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A FUNCTIONAL EQUATION RELATED TO AN EQUALITY OF MEANS PROBLEM

BY

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Abstract. The functional equation

$$\frac{F(x)-F(y)}{x-y} = \frac{G(x)+G(y)}{H(x)+H(y)}$$

where F, G, H are unknown functions is considered. Some motivations, coming from the equality problem for means, are presented.

Introduction. In this paper we consider the functional equation

$$\frac{F(x)-F(y)}{x-y} = \frac{G(x)+G(y)}{H(x)+H(y)}$$

where F, G, H are unknown real-valued functions defined in a real interval and H is positive (or negative). The main result of Section 1 (Theorem 1) gives a general solution. A motivation for these considerations is given in Section 2 where we show that some special equality of means problems lead to this equation.

1. A functional equation

THEOREM 1. *Let $I \subset \mathbb{R}$ be an interval. The functions $F, G : I \to \mathbb{R}$ and $H : I \to (0, \infty)$ satisfy the functional equation*

$$\frac{F(x)-F(y)}{x-y} = \frac{G(x)+G(y)}{H(x)+H(y)}, \quad x, y \in I,\ x \neq y, \tag{1}$$

if, and only if, one of the following cases occurs:

(i) *H is arbitrary and there are some constants $a, b \in \mathbb{R}$ such that*

$$G(x) = aH(x), \quad F(x) = ax + b, \quad x \in I;$$

(ii) *H is a constant function, i.e.*

$$H(x) = a, \quad x \in I,$$

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for some $a \in (0, \infty)$ *and there are constants* $b, c, d \in \mathbb{R}$, $b \neq 0$, *such that*

$$G(x) = bx + c, \qquad F(x) = \frac{b}{2a}x^2 + \frac{c}{a}x + d, \qquad x \in I;$$

(iii) *there are* $x_0 \in I$ *and* $k, m, p, q \in \mathbb{R}$, $m \neq 0$, *such that, for all* $x \in I$,

$$H(x) = kH_0(x - x_0), \quad G(x) = k[mG_0(x - x_0) + pH_0(x - x_0)],$$
$$F(x) = mF_0(x - x_0) + p(x - x_0) + q,$$

where the function $H_0 : I - x_0 \to (0, \infty)$ *is given by*

$$H_0(x) := \sqrt{1 + bx + ax^2}, \qquad x \in I - x_0,$$

for some $a, b \in \mathbb{R}$ *and* $a^2 + b^2 \neq 0$, *and the functions* $G_0, F_0 : I - x_0 \to \mathbb{R}$ *are given by*

$$G_0(x) := \frac{s[H_0(x) - 1] + x[H_0(s) + 1]}{x[H_0(s) - 1] + s[H_0(x) + 1]} \cdot \frac{[H_0(x) + 1]}{[H_0(s) + 1]}, \qquad x \in I - x_0;$$

$$F_0(x) := \frac{xG_0(x)}{H_0(x) + 1}, \qquad x \in I - x_0,$$

where $s \in I - x_0$, $s \neq 0$, *is fixed.*

Proof. Assume that the functions F, $G : I \to \mathbb{R}$ and $H : I \to (0, \infty)$ satisfy equation (1).

Suppose that there is a nontrivial subinterval $J \subset I$ and a constant $a \in \mathbb{R}$ such that $G(x) = aH(x)$ for all $x \in J$ (that is, G is proportional to H on J). Then, by (1), we have $F(x) = ax + b\,(x \in J)$ for some real b. Suppose that there is $y \in I \setminus J$ such that $F(y) \neq ay + b$. From (1) we get

$$\frac{ax + b - F(y)}{x - y} = \frac{aH(x) + G(y)}{H(x) + H(y)}, \qquad x \in J,$$

whence, for all $x \in J$,

$$[F(y) - ay - b]H(x) = [G(y) - aH(y)]x + [yG(y) + bH(y) - F(y)H(y)].$$

It follows that, for some real c, d,

$$H(x) = cx + d, \qquad x \in J.$$

Inserting this function in the previous equation we obtain

$$[cF(y) + aH(y) - G(y) - acy - bc]x$$
$$+ [F(y)H(y) + dF(y) - yG(y) - bH(y) - ady - bd] = 0$$

