **(iii):** $p = -1$ *(that is, $M^{[p]}$ is the harmonic mean) and for some $a, b, c, k \in \mathbb{R}$, $a \neq 0 \neq k$,*
$$f(x) = k\sqrt{ax + b} + c, \qquad x \in I.$$

Proof. First consider the case when $p = 0$. If f satisfies equation (8), then

$$\frac{f(x) - f(y)}{x - y} = \sqrt{f'(x)f'(y)}, \qquad x, y \in I,\ x \neq y,$$

whence

$$f(x) - f(y) = \sqrt{f'(x)f'(y)}\,(x - y), \qquad x, y \in I.$$

Since $f(x) - f(y) = [f(x) - f(z)] + [f(z) - f(y)]$ for all $x, y, z \in I$, we hence get

$$\sqrt{f'(x)f'(y)}\,(x - y) = \sqrt{f'(x)f'(z)}\,(x - z) + \sqrt{f'(z)f'(y)}\,(z - y), \qquad x, y, z \in I.$$

Setting here $z = \frac{x+y}{2}$, $x \neq y$, and then dividing both sides by $x - y$, we get

$$2\sqrt{f'(x)f'(y)} = \sqrt{f'(x)f'\left(\frac{x+y}{2}\right)} + \sqrt{f'\left(\frac{x+y}{2}\right)f'(y)},$$

which, obviously, also remains true for all $x, y \in I$. Dividing both sides of this equality by $2\sqrt{f'(x)f'\left(\frac{x+y}{2}\right)f'(y)}$ we hence get

$$\frac{1}{\sqrt{f'\left(\frac{x+y}{2}\right)}} = \frac{\frac{1}{\sqrt{f'(x)}} + \frac{1}{\sqrt{f'(y)}}}{2}, \qquad x, y \in I;$$

that is, $\frac{1}{\sqrt{f'\left(\frac{x+y}{2}\right)}}$ is the arithmetic mean of $\frac{1}{\sqrt{f'(x)}}$ and $\frac{1}{\sqrt{f'(y)}}$ (and $\sqrt{f'\left(\frac{x+y}{2}\right)}$ is the harmonic mean of $\sqrt{f'(x)}$ and $\sqrt{f'(y)}$). It follows that the function $\gamma := 1/\sqrt{f'}$ satisfies the Jensen functional equation

$$\gamma\left(\frac{x+y}{2}\right) = \frac{\gamma(x) + \gamma(y)}{2}, \qquad x, y \in I.$$

Since γ is Lebesgue measurable, there are $k, m \in \mathbb{R}$ such that $\gamma(x) = kx + m$, for all $x \in I$ (cf. M. Kuczma [2, Chapter XIII, Section 2]. Hence

$$f'(x) = \frac{1}{(kx+m)^2}, \qquad x \in I,$$

where $k \neq 0$, as f' is not constant, whence, for some real a,

$$f(x) = a - \frac{1}{k(kx+m)} = \frac{ax + \left(\frac{am}{k} - \frac{1}{k^2}\right)}{x + \frac{m}{k}}, \qquad x \in I.$$

Setting $b := \frac{am}{k} - \frac{1}{k^2}$, $c := \frac{m}{k}$ we get

$$f(x) = \frac{ax+b}{x+c}, \qquad x \in I,$$

and

$$ac - b = a\frac{m}{k} - \left(\frac{am}{k} - \frac{1}{k^2}\right) = \frac{1}{k^2} \neq 0.$$

It is easy to verify that f satisfies equation (8).

Now assume that $p \neq 0$. In this case,

$$\varphi(t) = At^p + B, \qquad t > 0,$$

for some $A, B \in \mathbb{R}$, $A \neq 0$, is a generator of the mean $M^{[p]}$, and equation (8) can be written in the form

$$\frac{f(x) - f(y)}{x - y} = \left(\frac{[f'(x)]^p + [f'(y)]^p}{2}\right)^{1/p}, \qquad x, y \in I, \ x \neq y.$$

Assume that $f : I \to \mathbb{R}$ satisfies this equation. Then, obviously, f is of the class C^∞ in I.

