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A differential equation related to the l^p-norms

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Abstract. Let $p \in (1, \infty)$. The question of existence of a curve in $\mathbb{R}^2_+$ starting at $(0,0)$ and such that at every point (x, y) of this curve, the l^p-distance of the points (x, y) and $(0, 0)$ is equal to the Euclidean length of the arc of this curve between these points is considered. This problem reduces to a nonlinear differential equation. The existence and uniqueness of solutions is proved and nonelementary explicit solutions are given.

1. Introduction. We consider the following problem. Let $p \in (1, \infty)$ be fixed. Does there exist a (regular) curve $\mathbb{R}_+ \ni t \mapsto (x(t), y(t)) \in \mathbb{R}^2_+$ starting at $(0,0)$ and such, for any $t \in \mathbb{R}_+$, the l^p-distance of the points $(x(t), y(t))$ and $(0,0)$ is equal to the Euclidean length of the arc of this curve between these points?

Of course, since the functional $[1, \infty) \ni p \mapsto \|\cdot\|_p$ is decreasing, such a curve may exist only if $1 < p \leq 2$. In Section 2 we show that in this case the problem leads to a nontrivial differential equation of the first order and we give its geometrical interpretation involving the scalar product of the tangent and radius vectors. In Section 3, considering this equation, we prove that for any point (x_0, y_0) with $x_0, y_0 > 0$, there are exactly two curves of class C^1

$$\mathbb{R}_+ \ni t \mapsto (x(t), y(t))$$

issuing from $(0, 0)$, passing through (x_0, y_0), contained in $[0, \infty)^2$ and such that the l^p-norm of any point $(x(t), y(t))$ of this curve coincides with the Euclidean length of the arc of the curve between $(0,0)$ and $(x(t), y(t))$. One of them is the graph of a function of the form

$$y = \begin{cases} 0 & \text{for } x \in [0, c], \\ y(x) & \text{for } x > c, \end{cases}$$

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where

$$c = c(x_0, y_0) > 0 \quad \text{and} \quad y(x) > 0 \text{ for } x > c,$$

that is tangent at $(0,0)$ to the x-axis. The other is symmetric with respect to the diagonal to the above curve with $c = c(y_0, x_0)$; so it is tangent at $(0,0)$ to the y-axis (Theorem 4.8).

Surprisingly enough there is no curve of this type such that, apart from the origin $(0,0)$, all its points belong to the interior of $\mathbb{R}^2_+$ (Theorem 4.5). In other words, to attain a point with both positive coordinates along such a curve, it is necessary to go for some positive time along one of the axes. It follows, in particular, that this curve is not analytic.

In Section 5, by means a numerical approach, the nonelementary explicit solutions are given and their graphs are presented.

At the end of the paper we consider the case $p = 1$.

The basic question considered here appeared in a discussion concerning some measurement problems between Peter Kahlig and the third name author about ten years ago.

2. Preliminaries. For $p \in [1, \infty]$, $\mathbf{l}^p$ denotes the plane $\mathbb{R}^2$ with the norm $\|\cdot\|_p : \mathbb{R}^2 \to [0, \infty)$, defined by

$$\|(x,y)\|_p := \begin{cases} (|x|^p + |y|^p)^{1/p} & \text{for } p \in [1, \infty), \\ \max(|x|, |y|) & \text{for } p = \infty. \end{cases}$$

Recall that all these norms are equivalent and, for each fixed $(x,y) \in \mathbb{R}^2$, the function

$$[1, \infty] \ni p \mapsto \|(x,y)\|_p \text{ is decreasing;}$$

in particular, for all $(x,y) \in \mathbb{R}^2$,

$$\|(x,y)\|_1 \le \|(x,y)\|_p \le \|(x,y)\|_\infty,$$

where $\|\cdot\|_1$ is the so called *taxi-driver norm* and $\|\cdot\|_\infty$, the *corridor norm.*

The extremal norms $\|\cdot\|_1$ and $\|\cdot\|_\infty$ are easy to calculate. However, the lack of uniqueness of segments joining points for these norms is sometimes disadvantageous.

Recall the following

REMARK 2.1. Let $p \in (1, \infty)$ be fixed. Then, for all $x_1, x_2, y_1, y_2 \in \mathbb{R}^2$,

$$\|(x_1 + x_2, y_1 + y_2)\|_p = \|(x_1, y_2)\|_p + \|(x_2, y_2)\|_p,$$

if, and only if, the vectors (x_1, y_1) and (x_2, y_2) are positively proportional, i.e., there is $\alpha > 0$ such that $(x_2, y_2) = \alpha(x_1, y_1)$.

The taxi-driver and corridor norms do not have this property.