for all $x \in J$. It follows that

$$cF(y) + aH(y) - G(y) - acy - bc = 0$$

and

$$F(y)H(y) + dF(y) - yG(y) - bH(y) - ady - bd = 0.$$

If $c = 0$ then from the first of these equations we get

$$G(y) = aH(y).$$

If $c \neq 0$ then again from the first equation,

$$F(y) = \frac{1}{c}[G(y) - aH(y) + acy + bc].$$

Inserting this value in the second equation we obtain

$$G(y) = aH(y).$$

Thus we have shown that if G is proportional to H on any nontrivial subinterval of I, then G is proportional to H on the whole I. It follows that $F(x) = ax + b$ and $G(x) = aH(x)$ for all $x \in I$. It is easy to see that for arbitrary $H : I \to (0, \infty)$ these functions satisfy (1).

Now we can assume that G is not proportional to H on any nontrivial subinterval of I.

Suppose that H is constant on some nontrivial subinterval $J \subset I$, i.e. there is $a > 0$ such that $H(x) = a$ for all $x \in J$. From (1) we get

$$2a[F(x) - F(y)] = [G(x) + G(y)](x - y), \qquad x, y \in J,\ x \neq y.$$

Since

$$2a[F(x) - F(y)] = 2a[F(x) - F(z)] + 2a[F(z) - F(y)],$$

we hence get

$$[G(x) + G(y)](x - y) = [G(x) + G(z)](x - z) + [G(z) + G(y)](z - y)$$

for all $x, y, z \in J$, $x \neq y \neq z \neq x$. Take arbitrary $x, y \in J$, $x \neq y$, and $t \in (0, 1)$. Setting in the above equation $z = tx + (1 - t)y$ we get

$$G(tx + (1 - t)y) = tG(x) + (1 - t)G(y),$$

whence $G(x) = bx + c$ ($x \in J$) for some real b, c. We may assume that $b \neq 0$ as, in the opposite case, G would be proportional to H on J. Equation (1) implies that $F(x) = \frac{b}{2a}x^2 + \frac{c}{a}x + d$ ($x \in J$) for some real d.

Now assume that there is $y \in I \setminus J$. From (1) we get

$$\frac{\frac{b}{2a}x^2 + \frac{c}{a}x + d - F(y)}{x - y} = \frac{bx + c + G(y)}{a + H(y)}, \qquad x \in I,$$

which can be written in the form

$$\frac{b[H(y) - a]}{2a}x^2 + \frac{aby - 2ac - 2cG(y) - cH(y)}{a}x + cy + yG(y) + ad - aF(y) + dH(y) - F(y)H(y) = 0.$$

Since $b \neq 0$, it follows that $H(y) = a$. Thus $H(x) = a$ for all $x \in I$, and consequently there are real b, c, d such that $F(x) = \frac{b}{2a}x^2 + \frac{c}{a}x + d$ and $G(x) = bx + c$ for all $x \in I$.

Now consider the case when on any nontrivial subinterval of I neither G is proportional to H, nor H is constant.

Assume first that $0 \in I$.

Let us remark that *if F, G and H satisfy equation* (1) *then, for all $m, p, q, r \in \mathbb{R}$, $r > 0$, the functions F_1, G_1 and H_1 defined by*

$$F_1(x) := mF(x) + px + q, \quad G_1(x) := r[mG(x) + pH(x)], \\ H_1(x) := rH(x) \tag{2}$$

also satisfy (1).