Let $I_0 \subset I$ be a maximal open and non-empty interval such that $f''(x) \neq 0$ for all $x \in I_0$. By the Darboux property of a derivative, f'' is of a constant sign in I_0. Since, by assumption, f'' does not vanish everywhere in I, such an interval exists. In view of Theorem 1, there is $C \neq 0$ such that

$$pA[f'(x)]^{p-1} = \frac{C}{|f''(x)|^{2/3}}, \qquad x \in I_0,$$

whence

$$[f'(x)]^{p-1}|f''(x)|^{2/3} = \frac{C}{pA}, \qquad x \in I_0. \tag{9}$$

If $p = 1$ we hence get

$$f''(x) = a, \qquad x \in I_0,$$

for some $a \in \mathbb{R}$, $a \neq 0$. Consequently

$$f(x) = \frac{a}{2}x^2 + bx + c, \qquad x \in I_0,$$

for some $b, c \in \mathbb{R}$ and, obviously, $I = I_0$. Since

$$\frac{f(x)-f(y)}{x-y} = a\frac{x+y}{2} + b = \frac{(ax+b)+(ay+b)}{2} = M^{[1]}\left(f'(x), f'(y)\right)$$

for all $x, y \in I$, $x \neq y$, equation (8) is satisfied.

In the sequel we assume that $p \neq 1$. Since f'' is of a constant sign in I_0, from (9) we get

$$[f'(x)]^q f''(x) = a, \qquad x \in I_0, \tag{10}$$

for some $a \neq 0$ and

$$q := \frac{3}{2}(p-1). \tag{11}$$

If $q = -1$, then (10) implies that

$$\log f'(x) = ax + b, \qquad x \in I_0,$$

for some $b \in \mathbb{R}$, whence

$$f(x) = \frac{1}{a}e^{ax+b} + \delta, \qquad x \in I_0,$$

and, of course, $I = I_0$. From (11) we get

$$p = \frac{1}{3}.$$

Since

$$\frac{f(x)-f(y)}{x-y} = \frac{e^{ax+b}-e^{ay+b}}{a\,(x-y)}, \qquad x, y \in I,\ x \neq y,$$

and

$$M^{[1/3]}\left(f'(x), f'(y)\right) = \left(\frac{\left[e^{ax+b}\right]^{1/3} + \left[e^{ay+b}\right]^{1/3}}{2}\right)^3, \qquad x, y \in I,\ x \neq y,$$

equation (8) is not fulfilled.

If $q \neq -1$, then, from (10),

$$\frac{1}{q+1}\left[f'(x)\right]^{q+1} = ax + b, \qquad x \in I_0;$$

that is,

$$f'(x) = \left[(q+1)\,(ax+b)\right]^{1/(q+1)}, \qquad x \in I_0. \tag{12}$$

For $q = -2$, we hence get

$$f'(x) = -\frac{1}{ax+b}, \qquad x \in I_0,$$

whence, for some $m \in \mathbb{R}$,

$$f(x) = -\log|ax+b| + m, \qquad x \in I_0, \tag{13}$$

and, of course, $I = I_0$. In this case, (11) implies that

$$p = -\frac{1}{3}.$$

From (13) we have

$$\frac{f(x)-f(y)}{x-y} = \frac{1}{x-y}\log\left|\frac{ay+b}{ax+b}\right|, \qquad x,y\in I,\ x\neq y,$$

$$M^{[-1/3]}\left(f'(x),f'(y)\right) = \frac{8}{\left(\sqrt[3]{ax+b}+\sqrt[3]{ay+b}\right)^3}, \qquad x,y\in I,$$

and, obviously, equation (8) is not satisfied.

Assume that $q\notin\{-1,-2\}$. From (12) we get

$$f(x) = \frac{1}{(q+2)\,a}\left[(q+1)\,(ax+b)\right]^{(q+2)/(q+1)} + m, \qquad x\in I_0,$$

for some $k,m,r\in\mathbb{R}$, $k\neq 0$. Thus

$$f(x) = k\,(ax+b)^r + m, \qquad x\in I_0, \tag{14}$$

where

$$r := \frac{q+2}{q+1}\neq 0, \qquad k := \frac{(q+1)^{(q+2)/(q+1)}}{(q+2)\,a}\neq 0. \tag{15}$$