3. Curves and equations related to l^p-norms. According to what we have mentioned in the Introduction, we look for a regular (say continuously differentiable curve) $[0,\infty) \ni t \mapsto (x(t), y(t)) \in \mathbb{R}^2_+$, $x(0) = 0$, $y(0) = 0$, $x'(t)^2 + y'(t)^2 > 0$ for all $t > 0$, and such that, at every time t, the Euclidean length of the arc of the curve between the points $(0,0)$ and $(x(t), y(t))$ is equal to the l^p-norm of the vector $(x(t), y(t))$, which means that

$$\int_0^t \sqrt{x'(s)^2 + y'(s)^2}\, ds = (x(t)^p + y(t)^p)^{1/p}, \quad t > 0. \tag{3.1}$$

We shall show the following

CLAIM. *If a curve $[0,\infty) \ni t \mapsto (x(t), y(t))$ satisfies the above conditions, then:*

(i) *for all $t > 0$,*

$$x'(t) \geq 0 \quad and \quad y'(t) \geq 0,$$
$$x'(t) = 0 \Rightarrow x(t) = 0 \quad and \quad y'(t) = 0 \Rightarrow y(t) = 0;$$

(ii) *if $y(t) = 0$ (resp. $x(t) = 0$) for some $t > 0$, then $y(s) = 0$ (resp. $x(s) = 0$) for all $s \in [0,t]$.*

Indeed, differentiating both sides of (3.1) we get

$$\sqrt{x'(t)^2 + y'(t)^2} = \frac{x(t)^{p-1}x'(t) + y(t)^{p-1}y'(t)}{(x(t)^p + y(t)^p)^{1-1/p}}, \quad t > 0. \tag{3.2}$$

As the left-hand side is positive, there is no $t > 0$ such that, simultaneously, both $x'(t)$ and $y'(t)$ are negative. Assume that $x'(t) > 0$ and $y'(t) < 0$ for some $t > 0$. Then, applying (3.2), we would get

$$0 < x'(t) < \sqrt{x'(t)^2 + y'(t)^2} < \frac{x(t)^{p-1}x'(t)}{(x(t)^p + y(t)^p)^{1-1/p}},$$

whence

$$1 < \frac{x(t)^{p-1}}{(x(t)^p + y(t)^p)^{1-1/p}} \leq \frac{x(t)^{p-1}}{(x(t)^p)^{1-1/p}} = 1,$$

a contradiction. In the same way we can show that the case $x'(t) < 0$ and $y'(t) > 0$ for some $t > 0$ cannot occur. The remaining statement of part (i) is an immediate consequence of (3.2).

To show (ii) assume that $y(t) = 0$ for some $t > 0$. If the function $[0,t] \ni s \mapsto y(s)$ were not constant we would have $y'(s) > 0$ for some $s > 0$, whence

$$x(t) = (x(t)^p + y(t)^p)^{1/p} = \int_0^t \sqrt{x'(s)^2 + y'(s)^2}\, ds > \int_0^t \sqrt{x'(s)^2}\, ds = x(t).$$

This contradiction completes the proof.

This claim implies that, without any loss of generality, we may assume that either $x'(t) > 0$ for all $t > 0$ or $y'(t) > 0$ for all $t > 0$. If, for instance, $x'(t) > 0$ for all $t > 0$, from the equation $x = x(t)$ $(t \geq 0)$ we can determine uniquely $t = t(x)$. Then the graph of the curve coincides with the graph of the function $y = y(t(x))$, $x \geq 0$, the curve can be written in the form

$$x = t, \quad y = y(t), \quad t \geq 0,$$

where $y = y(x)$, $x > 0$, is continuously differentiable, and (3.1) reduces to the equation

$$\int_0^x \sqrt{1 + y'(t)^2}\, dt = (x^p + y(x)^p)^{1/p}, \quad x > 0. \tag{3.3}$$

For obvious reasons, we have to admit functions $y = y(t)$, $t \geq 0$, such that $y'(0) = \infty$. In this case, by symmetry, if $y'(t) > 0$ for all $t > 0$, we could consider the curves in the form

$$x = x(t), \quad y = t, \quad t \geq 0.$$

In this connection let us make the following

REMARK 3.1. By the definition of l^p-norm, the curves $y = 0$ and $x = 0$, i.e. the curves coinciding with each of the axes, have the required property. Thus, if starting at $(0, 0)$ we go along the first axis up to the point $(c, 0)$, for some $c > 0$, then equation (3.2) takes the form

$$c + \int_c^x \sqrt{1 + y'(x)^2}\, dx = (x^p + y(x)^p)^{1/p}, \quad x \geq 0. \tag{3.4}$$

Differentiating both sides of the integro-differential equation (3.3) (or (3.4)) with respect to x we get the differential equation

$$\sqrt{1 + y'(x)^2} = \frac{x^{p-1} + y(x)^{p-1} y'(x)}{(x^p + y(x)^p)^{1-1/p}}. \tag{3.5}$$