Therefore, for convenience, we may also assume that

$$F(0) = G(0) = 0 \quad \text{and} \quad H(0) = 1.$$

Hence, setting $y = 0$ in (1) we obtain

$$F(x) = \frac{xG(x)}{H(x)+1}, \qquad x \in I, \tag{3}$$

whence, from (1), for all $x, y \in I$, $x \neq y$,

$$\left[\frac{x[H(x)+H(y)]}{H(x)+1} - x + y\right] G(x) = \left[\frac{y[H(x)+H(y)]}{H(y)+1} + x - y\right] G(y). \tag{4}$$

Replacing y in (4) by $z \in I$, $z \neq y$, we get

$$\left[\frac{x[H(x)+H(z)]}{H(x)+1} - x + z\right] G(x) = \left[\frac{z[H(x)+H(z)]}{H(z)+1} + x - z\right] G(z),$$

whence

$$G(x) = G(z)\frac{z[H(x)-1] + x[H(z)+1]}{x[H(z)-1] + z[H(x)-1]} \cdot \frac{H(x)+1}{H(z)+1}. \tag{5}$$

Replacing $G(x)$ in (4) by the right-hand side of (5), and simultaneously $G(y)$ by the right-hand side of (5) with x replaced by y, we obtain an equality which after simple calculations gives

$$yz(z-y)H(x)^2 + zx(x-z)H(y)^2 + xy(y-x)H(z)^2 \\ = yz(z-y) + zx(x-z) + xy(y-x) \tag{6}$$

for all $x, y, z \in I$, $x \neq y \neq z \neq x$. Obviously, (6) is satisfied for all $x, y, z \in I$. Taking any fixed $y, z \in I$, $y \neq z$, we infer that H has to be of the form

$$H(x) = \sqrt{ax^2 + bx + c}, \qquad x \in I,$$

for some $a, b, c \in \mathbb{R}$. Since $H(0) = 1$ we hence get

$$H(x) = H_0(x) = \sqrt{ax^2 + bx + 1}, \qquad x \in I, \tag{7}$$

and $a^2 + b^2 \neq 0$ as H is not constant.

Taking a fixed $z := s \in I$, $s \neq 0$, in (5) we get

$$G(x) = G(s)\frac{s[H_0(x)-1] + x[H_0(s)+1]}{x[H_0(s)-1] + s[H_0(s)-1]} \cdot \frac{H_0(x)+1}{H_0(s)+1}$$

for $x \in I$, whence

$$(8) \qquad \begin{aligned} &G(x) = qG_0(x) \quad \text{where} \\ &G_0(x) := \frac{s[H_0(x)-1] + x[H_0(s)+1]}{x[H_0(s)-1] + s[H_0(s)-1]} \cdot \frac{H_0(x)+1}{H_0(s)+1}, \quad x \in I, \end{aligned}$$

and, by (3),

$$(9) \qquad F(x) = qF_0(x) \quad \text{where} \quad F_0(x) := \frac{xG_0(x)}{H_0(x)+1}, \quad x \in I,$$

with $q := G(s) \neq 0$.

We have additionally assumed that $F(0) = G(0) = 0$ and $H(0) = 1$. In view of (2) and the relevant remark we conclude that the functions

$$H(x) = kH_0(x), \quad G(x) = k[mG_0(x) + pH_0(x)],$$
$$F(x) := mF_0(x) + px + q, \quad x \in I,$$

where $k, m, p, q \in \mathbb{R}$, $am \neq 0$, form a general solution of equation (1) if H is not constant and $0 \in I$.

Assume that $0 \notin I$.

If the functions $F, G : I \to \mathbb{R}$ and a nonconstant function $H : I \to (0, \infty)$ satisfy (1), then, for each $x_0 \in I$, the functions $F_1, G_1 : I - x_0 \to \mathbb{R}$ and $H_1 : I - x_0 \to (0, \infty)$ defined by

$$F_1(x) := F(x + x_0), \quad G_1(x) := G(x + x_0),$$
$$H_1(x) := H(x + x_0), \quad x \in I - x_0,$$

satisfy (1) in the interval $J := I - x_0$, and $0 \in J$.

Hence, applying the previous step to the functions F_1, G_1, H_1 we infer that

$$H(x) = kH_0(x - x_0), \quad G(x) = k[mG_0(x - x_0) + pH_0(x - x_0)],$$
$$F(x) = mF_0(x - x_0) + p(x - x_0) + q$$

for all $x \in I$.