Making use of (14) and setting $u:=ax+b$, $v:=ay+b$ for $x,y\in I_0$, $x\neq y$, we get

$$\frac{f(x)-f(y)}{x-y} = k\frac{(ax+b)^r-(ay+b)^r}{x-y} = k\frac{u^r-v^r}{\frac{u-b}{a}-\frac{v-b}{a}} = ka\frac{u^r-v^r}{u-v},$$

and

$$f'(x) = kar\,(ax+b)^{r-1},\ f'(y) = kar\,(ay+b)^{r-1}.$$

Thus

$$M^{[p]}\left(f'(x),f'(y)\right) = kar\left(\frac{(ax+b)^{p(r-1)}+(ay+b)^{p(r-1)}}{2}\right)^{1/p}$$
$$= kar\left(\frac{u^{p(r-1)}+v^{p(r-1)}}{2}\right)^{1/p},$$

whence, by (8),

$$\frac{u^r-v^r}{u-v} = r\left(\frac{u^{p(r-1)}+v^{p(r-1)}}{2}\right)^{1/p}$$

for all $u,v\in J_0:=aI_0+b, u\neq v$. The right side,

$$R(u,v) := r\left(\frac{u^{p(r-1)}+v^{p(r-1)}}{2}\right)^{1/p},$$

is an analytic function (in the real sense) in $(0,\infty)^2$. This equality implies that the function on the left side extended to the diagonal by the formula

$$L(u,v) := \begin{cases} \frac{u^r-v^r}{u-v} & \text{for } u\neq v,\\ ru^{r-1} & \text{for } u=v\end{cases}$$

is also analytic in $(0,\infty)^2$ (cf. also Remark 3 below) and we have

$$R(u,v)-L(u,v)=0, \qquad u,v>0.$$

Setting $v = 1$ we get

$$g(u) := R(u,1) - L(u,1) = 0, \qquad u > 0,$$

and, of course,

$$g^{(k)}(1) = 0 \text{ for all } k \in \{0, 1, ...\}.$$

After some calculations we get

$$g''(1) = \frac{1}{12} r(1-r)[3p(r-1) - (r+1)]$$

and

$$g^{(4)}(1) = \frac{1}{80} r(r-1)[10p^3(r-1)^3 + 15p^2(1-r)^3 + 10p(1-r)(3r^2 - 24r + 43) + 11r^3 - 69r^2 + 61r + 141].$$

Since $p \neq 0$ and $p \neq 1$, from the equality $g''(1) = 0$ we get

$$p = \frac{r+1}{3(r-1)}. \tag{16}$$

Setting this value into the equality $g^{(4)}(1) = 0$ we get

$$g^{(4)}(1) = \frac{r(2-r)(r-1)(r+1)(2r-1)}{540} = 0.$$

By (15) and (16), respectively, we can omit the cases when $r = 0$ or $r = 1$. If $r = -1$, then, by (16), we get $p = 0$, the case already considered. Therefore, applying (16), we conclude that either $r = 2$ and $p = 1$ or $r = \frac{1}{2}$ and $p = -1$.

If $r = 2$ and $p = 1$, then, by (14), we get

$$f(x) = k(ax+b)^2 \text{ for } x \in I_0 \quad \text{and} \quad M^{[1]}(u,v) = \frac{u+v}{2}, \quad u, v > 0,$$

and, if $r = \frac{1}{2}$ and $p = -1$, we get

$$f(x) = k\sqrt{ax+b} \text{ for } x \in I_0 \quad \text{and} \quad M^{[-1]}(u,v) = \frac{2uv}{u+v}, \quad u, v > 0.$$

It is easy to verify that in both cases equation (8) is fulfilled. Moreover, in each of these cases, the regularity of the solutions implies that $I = I_0$. This completes the proof. □

Remark 3. The analyticity of the function L can also be obtained as follows. Treating L as a function of two complex variables, u and v, it is easy to see that, at any point of the diagonal points $(u,u) \neq (0,0)$, the function L is separately analytic (holomorphic) at (u,u) with respect to each variable. Therefore, by the famous theorem of Hartogs [1], L is analytic at (u,u) with respect to both variables.

Remark 4. The necessity of the positivity of f' in Theorem 2 follows from the definition of the power means. Defining $M^{[p]} : (-\infty, 0)^2 \to (-\infty, 0)$ by the formula

$$M^{[p]}(u,v) := \begin{cases} -\left(\frac{(-u)^p + (-v)^p}{2}\right)^{1/p} & \text{if } p \neq 0, \\ -\sqrt{uv} & \text{if } p = 0, \end{cases}$$

we can formulate the counterpart of Theorem 2 for f such that $f' < 0$.