Geometrical interpretation of equation (3.5). Let p^* be the conjugate to p, that is, $1/p + 1/p^* = 1$. Note that this differential equation can be written in the form

$$(1, y'(x)) \circ (x^{p-1}, [y(x)]^{p-1}) = \|(1, y'(x))\|_2 \|(x^{p-1}, [y(x)]^{p-1})\|_{p^*},$$

where "$\circ$" stands for the scalar product. Thus, a function $y = y(x)$ satisfies equation (3.5) if, and only if, the scalar product of the vector $(1, y'(x))$, which is tangent to the graph of the function $y = y(x)$ at the point $(x, y(x))$, and the vector $(x^{p-1}, [y(x)]^{p-1})$, is equal to the product of the Euclidean norm of the tangent vector and the l^{p^*}-norm (conjugate to the l^p-norm) of the vector $(x^{p-1}, [y(x)]^{p-1})$.

4. Main results

REMARK 4.1. For $p = 2$ equation (3.5) reduces to

$$(xy'(x) - y(x))^2 = 0, \quad x \geq 0,$$

and its solution for $x \geq 0$ is $y(x) = ax$. Indeed, we have

$$\int_0^x [1 + y'(t)^2]^{1/2}\, dt = \int_0^x \sqrt{1 + a^2}\, dt = \sqrt{x^2 + (ax)^2} = \sqrt{x^2 + [y(x)]^2}, \quad x \geq 0.$$

Therefore, in what follows we deal with the case $p \in (1, 2)$.

LEMMA 4.2. *Let $1 < p < 2$. If a function $y = y(x) \geq 0$ for $x \geq 0$ satisfies equation* (3.5) *and*

$$a := y'(0)$$

then either $a = 0$ or $a = +\infty$.

Proof. Note that (3.5) can be written in the form

$$\text{(4.1)} \quad \sqrt{1 + y'(x)^2} = \left(\frac{x}{(x^p + y(x)^p)^{1/p}}\right)^{p-1} + \left(\frac{y(x)}{(x^p + y(x)^p)^{1/p}}\right)^{p-1} y'(x), \quad x > 0.$$

Since

$$A := \lim_{x \to 0+} \frac{x}{(x^p + y(x)^p)^{1/p}} = \lim_{x \to 0+} \frac{1}{(1 + (y(x)/x)^p)^{1/p}} = \frac{1}{(1 + a^p)^{1/p}},$$

$$B := \lim_{x \to 0+} \frac{y(x)}{(x^p + y(x)^p)^{1/p}} = \lim_{x \to 0+} \frac{y(x)/x}{(1 + (y(x)/x)^p)^{1/p}} = \frac{a}{(1 + a^p)^{1/p}},$$

we have

$$B = aA.$$

Of course

$$0 \leq A \leq 1.$$

Assume, on the contrary, that $0 < a < \infty$. Letting $x \to 0$ from the right in (4.1), we get

$$\sqrt{1 + a^2} = A^{p-1} + B^{p-1}a = A^{p-1} + (aA)^{p-1}a = A^{p-1}(1 + a^p)$$
$$= \left(\frac{1}{(1 + a^p)^{1/p}}\right)^{p-1} (1 + a^p) = (1 + a^p)^{1/p},$$

whence $p = 2$. This contradiction completes the proof. ∎

Squaring (3.5) we obtain, for $y = y(x)$, the differential equation

$$\text{(4.2)} \quad [(x^p + y^p)^{2-2/p} - y^{2p-2}](y')^2 - 2x^{p-1}y^{p-1}y' + [(x^p + y^p)^{2-2/p} - x^{2p-2}] = 0.$$

Now we prove the following

LEMMA 4.3. *If $p \in (1,2)$ then, for all $x, y > 0$,*

$$x^{2p-2} + y^{2p-2} > (x^p + y^p)^{2-2/p} > 0$$

and

$$(xy)^{p-1} > (x^p + y^p)^{1-1/p} \sqrt{[x^{2p-2} + y^{2p-2} - (x^p + y^p)^{2-2/p}]}.$$

Proof. Since $0 < 2 - 2/p < 1$, the function $\varphi(t) = t^{2-2/p}$ is subadditive in $(0, \infty)$. Thus, for all $x, y > 0$,

$$(x^p + y^p)^{2-2/p} < (x^p)^{2-2/p} + (y^p)^{2-2/p} = x^{2p-2} + y^{2p-2},$$

which proves the first inequality. From this inequality, for all $x, y > 0$,

$$\begin{aligned}(xy)^{2(p-1)} &= (xy)^{2(p-1)} - ((x^p + y^p)^{2-2/p})^2 + ((x^p + y^p)^{2-2/p})^2 \\ &> (xy)^{2(p-1)} - (x^p + y^p)^{2-2/p}(x^{2p-2} + y^{2p-2}) + ((x^p + y^p)^{2-2/p})^2 \\ &= ((xy)^{p-1})^2 \\ &\quad - \left((x^p + y^p)^{1-1/p}\sqrt{[x^{2p-2} + y^{2p-2} - (x^p + y^p)^{2-2/p}]}\right)^2 > 0,\end{aligned}$$

