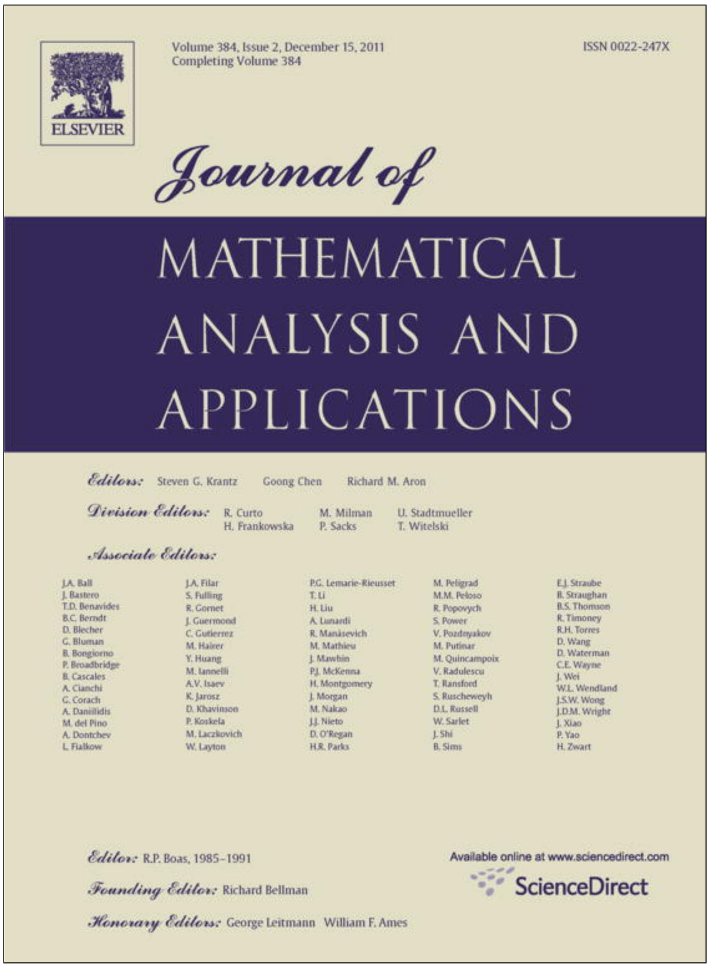
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Mean-type mappings and invariant curves

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ABSTRACT

The problem of invariant curves for continuous mean-type mappings is considered. The obtained results are applied in solving some functional equations.

1. Introduction

To recall the problem of invariant curves, assume that $D \subset \mathbb{R}^2$ is a domain, and $T: D \to \mathbb{R}^2$, $T = (f, g)$ – a mapping. Let $I \subset \mathbb{R}$ be an interval and $\varphi: I \to \mathbb{R}$ a continuous function such that $C := \{(x, \varphi(x)):\ x \in I\}$, the graph of φ, is contained in D. If the transform T maps C into itself, i.e. if

$$\varphi\big[f\big(x, \varphi(x)\big)\big] = g\big(x, \varphi(x)\big), \quad x \in I, \tag{1.1}$$

the graph C is called an *invariant* (or T*-invariant*) *curve*. Eq. (1.1), where φ is unknown, is called a *functional equation on invariant curves*.

The Lipschitzian solutions of this equation, by means of the Banach contraction principle, were considered by Montel [17] (cf. also Hadamard [7], Kuczma [9], Lattès [10]).

The continuous solutions of Eq. (1.1), in various special cases, were considered, for instance, by Dhombres [5], Jarczyk [8], Matkowski and Okrzesik [15,16] (cf. Final remarks).

In this paper we examine the continuous solutions of Eq. (1.1) assuming that T is a continuous mean-type mapping, that is $T = (M, N)$, where M and N are continuous means. The key tool, Theorem 1 on convergence of the sequence of iterates of mean-type mapping (M, N) to a unique continuous (M, N)-invariant mean, is presented in Section 2. The main results on invariant curves, contained in Section 3, are proved without any additional regularity conditions. In Section 4 we apply them to solve some composite functional equations.