Remark 5. Equation (8) for $p = 0$ has appeared to be useful in solving a problem of Zs. Páles concerning existence of discontinuous Jensen affine (convex and concave) functions in the sense of Beckenbach with respect to the two-parameter family of curves generated by a Tchebycheff system [3].

Applying Theorem 2 we can prove the following:

Theorem 3. *Let $I \subset \mathbb{R}$ be an interval. Suppose that $f, g : I \to \mathbb{R}$ are differentiable, $f'(x)g'(x) > 0$ for $x \in I$ and f'/g' is not constant in I. If M is a power mean, then*

$$\frac{f(x) - f(y)}{g(x) - g(y)} = M\left(\frac{f'(x)}{g'(x)}, \frac{f'(y)}{g'(y)}\right), \qquad x, y \in I,\ x \neq y, \tag{17}$$

if, and only if, one of the following cases occurs:

(i): *M is the geometric mean, g is arbitrary, and for some $a, b, c \in \mathbb{R}$, $ac - b \neq 0$,*

$$f(x) = \frac{ag(x) + b}{g(x) + c}, \qquad x \in I;$$

(ii): *M is the arithmetic mean, g is arbitrary, and for some $a, b, c \in \mathbb{R}$, $a \neq 0$,*

$$f(x) = \frac{a}{2} g(x)^2 + bg(x) + c, \qquad x \in I;$$

(iii): *M is the harmonic mean, g is arbitrary, and for some $a, b, c, k \in \mathbb{R}$, $a \neq 0 \neq k$,*

$$f(x) = k\sqrt{ag(x) + b} + c, \qquad x \in I.$$

Proof. Suppose that the functions f and g satisfy the assumptions and equation (17) where $M = M^{[p]}$ for some $p \in \mathbb{R}$. Obviously $g(I)$ is an interval and the function $f \circ g^{-1} : g(I) \to \mathbb{R}$ satisfies all assumptions of Theorem 2. Take arbitrary $u, v \in g(I)$, $u \neq v$. Setting $x := g^{-1}(u)$, $y := g^{-1}(v)$ in (17) we get

$$\begin{aligned} \frac{f \circ g^{-1}(u) - f \circ g^{-1}(v)}{u - v} &= M^{[p]}\left(\frac{f' \circ g^{-1}(u)}{g' \circ g^{-1}(u)}, \frac{f' \circ g^{-1}(v)}{g' \circ g^{-1}(v)}\right) \\ &= M^{[p]}\left(\left(f \circ g^{-1}\right)'(u), \left(f \circ g^{-1}\right)'(v)\right), \end{aligned}$$

which means that the function $f \circ g^{-1}$ satisfies equation (8). Theorem 2 implies that either $p = 0$ and for some $a, b, c \in \mathbb{R}$, $ac - b \neq 0$,

$$f \circ g^{-1}(u) = \frac{au + b}{u + c}, \qquad u \in g(I),$$

or $p = 1$ and for some $a, b, c \in \mathbb{R}$, $a \neq 0$,

$$f \circ g^{-1}(u) = \frac{a}{2}u^2 + bu + c, \qquad u \in g(I),$$

or $p = -1$ and for some $a, b, c, k \in \mathbb{R}$, $a \neq 0 \neq k$,

$$f \circ g^{-1}(u) = k\sqrt{au + b} + c, \qquad u \in g(I).$$

Setting in these formulas $u = g(x)$ for $x \in I$, we obtain the result. $\square$

3. Final remark

Replacing f' by h of the equation for $p=0$, $p=1$, and $p=-1$ (the cases itemized in Theorem 2) we get three functional equations:

$$\frac{f(x)-f(y)}{x-y}=\sqrt{h(x)h(y)}, \qquad x,y\in I, x\neq y,$$
$$\frac{f(x)-f(y)}{x-y}=\frac{h(x)+h(y)}{2}, \qquad x,y\in I, x\neq y,$$
$$\frac{f(x)-f(y)}{x-y}=\frac{2h(x)h(y)}{h(x)+h(y)}, \qquad x,y\in I, x\neq y.$$

It is not difficult to show that, without any regularity assumptions on f and h, each of these equations characterizes the respective function f and its derivative h.

It is interesting that this fact remains true if we "pexiderize" these equations by replacing $f(y)$ by $\phi(y)$ and $h(y)$ by $\gamma(y)$.

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