that is, the second inequality holds true. ∎

This lemma implies that a function $y = y(x)$ satisfies equation (4.2) iff either

$$y' = f_1(x, y), \tag{4.3}$$

where

$$f_1(x,y) := \frac{(xy)^{p-1} + (x^p + y^p)^{1-1/p}\sqrt{x^{2p-2} + y^{2p-2} - (x^p + y^p)^{2-2/p}}}{(x^p + y^p)^{2-2/p} - y^{2p-2}};$$

or

$$y' = f_2(x, y), \tag{4.4}$$

where

$$f_2(x,y) := \frac{(xy)^{p-1} - (x^p + y^p)^{1-1/p}\sqrt{x^{2p-2} + y^{2p-2} - (x^p + y^p)^{2-2/p}}}{(x^p + y^p)^{2-2/p} - y^{2p-2}}.$$

Applying Lemma 4.3 and making some easy calculations, we obtain

LEMMA 4.4. *Let $p \in (1,2)$. The functions $f_i : \mathbb{R}^2_+ \setminus (0,0) \to \mathbb{R}$, $i = 1, 2$, defined above are:*

1. *continuous, positive in $(0,\infty)^2$, and*
$$f_i(x, 0) = 0 \quad \text{for all } x > 0$$
(so f_i are non-negative in $\mathbb{R}^2_+ \setminus (0,0)$);
2. *of class C^∞ in $(0,\infty)^2$;*
3. *homogeneous of order 0, that is,*
$$f_i(x,y) = g_i(y/x), \quad x, y \geq 0,$$

where $g_i : \mathbb{R}_+ \to \mathbb{R}_+$ *given by*

$$g_1(u) := f_1(1,u) = \frac{u^{p-1} + (1+u^p)^{1-1/p}\sqrt{1+u^{2p-2}-(1+u^p)^{2-2/p}}}{(1+u^p)^{2-2/p}-u^{2p-2}},$$

$$g_2(u) := f_2(1,u) = \frac{u^{p-1} - (1+u^p)^{1-1/p}\sqrt{1+u^{2p-2}-(1+u^p)^{2-2/p}}}{(1+u^p)^{2-2/p}-u^{2p-2}},$$

is onto, strictly increasing,

$$0 < g_2(u) < u < g_1(u) \quad \text{for } u > 0, \quad g_1(0) = g_2(0) = 0,$$

and for all $u > 0$,

$$g_1(u)g_2(u) = 1.$$

THEOREM 4.5. *Let* $p \in (1,2)$. *There is no solution* $y = y(x)$, $x \geq 0$, *of equation* (3.5) *such that* $y(0) = 0$ *and* $y(x) > 0$ *for all* $x > 0$.

Proof. Assume that $\mathbb{R}_+ \ni x \mapsto y(x)$ is such a solution. Then, for any point $(x, y(x))$ of its graph we would have

$$\int_0^x \sqrt{1+y'(t)^2}\,dt = (x^p + y(x)^p)^{1/p} = \|(x,y(x))\|_p, \qquad x > 0.$$

Hence for all $c > 0$ and $x > 0$,

$$c + \int_0^x \sqrt{1+y'(t)^2}\,dt = c + \int_c^{x+c} \sqrt{1+y'(t-c)^2}\,dt = \|(x+c, y(x+c))\|_p,$$

which means that, for any $c > 0$, the function

$$y_c(x) := \begin{cases} 0 & \text{for } x \in [0,c), \\ y(x-c) & \text{for } x \geq c \end{cases}$$

would also be a solution of (4.3). Thus the field of linear elements determined

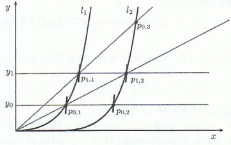

Fig. 1

by (4.3) would be such that on half-lines starting at $(x, y(x))$ and parallel to the x-axis the linear elements would have the same slope. This is impossible because, being homogeneous (part 3 of Lemma 4.4), equation (4.3) has parallel linear elements on each ray starting at $(0,0)$ (see Figure 1). ■