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2. Preliminaries

Let $I \subset \mathbb{R}$ be an interval. A function $M : I^2 \to \mathbb{R}$ is said to be a *mean* on I if, for all $x, y \in I$,

$$\min(x, y) \leqslant M(x, y) \leqslant \max(x, y).$$

A mean M in I is called *strict* if these inequalities are sharp whenever $x \neq y$.

If $I = (0, \infty)$ we say that a mean M in I is *positively homogeneous* if

$$M(tx, ty) = tM(x, y), \quad t, x, y > 0.$$

Note the following easy to verify

Remark 1. Let $M : I^2 \to \mathbb{R}$ be an arbitrary function. Then the following conditions are equivalent

(i) M is a mean;
(ii) $M(J^2) \subset J$ for every subinterval $J \subset I$;
(iii) $M(J^2) = J$ for every subinterval $J \subset I$.

Hence we have

Remark 2. If $M : I^2 \to \mathbb{R}$ is a mean then M maps I^2 onto I and, moreover, M is reflexive, that is, for all $x \in I$,

$$M(x, x) = x.$$

Let us also note the following

Remark 3. If a function $M : I^2 \to \mathbb{R}$ is reflexive and (strictly) increasing with respect to each variable, then M is a (strict) mean I.

In the sequel a mean $M : I^2 \to I$ is called *increasing* (*strictly increasing*) if it is increasing (resp. strictly increasing) with respect to each variable.

A mapping $\mathbf{M} : I^2 \to I^2$ is referred to as *mean-type*, if $\mathbf{M} = (M, N)$ for some means $M, N : I^2 \to I$. We say that the mean-type mapping is *strict* (*positively homogeneous*) if both its coordinate means M and N are strict (positively homogeneous).

Put $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$.

If $\mathbf{M} : I^2 \to I^2$ is a mean-type mapping then, clearly, the sequence $(\mathbf{M}^n)_{n=0}^{\infty}$ of the iterates of $\mathbf{M}$,

$$\mathbf{M}^0 := \mathrm{Id}|_{I^p}; \qquad \mathbf{M}^{n+1} := \mathbf{M} \circ \mathbf{M}^n \quad \text{for } n \in \mathbb{N}_0,$$

is well defined.

Given a mean-type mapping $\mathbf{M} : I^2 \to I^2$, $\mathbf{M} = (M, N)$, and a mean $K : I^2 \to I$, we say that K is *invariant with respect to the mean-type mapping* $\mathbf{M}$, briefly, $\mathbf{M}$-*invariant*, if

$$K \circ \mathbf{M} = K,$$

or, equivalently, if

$$K\big(M(x, y), N(x, y)\big) = K(x, y), \quad x, y \in I.$$

Let us quote the following two-dimensional counterpart of a more general result proved in [14].

Theorem 1. *Let $I \subset \mathbb{R}$ be an interval. Suppose that $\mathbf{M} : I^2 \to I^2$, $\mathbf{M} = (M, N)$, is a continuous mean-type mapping such that, for all $x, y \in I$, $x \neq y$,*

$$\max\big(M(x, y), N(x, y)\big) - \min\big(M(x, y), N(x, y)\big) < \max(x, y) - \min(x, y).$$

Then

(i) *for every $n \in \mathbb{N}$, the n-th iterate $\mathbf{M}^n$ is a mean-type mapping of I^2*;
(ii) *there is a continuous mean $K : I^2 \to I$ such that the sequence of iterates $(\mathbf{M}^n)_{n=0}^{\infty}$ converges, uniformly on compact subsets of I^2, to the mean-type mapping $\mathbf{K} : I^2 \to I^2$, $\mathbf{K} = (K, K)$*;
(iii) *K is $\mathbf{M}$-invariant*;
(iv) *a continuous $\mathbf{M}$-invariant mean is unique*;

(v) *if* **M** *is strict then so is* K;
(vi) *if* M *and* N *are (strictly) increasing with respect to each variable then so is* K;
(vii) *if* $I = (0, \infty)$ *and* M, N *are positively homogeneous, then every iterate of* **M** *and* K *are positively homogeneous.*

Remark 4. This theorem generalizes the earlier results where beside the continuity, some additional conditions are assumed (cf. Borwein and Borwein [1, Theorem 8.2, p. 244], where it is assumed that the means M and N are strict and comparable, also Bullen [2, p. 414]; Z. Daróczy and Zs. Páles [3,4], where both means are strict; and [11–13] where it is assumed that at most one of the means is not strict).

Note that in Theorem 1 neither M nor N must be strict.

Recall also that Gauss cf. [6] was the first who considered the iterations of the mean-type mapping in a special case when M is the arithmetic mean and N is the geometric mean.

3. Main results

Theorem 2. *Let* $I, J \subset \mathbb{R}$, $J \subset I$, *be intervals. Suppose that* $M, N : I^2 \to I$ *are continuous means such that, for all* $x, y \in I$, $x \neq y$,

$$\max\bigl(M(x,y), N(x,y)\bigr) - \min\bigl(M(x,y), N(x,y)\bigr) < \max(x,y) - \min(x,y).$$

If the graph of a continuous function $\varphi : J \to I$ *is an invariant curve under the mean-type mapping* (M, N), *that is, if*

$$\varphi\bigl[M\bigl(x, \varphi(x)\bigr)\bigr] = N\bigl(x, \varphi(x)\bigr), \quad x \in J, \tag{3.1}$$

then φ *satisfies the functional equation*

$$\varphi\bigl[K\bigl(x, \varphi(x)\bigr)\bigr] = K\bigl(x, \varphi(x)\bigr), \quad x \in J, \tag{3.2}$$

where K *is a unique continuous* (M, N)*-invariant mean.*

Proof. By Theorem 1, there is a unique continuous (M, N)-invariant mean K in I. Put $(M_n, N_n) := (M, N)^n$ for the n-th iterate of (M, N) ($n \in \mathbb{N}$), and assume that the graph of a continuous function $\varphi : J \to I$ is an invariant curve under the transformation (M, N), that is (3.1) holds true. Note that

$$\varphi\bigl[M_n\bigl(x, \varphi(x)\bigr)\bigr] = N_n\bigl(x, \varphi(x)\bigr), \quad x \in J,\ n \in \mathbb{N}. \tag{3.3}$$

Indeed, by (3.1) this equality is true for $n = 1$. Assume that (3.3) holds true for some $n \in \mathbb{N}$. Since

$$(M_{n+1}, N_{n+1}) = \bigl(M \circ (M_n, N_n), N \circ (M_n, N_n)\bigr), \quad n \in \mathbb{N},$$

making use of (3.3) and (3.1) we get, for all $x \in J$,

$$\begin{aligned}
\varphi\bigl[M_{n+1}\bigl(x, \varphi(x)\bigr)\bigr] &= \varphi\bigl[M\bigl(M_n\bigl(x, \varphi(x)\bigr), N_n\bigl(x, \varphi(x)\bigr)\bigr)\bigr] \\
&= \varphi\bigl[M\bigl(M_n\bigl(x, \varphi(x)\bigr), \varphi\bigl[M_n\bigl(x, \varphi(x)\bigr)\bigr]\bigr)\bigr] \\
&= N\bigl(M_n\bigl(x, \varphi(x)\bigr), N_n\bigl(x, \varphi(x)\bigr)\bigr) \\
&= N_{n+1}\bigl(x, \varphi(x)\bigr),
\end{aligned}$$

and the induction completes the proof of (3.3). Now, by the continuity of φ, letting $n \to \infty$ and applying Theorem 1, we obtain (3.2). □

Remark 5. In general, the mean-type mappings are not nonexpansive. Therefore this result cannot be deduced from the Montel theorem [17] (cf. also Kuczma [9]). Moreover, we assume the continuity of the map T and the invariant curve. In [17,9], under the assumption of the Lipschitz-continuity of T, the Lipschitz-continuous invariant curves are considered.