LEMMA 4.6. *Let $p \in (1,2)$. For the function $g_2(u) := f_2(1,u)$, $u > 0$, we have*

$$\lim_{u\to\infty} \frac{u}{g_2(u)} = \infty.$$

Proof. Note that, for all $p \in (1,2)$ and $u > 0$,

$$(1+u^p)^{1-1/p} > u^{p-1},$$

whence

$$\begin{aligned}\frac{u}{g_2(u)} &= \frac{u((1+u^p)^{2-2/p} - u^{2-2/p})}{u^{p-1} - (1+u^p)^{1-1/p}\sqrt{1+u^{2p-2} - (1+u^p)^{2-2/p}}}\\ &> \frac{u((1+u^p)^{2-2/p} - u^{2-2/p})}{u^{p-1} - u^{p-1}\sqrt{1+u^{2p-2} - (1+u^p)^{2-2/p}}}\\ &= \frac{u((1+u^p)^{2-2/p} - u^{2-2/p})(1+\sqrt{1+u^{2p-2} - (1+u^p)^{2-2/p}})}{u^{p-1}((1+u^p)^{2-2/p} - u^{2p-2})}\\ &= u^{2-p}\Big(1+\sqrt{1+u^{2p-2} - (1+u^p)^{2-2/p}}\Big) =: h(u).\end{aligned}$$

Since, obviously,

$$\lim_{u\to\infty} \Big(1+\sqrt{1+u^{2p-2} - (1+u^p)^{2-2/p}}\Big) = 2,$$

we have

$$\lim_{u\to\infty} h(u) = \infty,$$

which yields the conclusion. ■

To extend any local solution of equations (4.3) and (4.4) onto the whole interval $(0,\infty)$ we need the following

LEMMA 4.7. *Let $p \in (1,2)$. Equations (4.3) and (4.4) do not have blow-up solutions.*

Proof. By Lemma 4.6, there exists u_0 such that $u > g_2(u)$ for all $u > u_0$. Hence, putting

$$\Omega := \{(x,y) \in (0,\infty)^2 : y/x > u_0\},$$

we have $y/x > g_2(y/x)$ for all $(x,y) \in \Omega$.

Now, assume, on the contrary, that there exists a solution y^* of (4.4) (which, by Lemma 4.4, is increasing) that explodes at some $x_b \in (0,\infty)$, i.e.

$$\lim_{x\to x_b-} y^*(x) = \infty.$$

It follows that there exists an $x_0 \in (0, x_b)$ such that $(x, y(x)) \in \Omega$ for all $x \in (x_0, x_b)$. Take an $x_1 \in (x_0, x_b)$ and put $y_1 := y^*|_{[x_1,x_b)}$. The function y_1 is, on the interval $[x_1, x_b)$, a solution of the following Cauchy problem:

$$y' = g_2(y/x), \quad y(x_1) = y^*(x_1).$$

Note that another Cauchy problem,

$$y' = y/x, \quad y(x_1) = y^*(x_1) + 1,$$

has a unique solution y_2 on $[x_1, x_b)$ and

$$y_2(x) = \frac{y^*(x_1) + 1}{x_1}x \quad \text{for } x \in [x_1, x_b).$$

Since $(y^*(x_1) + 1)/x_1 > u_0$, we have $(x, y_2(x)) \in \Omega$ for all $x \in [x_1, x_b)$. The continuity of y_1 and y_2, the inequality $y_1(x_1) < y_2(x_1)$ and the assumed blowing-up of y_1 imply that there is an $x^* \in (x_1, x_b)$ such that $y_1(x^*) = y_2(x^*)$ and $y_1(x) < y_2(x)$ for all $x \in (x_1, x^*)$. In this case, by the respective differential equations, we obtain

$$y_1'(x^*) = g_2(y_1(x^*)/x^*) \quad \text{and} \quad y_2'(x^*) = y_1(x^*)/x^*.$$

By the relation between y_1 and y_2 we have $y_1'(x^*) \geq y_2'(x^*)$. Hence

$$g_2(y_1(x^*)/x^*) \geq y_1(x^*)/x^*.$$

On the other hand, as $(x^*, y_1(x^*)) \in \Omega$, we deduce that

$$g_2(y_1(x^*)/x^*) < y_1(x^*)/x^*.$$

This contradiction proves that equation (4.4) has no blow-up solutions.

From Lemma 4.4 we have $g_1(u)g_2(u) = 1$ for $u > 0$. It follows that (4.3) does not have blow-up solutions either. ∎

In the following, we confine our considerations to equation (4.3).