Theorem 3. *Let* $I \subset \mathbb{R}$ *be an open interval,* $K : I^2 \to I$ *a continuous strictly increasing mean, and* $M, N : I^2 \to I$ *continuous and strict means. Then*:

(i) *For every* $c \in I$, *the set*

$$I_K(c) := \bigl\{x \in I:\ K(x, y) = c \text{ for some } y \in I\bigr\}$$

is a nontrivial subinterval of I,

$$c \in \operatorname{Int} I_K(c),$$

and there exists exactly one function $\varphi_{K,c} : I_K(c) \to I$ such that

$$K\big(x, \varphi_{K,c}(x)\big) = c, \quad x \in I_K(c); \tag{3.4}$$

moreover $\varphi_{K,c}$ is continuous, strictly decreasing, and

$$\varphi_{K,c}(c) = c.$$

(ii) *If K is invariant with respect to the mean-type mapping (M, N), then for every $c \in I$, the graph of the function $\varphi_{K,c}$ is an invariant curve with respect to the map (M, N).*

(iii) *Suppose that K is (M, N)-invariant. If $J \subset I$ is an interval and the graph of a continuous function $\varphi : J \to I$ is an invariant curve with respect to the map (M, N), that is if* (3.2):

$$\varphi\big[M\big(x, \varphi(x)\big)\big] = N\big(x, \varphi(x)\big), \quad x \in J,$$

then either there is a unique $c \in I$ such that $\varphi = \varphi_{K,c}|_J$, that is φ is the restriction of $\varphi_{K,c}$ described above to J, where K is a unique continuous (M, N)-invariant mean, or there are $a, b \in \operatorname{cl} I$, $-\infty \leqslant a \leqslant b \leqslant +\infty$, such that

$$\varphi(x) = \begin{cases} \varphi_{K,a}(x) & \text{for } x \in (-\infty, a] \cap J, \\ x & \text{for } x \in (a, b) \cap J, \\ \varphi_{K,b}(x) & \text{for } x \in [b, \infty) \cap J. \end{cases} \tag{3.5}$$

Proof. Ad. (i). Take $c \in I$. Since $K(c, c) = c$, we conclude that $c \in I_K(c)$, so $I_K(c)$ is non-empty. There are $x, y \in I_K(c)$ such that $x < c < y$. In the opposite case we would have either $K(x, y) < c$ for all $x, y \in I$, $x < y$, or $K(x, y) > c$ for all $x, y \in I$, $x < y$. Indeed, the set $K(\{(x, y) \in I^2\colon\ x < y\})$, as a continuous image of the connected set, is an interval. Hence, by the continuity of K, we would get that either $K(x, x) = x \leqslant c$ for all $x \in I$ or $K(x, x) = x \geqslant c$ for all $x \in I$. This is impossible as $K(x, x) = x$ for all $x \in I$ and $c \in I = \operatorname{Int} I$. This proves that $I_K(c)$ is not a singleton.

To show that $I_K(c)$ is an interval take arbitrary points $x_1 < x_2$ of $I_K(c)$ and fix $x \in]x_1, x_2[$. Then there exist elements y_1 and y_2 of I such that

$$K(x_1, y_1) = c = K(x_2, y_2).$$

Clearly, $y_1 > y_2$ by the strict monotonicity of K. Consider now the continuous function $h : I \to \mathbb{R}$ defined by $h(y) := h(x, y)$. Applying again the strict monotonicity of K, we have

$$h(y_1) = K(x, y_1) > K(x_1, y_1) = c;$$
$$h(y_2) = K(x, y_2) > K(x_1, y_1) = c.$$

The Darboux property of h guarantees the existence of an element $y \in I$ such that $h(y) = c$; that is $K(x, y) = c$. This means that $x \in I_K(c)$, which was to be proved.

By the strict monotonicity of K, for every $x \in I_K(c)$ there is a unique $y \in I$ such that $K(x, y) = c$. Thus there exists a unique function $\varphi_{K,c} : I_K(c) \to I$ such that

$$K\big(x, \varphi_{K,c}(x)\big) = c, \quad x \in I_K(c).$$

The assumed increasing monotonicity of K implies that $\varphi_{K,c}$ is strictly decreasing. Take $x_0 \in I_K(c)$, $x_0 > \inf I_K(c)$, and an increasing sequence (x_n) such that $\lim_{n\to\infty} x_n = x_0$. By the monotonicity of $\varphi_{K,c}$,

$$\lim_{n\to\infty} \varphi_{K,c}(x_n) = \varphi_{K,c}(x_0-).$$

Hence, by the continuity of K and the definition of $\varphi_{K,c}$,

$$K\big(x_0, \varphi_{K,c}(x_0-)\big) = \lim_{n\to\infty} K\big(x_n, \varphi_{K,c}(x_n)\big) = c,$$

whence $\varphi_{K,c}(x_0-) = \varphi_{K,c}(x_0)$. Thus $\varphi_{K,c}$ is left continuous at x_0. In the same way we can show that $\varphi_{K,c}$ is right continuous at every point $x_0 \in I_K(c)$ such that $x_0 < \sup I_K(c)$.

Ad. (ii). In view of (M, N)-invariance of K and (3.4) we have

$$K\big(M\big(x, \varphi_{K,c}(x)\big), N\big(x, \varphi_{K,c}(x)\big)\big) = K\big(x, \varphi_{K,c}(x)\big) = c,$$

for all $x \in I_K(c)$, whence, by the uniqueness of $\varphi_{K,c}$ proved in part (i),

$$\varphi_{K,c}\big(M\big(x, \varphi_{K,c}(x)\big)\big) = N\big(x, \varphi_{K,c}(x)\big), \quad x \in I_K(c).$$

Ad. (iii). Let $J \subset I$ be an interval and suppose that a continuous function $\varphi : J \to I$ satisfies Eq. (3.1). In view of Theorem 1, the mean K is a unique continuous (M, N)-invariant mean. By Theorem 2 the function φ satisfies the functional equation (3.2):

$$\varphi\big[K\big(x, \varphi(x)\big)\big] = K\big(x, \varphi(x)\big), \quad x \in J.$$

Clearly, the range of the function $J \ni x \to K(x, \varphi(x))$ is an interval of the endpoints $a, b \in \mathbb{R} \cup \{-\infty, +\infty\}$, $a \leqslant b$. If $a = b$, then this function is constant, and setting $c := a$, we obtain

$$K\big(x, \varphi(x)\big) = c, \quad x \in J.$$

Since K is strictly increasing, according to part (i) of our result, we have $\varphi(x) = \varphi_{K,c}(x)$ for all $x \in J \cap I_K(c)$ and, obviously, $J \subset I_K(c)$.

If $a < b$ then, setting $z := K(x, \varphi(x))$ in Eq. (3.2), we get

$$\varphi(z) = z, \quad z \in J \cap (a, b).$$

Now it is obvious that the function $\varphi : J \to I$ defined by formula (3.5) satisfies Eq. (3.2). This completes the proof. □

As an immediate consequence of the above theorem we obtain

Corollary 1. *Let $I \subset \mathbb{R}$ be an open interval and $M, N : I^2 \to I$ be means which are continuous and strictly increasing in each variable. Suppose that $\varphi : J \to I$ is monotonic and continuous.*

Then the graph of the function φ is an (M, N)-invariant curve if, and only if, either there is a unique $c \in I$ such that $\varphi = \varphi_{K,c}|_J$, where K is a unique continuous (M, N)-invariant mean, or

$$\varphi(x) = x, \quad x \in J.$$

We end this section with the following easy to verify

Remark 6. Using Theorem 3 one can easily find all (M, N)-invariant curves the graphs of which need not be the graphs of functions.

4. Application in solving some functional equations in a single variable

Proposition 1. *Let $f, g : (0, \infty)^2 \to (0, \infty)$ be given continuous functions and let $J \subset (0, \infty)$ be an interval.*

(i) *If a continuous function $\varphi : J \to (0, \infty)$ satisfies the functional equation*

$$\varphi\left(\frac{xf(x, \varphi(x)) + \varphi(x)g(x, \varphi(x))}{f(x, \varphi(x)) + g(x, \varphi(x))}\right) = \frac{x\varphi(x)[f(x, \varphi(x)) + g(x, \varphi(x))]}{xf(x, \varphi(x)) + \varphi(x)g(x, \varphi(x))}, \quad x \in J, \tag{4.1}$$

then

$$\varphi\big(\sqrt{x\varphi(x)}\big) = \sqrt{x\varphi(x)}, \quad x \in J. \tag{4.2}$$