From the theory of differential equations (cf. for instance [2]) we deduce

THEOREM 4.8. *Assume that $1 < p < 2$. Let $x_0, y_0 > 0$ be fixed. Then there exists exactly one solution $y : [0,\infty) \to \mathbb{R}_+$ of equation (4.3) such that $y(x_0) = y_0$. Moreover there is a unique $c = c(x_0, y_0) > 0$ such that $y(x) = y_c(x)$ for all $x \geq 0$, where*

$$y_c(x) := \begin{cases} 0 & \text{for } x \in [0, c], \\ y(x) & \text{for } x > c, \end{cases} \qquad y'(c) = 0, \quad y(x) > 0 \text{ for } x > c;$$

y_c is strictly increasing in $[c, \infty)$, and, for every $x \geq c$,

$$\int_0^x \sqrt{1 + y_c'(t)^2}\,dt = c + \int_c^x \sqrt{1 + y'(t)^2}\,dt = (x^p + y_c(x)^p)^{1/p}.$$

Proof. Take arbitrary $x_0, y_0 > 0$. According to the theory of differential equations ([1], [2]) there exists a unique local solution $y = y(x)$ of (4.3) such that $y(x_0) = y_0$. The regularity of f_1 in $(0,\infty)^2$ (see Lemma 4.4) ensures the existence of its unique maximal extension with the graph contained in $(0,\infty)^2$. We denote it by $y = y(x)$ and its domain by I. By the first part of Lemma 4.4 we have $f_1 > 0$ in $(0,\infty)^2$, so y is strictly increasing in I. Since, by Lemma 4.7, equation (4.3) has no blow-up solutions, there is a $c = c(x_0, y_0) \geq 0$ such that $I = (c, \infty)$. Then either $c > 0$ or $c = 0$.

Assume first that $c > 0$. Then $y(c+) = 0$, as in the opposite case we would have $y(c) > 0$, and the solution $y = y(x)$ could be extended to an interval $(c - \varepsilon, \infty)$ for some $\varepsilon > 0$, contradicting the definition of c. Moreover, by part 1 of Lemma 4.4 we have $y'(c+) = 0$. Now the function $y_c : (0,\infty) \to \mathbb{R}_+$ given by

$$y_c(x) := \begin{cases} 0 & \text{for } x \in [0, c], \\ y(x) & \text{for } x > c, \end{cases}$$

is the required solution.

Assume now that $c = 0$. Then either $y(0+) = 0$ or $y(0+) > 0$. In view of Theorem 4.5 the first case cannot happen. Hence $y(0+) > 0$. Then $y(x) > 0$ for all $x > 0$ and

$$y'(x) = g_1(y(x)/x), \quad x > 0.$$

Differentiating both sides of this equality (by Lemma 4.4 we can do it), and then making use of this equality again, we get, for all $x > 0$,

$$y''(x) = g_1'\left(\frac{y(x)}{x}\right)\frac{xy'(x) - y(x)}{x^2} = \frac{1}{x}g_1'\left(\frac{y(x)}{x}\right)\left[g_1\left(\frac{y(x)}{x}\right) - \frac{y(x)}{x}\right].$$

Since, by part 3 of Lemma 4.4, $g_1(u) - u > 0$ for all $u > 0$, we hence get $y''(x) > 0$ for all $x > 0$, which proves that $y = y(x)$ is convex in $(0,\infty)$. Since it is strictly increasing, it follows that $y'(0+)$ is finite. Hence, applying Lemma 4.2, we get $y'(0+) = 0$. Making use of the equality

$$\sqrt{1 + y'(x)^2} = \left(\frac{x}{(x^p + y(x)^p)^{1/p}}\right)^{p-1} + \left(\frac{y(x)}{(x^p + y(x)^p)^{1/p}}\right)^{p-1} y'(x), \quad x > 0,$$

applied in the proof of Lemma 4.2, and letting $x \to 0$, we get a contradictory relation $1 = 0$. This completes the proof. ∎

REMARK 4.9. Clearly, $y(x) \equiv 0$ is a singular solution of (4.3), and for any $c > 0$, the solution y_c coincides with y on $[0, c]$.

5. Finding the solutions and a numerical approach. To find the desired curve issuing from $(0,0)$, or from any point $(r_0, 0)$ for $r_0 > 0$ (in the latter case the curve should go immediately into the open first quarter), is difficult even numerically. The reason is that each point $(r_0, 0)$, $r_0 \geq 0$, is singular for equation (3.1). In this section we discuss this problem.

Take $p \in (1,2)$ and fix $r_0 \geq 0$. It will be convenient to look for the curve (trajectory) starting from $(r_0, 0)$ in the form

$$(5.1) \qquad \begin{cases} x(t) = (r_0 + t)\mathbf{I}_{[-r_0,0]}(t) + r_0 r(t)[\cos\alpha(t)]^{2/p}\mathbf{I}_{(0,\infty)}(t), \\ y(t) = r_0 r(t)[\sin\alpha(t)]^{2/p}\mathbf{I}_{(0,\infty)}(t), \end{cases} \qquad t \geq -r_0,$$

where $r(t) > 1$, $0 < \alpha(t) < \pi/2$, the unknown functions $t \mapsto \alpha(t)$, $t \mapsto r(t)$ are of class C^1, and $\mathbf{I}_A$ denotes the indicator function of a set A. We assume that both α and r (which, of course, depend on the parameter p) are strictly increasing.