(ii) *A continuous function $\varphi : J \to (0, \infty)$ satisfies Eq. (4.2) if, and only if, either there is $c > 0$ such that*

$$\varphi(x) = \frac{c^2}{x}, \quad x \in J,$$

or there are $a, b \in \mathbb{R} \cup \{\infty\}$, $0 \leqslant a < b \leqslant +\infty$, such that

$$\varphi(x) = \begin{cases} \frac{a^2}{x} & \text{for } x \in (-\infty, a] \cap J, \\ x & \text{for } x \in (a, b) \cap J, \\ \frac{b^2}{x} & \text{for } x \in [b, \infty) \cap J. \end{cases} \tag{4.3}$$

(iii) *A continuous and monotonic function $\varphi : J \to (0, \infty)$ satisfies Eq. (4.1) if, and only if, either there is $c > 0$ such that*

$$\varphi(x) = \frac{c^2}{x}, \quad x \in J,$$

or

$$\varphi(x) = x, \quad x \in J.$$

Proof. Put

$$M(x,y) := \frac{xf(x,y)+yg(x,y)}{f(x,y)+g(x,y)}, \qquad N(x,y) := \frac{xy[f(x,y)+g(x,y)]}{xf(x,y)+yg(x,y)}, \quad x,y>0.$$

It is easily seen that $M, N : (0,\infty)^2 \to (0,\infty)$ are continuous and strict means. Since

$$\sqrt{M(x,y)N(x,y)} = \sqrt{xy}, \quad x,y>0,$$

the geometric mean $K(x,y) := \sqrt{xy}$ is (M,N)-invariant. Suppose that $\varphi : J \to (0,\infty)$ satisfies Eq. (4.1). Then the graph of a continuous function $\varphi : J \to I$ is an invariant curve under the mean-type mapping (M,N). Since K is strictly increasing, in view of Theorem 2, the function φ satisfies Eq. (3.2) with the geometric mean K, that is

$$\varphi\big(\sqrt{x\varphi(x)}\big) = \sqrt{x\varphi(x)}, \quad x \in J,$$

which completes the proof of the first part. The second part and formula (4.3) follow from part (iii) of Theorem 3.

If φ is monotonic, then the domains of the functions φ_a and φ_b in formula (3.5) are at most the end-points of J. Part (ii) and the continuity of φ imply that either $\varphi(x) = \frac{c^2}{x}$ for all $x \in J$ or $\varphi(x) = x$ for all $x \in J$. A simple verification shows that these functions satisfy Eq. (4.1). The proof is completed. □

Proposition 2. *Let $f : \mathbb{R}^2 \to (0,1)$ be a continuous function and let $J \subset \mathbb{R}$ be an interval.*

(i) *If a continuous function $\varphi : J \to \mathbb{R}$ satisfies the functional equation*

$$\varphi\big(xf\big(x,\varphi(x)\big) + \varphi(x)\big(1 - f\big(x,\varphi(x)\big)\big)\big) = x\big(1 - f\big(x,\varphi(x)\big)\big) + \varphi(x) f\big(x,\varphi(x)\big), \quad x \in J, \tag{4.4}$$

then

$$\varphi\left(\frac{x+\varphi(x)}{2}\right) = \frac{x+\varphi(x)}{2}, \quad x \in J. \tag{4.5}$$

(ii) *A continuous function $\varphi : J \to \mathbb{R}$ satisfies Eq.* (4.5) *if, and only if, either there is $c \in \mathbb{R}$ such that*

$$\varphi(x) = 2c - x, \quad x \in J,$$

or there are $a, b \in \mathbb{R} \cup \{\infty\}$, $0 \leqslant a < b \leqslant +\infty$, such that

$$\varphi(x) = \begin{cases} \varphi_a(x) & \text{for } x \in (-\infty, a] \cap J, \\ x & \text{for } x \in (a,b), \\ \varphi_b(x) & \text{for } x \in [b,\infty) \cap J, \end{cases}$$

where $\varphi_a : ((0,a] \cap J) \to (0,\infty)$ and $\varphi_b : ([b,\infty) \cap J) \to (0,\infty)$ are some continuous functions such that

$$2a - x \leqslant \varphi_a(x) \leqslant 2b - x \quad \text{for } x \in (-\infty, a] \cap J; \qquad 2a - x \leqslant \varphi_b(x) \leqslant 2b - x \quad \text{for } x \in [b, +\infty) \cap J,$$

and

$$\varphi_a(a) = a, \qquad \varphi_b(b) = b.$$

(iii) *A continuous and monotonic function $\varphi : J \to (0,\infty)$ satisfies Eq.* (4.4) *if, and only if, either there is $c > 0$ such that*

$$\varphi(x) = 2c - x, \quad x \in J,$$

or

$$\varphi(x) = x, \quad x \in J.$$

Proof. Put

$$M(x,y) := xf(x,y) + \big(1 - f(x,y)\big)y, \qquad N(x,y) := \big(1 - f(x,y)\big)x + f(x,y)y, \quad x,y \in \mathbb{R}.$$

It is easily seen that $M, N : \mathbb{R}^2 \to \mathbb{R}$ are continuous and strict means. Since

$$M(x,y) + N(x,y) = x + y, \quad x,y \in \mathbb{R},$$

the arithmetic mean $K(x,y) := \frac{x+y}{2}$ is (M,N)-invariant. Now we can argue similarly as in the proof of Proposition 2. □

Remark 7. In the same way we can solve the functional equations

$$\varphi\big(M\big(x,\varphi(x)\big)\big) = x + \varphi(x) - M\big(x,\varphi(x)\big), \quad x \in J,$$

where M is a continuous strict mean in an interval I, $J \subset I$, and $\varphi : J \to I$ is a continuous function.

Example 1. Consider the functional equation

$$\varphi\left(\frac{x+\varphi(x)}{2}\right) = \sqrt{x\varphi(x)}, \quad x > 0, \tag{4.6}$$

where $\varphi : (0,\infty) \to (0,\infty)$ is an unknown continuous function.

Putting $A(x,y) := \frac{x+y}{2}$ and $G(x,y) := \sqrt{xy}$, we can write this equation in the form

$$\varphi\big(A\big(x,\varphi(x)\big)\big) = G\big(x,\varphi(x)\big), \quad x > 0,$$

which means that the graph of φ is an invariant curve under the mean mapping $(A, G) : (0,\infty)^2 \to (0,\infty)$. According to a theorem of Gauss (cf. [6], cf. also [1]), the mean $K : (0,\infty)^2 \to (0,\infty)$,

$$K(x,y) := \left(\frac{2}{\pi}\int_0^{\pi/2} \frac{dt}{\sqrt{x^2(\cos t)^2 + y^2(\sin t)^2}}\right)^{-1}, \quad x, y > 0,$$

is (A, G)-invariant. By Theorem 2 the function φ satisfies the functional equation

$$\varphi\big[K\big(x,\varphi(x)\big)\big] = K\big(x,\varphi(x)\big), \quad x > 0.$$

Since A and G are strictly increasing, in view of Theorem 1, so is K. Thus to determine all continuous solutions φ of Eq. (4.6), in view of Theorem 3, it is enough to find the continuous and decreasing functions φ satisfying the equation $K(x,\varphi(x)) = c$ for $x > 0$, that is the equation

$$\int_0^{\pi/2} \frac{dt}{\sqrt{x^2(\cos t)^2 + [\varphi(x)]^2(\sin t)^2}} = \frac{\pi}{2c}, \quad x > 0,$$

where $c > 0$.

5. Final remarks

The continuous solutions of the functional equation

$$\varphi\big(x\varphi(x)\big) = \big[\varphi(x)\big]^2 \quad \text{for } x \in [0,\infty)$$

(called a division model of population) were established by Dhombres [5]. Note that this is a T-invariant curve equation where the mapping $T = (f, g)$ is such that $f(x,y) = xy$ and $g(x,y) = y^2$.

The continuous solutions of the functional equation

$$\varphi\big(x + \varphi(x)\big) = P\big(\varphi(x)\big),$$

stemming from the problem of invariant curves for the mapping $T = (f, g)$ with $f(x,y) = x + y$ and $g(x,y) = P(y)$ were considered by Jarczyk [8].

The continuous solutions of the invariant curve equation for the mappings $T(x,y) = (xG(y), [G(y)])$ and $T(x,y) = (x^r G(y), x^{p(r-1)}[G(y)]^p)$ where $p, r \in \mathbb{R}$ are fixed and $G : (0,\infty) \to (0,\infty)$ is a given continuous function, have been established, respectively, in [15] and [16].

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