REMARK 5.1. It is easy to see that the trajectory (5.1) is continuous at the point r_0 iff

$$\lim_{t\to 0+} r(t) = 1 \quad \text{and} \quad \lim_{t\to 0+} \alpha(t) = 0.$$

Since (5.1) satisfies the relevant equation for $t \in (-r_0, 0]$ (cf. Remark 5.1), without any loss of generality we can confine our considerations to the trajectory

$$(5.2) \qquad \begin{cases} x(t) = r_0 r(t)[\cos\alpha(t)]^{2/p}, \\ y(t) = r_0 r(t)[\sin\alpha(t)]^{2/p}, \end{cases} \qquad t > 0.$$

Note that the l^p-distance of points of the curve (5.2) from $(0,0)$ depends only on $r_0 r(t)$ and

$$[x(t)^p + y(t)^p]^{1/p} = r_0 r(t), \quad t > 0.$$

It follows that equation (3.1) takes the form

$$\int_0^t \sqrt{x'(s)^2 + y'(s)^2}\, ds = r_0 r(t), \quad t > 0.$$

Differentiating both sides with respect to t we obtain

$$(5.3) \qquad x'(t)^2 + y'(t)^2 = (r_0 r(t))^2, \quad t > 0.$$

REMARK 5.2. If the trajectory (5.2) satisfies (5.3), then the functions α and r satisfy the differential equation

$$(5.4) \qquad r'(t) = g(\alpha(t))\alpha'(t) r(t), \quad t > 0,$$

with

$$g(z) = g_1(z) = \frac{\sqrt{c(z)^2 - 4b(z)d(z)} - c(z)}{2b(z)}$$

or

$$g(z) = g_2(z) = \frac{-\sqrt{c(z)^2 - 4b(z)d(z)} - c(z)}{2b(z)}$$

where

$$b(z) = (\sin z)^{2/p} + (\cos z)^{2/p} - 1,$$
$$c(z) = \frac{4}{p}(\cot z\,(\sin z)^{4/p} - \tan z\,(\cos z)^{4/p}),$$
$$d(z) = \frac{4}{p^2}(\cot^2 z\,(\sin z)^{4/p} + \tan^2 z\,(\cos z)^{4/p}).$$

Proof. Differentiating both sides of (5.3), after simplification, we get

$$((\sin\alpha(t))^{2/p} + (\cos\alpha(t))^{2/p} + 1)r'(t)^2$$
$$+\frac{4}{p}(\cot\alpha(t)\,(\sin\alpha(t))^{4/p} - \tan\alpha(t)\,(\cos\alpha(t))^{4/p})r(t)r'(t)\alpha'(t)$$
$$+\frac{4}{p^2}(\cot^2\alpha(t)\,(\sin\alpha(t))^{4/p} + \tan^2\alpha(t)\,(\cos\alpha(t))^{4/p})(r(t)\alpha'(t))^2 = 0,$$

which is quadratic with respect to $r'(t)$. Hence, as $r'(t) > 0$ for $t > 0$, we obtain (5.4). ■

REMARK 5.3. The functions g_1 and g_2 defined in Remark 5.2 are related by the equality

$$g_1(z) = -g_2(\pi/2 - z), \quad z \in (0, \pi/2).$$

Let us note that equation (5.4), and hence (5.3), has two solutions though (5.1) is the unique solution of (3.1). This follows from the fact that the curve symmetric to (5.1) with respect to the diagonal generates the same equation (5.3). One of these curves is related to g_1 and the other to g_2. Therefore, in the following, we confine our considerations to $g = g_1$.

Note also that the function g in equation (5.4) depends on an unknown function α which, to a large extent, can be arbitrarily chosen.

Figure 2 presents the graph of g_1 depending on $\alpha(t)$ for several values of p.

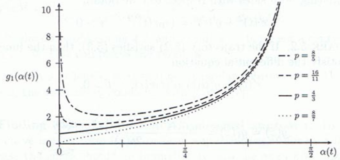

Fig. 2

Note that the functions α and r in (5.4) are closely related. For a particular α, equation (5.4) becomes a differential equation for the function r, and vice versa, for a chosen r, equation (5.4) is a differential equation for α. Thus, for an arbitrary α satisfying the above mentioned conditions there exists a function r such that equation (5.4) is satisfied; and vice versa.

The mutual dependence between $r(t)$ and $\alpha(t)$ for $t > 0$ is shown by the following

REMARK 5.4. Let $G : (0, \pi/2) \to \mathbb{R}_+$ be given by

$$G(z) := \lim_{\varepsilon \to 0+} \int_{\varepsilon}^{z} g_1(\tau)\, d\tau. \tag{5.5}$$

Then G is strictly increasing and onto, that is, $G((0, \pi/2)) = \mathbb{R}_+$ (of course, G is a primitive of g_1). Moreover, the functions α and r satisfy equation (5.2) iff

$$G \circ \alpha = \log \circ\, r$$

that is, iff

$$G(\alpha(t)) = \log(r(t)), \quad t > 0, \tag{5.6}$$

or, equivalently,

$$r(t) = \exp G(\alpha(t)), \quad t > 0. \tag{5.7}$$

Since the function G has no effective elementary representation, we have to use some numerical methods to find the trajectory in question.

Applying the above remark we obtain

THEOREM 5.5. *For $p \in (1, 2)$ and $r_0 > 0$, the graph of the curve*

$$\begin{cases} x(t) = (r_0 + t)\mathbf{I}_{[-r_0,0]}(t) + r_0 \exp(G(t))[\cos t]^{2/p}\mathbf{I}_{(0,\pi/2)}(t), \\ y(t) = r_0 \exp(G(t))[\sin t]^{2/p}\mathbf{I}_{(0,\infty)}(t), \end{cases} \tag{5.8}$$

for $-r_0 \le t < \pi/2$, where G is given by (5.5), is the graph of a solution of equation (4.3).

Proof. Let $\alpha(t) := t$ for $t \in (0, \pi/2)$. Then, from (5.7), $r(t) = \exp(G(t))$ for $t \in (0, \pi/2)$, whence, taking into account (5.1), we get (5.8). ∎

Note that the function G is of universal character as it depends only on p. This fact has here an essential practical meaning: once fixed, G can be used to get the trajectories emanating from any chosen point $(r_0, 0)$.

Figure 3 shows the shape of the curves in question starting from $(1, 0)$ for different values of p.

With this background the following natural question arises. From which point of the nonnegative part of the x-axis should one start (into the open

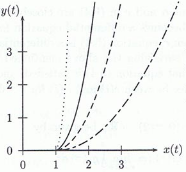

Fig. 3

first quarter) to get to a given point $(x_0, y_0) \in (0,\infty)^2$ along the trajectory in question?

An answer is given by the following

THEOREM 5.6. *If $(x_0, y_0) \in (0,\infty)^2$ then the curve starting from $(r_0, 0)$ such that*

$$r_0 = \frac{\sqrt{x_0^{2/p} + y_0^{2/p}}}{G((\arctan(y_0/x_0))^{p/2})}$$

passes through (x_0, y_0).

Proof. Applying (5.8) with the above r_0 and $t = (\arctan(y_0/x_0))^{p/2}$ we obtain $x(t) = x_0$ and $y(t) = y_0$. ∎

6. Case $p = 1$. Take arbitrary nonnegative strictly increasing sequences $x_i, y_i \in \mathbb{R}_+$, $i = 0, 1, \ldots$, such that $x_0 = y_0 = 0$ and

$$0 = x_0 < \cdots < x_k < \cdots, \quad 0 = y_0 < y_1 < \cdots < y_k < \cdots.$$

Denote by R_i the line segment joining (x_i, y_i) and (x_{i+1}, y_i), and by U_i the segment joining (x_{i+1}, y_i) and (x_{i+1}, y_{i+1}) for $i = 0, 1, \ldots$. Let (x, y) be an arbitrary point on the piecewise linear curve consisting of the consecutive segments

$$R_0, U_0, R_1, U_1, \ldots.$$

Then, obviously, the Euclidean length of this curve joining $(0,0)$ and (x, y) is equal to $\|(x,y)\|_1$. Thus, in this case, there are a lot of curves having the property under study. These curves are not differentiable at any of their vertices. However we have the following obvious

REMARK 6.1. Let $p = 1$ and fix $(x_0, y_0) \in (0,\infty)^2$. There exist exactly two curves

$$(0,\infty) \ni t \mapsto (x(t), y(t))$$

in $\mathbb{R}^2$ with the property considered, passing through (x_0, y_0) with only one non-differentiability point: either

$$x(t) = \begin{cases} t & \text{for } t \in (0, x_0], \\ x_0 & \text{for } t \in (x_0, \infty), \end{cases} \qquad y(t) = \begin{cases} 0 & \text{for } t \in (0, x_0], \\ t - x_0 & \text{for } t \in (x_0, \infty), \end{cases}$$

or

$$x(t) = \begin{cases} 0 & \text{for } t \in (0, y_0], \\ t - y_0 & \text{for } t \in (y_0, \infty), \end{cases} \qquad y(t) = \begin{cases} t & \text{for } t \in (0, y_0], \\ y_0 & \text{for } t \in (y_0, \infty). \end{cases}$$

7. Final remarks. The problem being considered has a kinematical interpretation. If a material point located at $(0, 0)$, moving only along the trajectories described above, is going to meet a point (x_0, y_0) (say, a rocket), then it has to go along the x-axis and into the open first quarter at the point $(0, c(y_0, x_0))$.

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