It is easy to see that the functions given in (i) and (ii) satisfy (1). So, to finish the proof it is enough to verify that the functions F, G and H given in (iii) satisfy (1). In the calculations, without any loss of generality, we may assume that $x_0 = 0$, that is, H, G, F are given by (7)–(9) with arbitrary fixed $q, s \in \mathbb{R}$, $q \neq 0$. For $x, y \in I$, $x \neq y$, we have

$$\begin{aligned}
\frac{F(x)-F(y)}{x-y} - \frac{G(x)+G(y)}{H(x)+H(y)}
&= \frac{[mF_0(x)+px+q]-[mF_0(y)+py+q]}{x-y} \\
&\quad - \frac{k[mG_0(x)+pH_0(x)]+k[mG_0(y)+pH_0(y)]}{kH_0(x)+kH_0(y)} \\
&= \left(m\frac{F_0(x)-F_0(y)}{x-y}+p\right) - \left(m\frac{G_0(x)+G_0(y)}{H_0(x)+H_0(y)}+p\right) \\
&= m\left(\frac{F_0(x)-F_0(y)}{x-y} - \frac{G_0(x)+G_0(y)}{H_0(x)+H_0(y)}\right).
\end{aligned}$$

Since $F_0(x) = xG_0(x)/(H_0(x)+1)$, the numerator of the expression

$$\frac{F_0(x)-F_0(y)}{x-y} - \frac{G_0(x)+G_0(y)}{H_0(x)+H_0(y)}$$

is

$$\begin{aligned}
&G_0(x)(H_0(y)+1)[x(H_0(y)-1)+y(H_0(x)+1)] \\
&\qquad - G_0(y)(H_0(x)+1)[x(H_0(y)+1)+y(H_0(x)-1)].
\end{aligned}$$

Setting here, according to (8),

$$G_0(x) := \frac{s[H_0(x)^2-1]+[H_0(s)+1]x[H_0(x)+1]}{x[H_0(s)^2-1]+[H_0(x)+1]s[H_0(s)+1]}, \quad x \in I,$$

we see that the numerator is

$$\begin{aligned}
&ys(s-y)H_0(x)^2 + sx(x-s)H_0(y)^2 + xy(y-x)H_0(s)^2 \\
&\qquad - ys(s-y) - sx(x-s) - xy(y-x),
\end{aligned}$$

which, in view of (6), vanishes for all $x, y \in I$ and $s \in I, s \neq 0$. This completes the proof. ∎

REMARK 1. If $b^2 - 4a = 0$ in the third part of the theorem then $H_0(x) = 1 + \frac{b}{2}x$ for $x \in I - x_0$. In this case the function H is affine, while G and F are rational functions of degree 2 and 3, respectively.

If $a = 0$ then $H_0(x) = \sqrt{1+bx}$ for $x \in I - x_0$.

REMARK 2. Let $I, J \subset \mathbb{R}$ be intervals and $\gamma : J \to I$ be one-to-one and onto. The functional equation

$$\frac{f(x)-f(y)}{\gamma(x)-\gamma(y)} = \frac{g(x)+g(y)}{h(x)+h(y)}, \quad x, y \in J,\ x \neq y,$$

with unknown $f, g, h : I \to \mathbb{R}$ and a bijective function $\gamma : J \to I$ is equivalent to equation (1) with $F := f \circ \gamma^{-1}$, $G := g \circ \gamma^{-1}$, and $H := h \circ \gamma^{-1}$.

REMARK 3. In part (iii) of Theorem 1, the function $G_0 : I - x_0 \to \mathbb{R}$ can be written in the form

$$G_0(x) := x\frac{H_0(x) + \frac{s}{r+1}(ax+b) + 1}{sH_0(x) + (r-1)x + s}, \quad x \in I - x_0,$$

where $r = \sqrt{1 + bs + as^2}$ for some $s \in \mathbb{R}$, $s \neq 0$, such that $1 + bs + as^2 \geq 0$. In this formula the function H_0 occurs only two times.

REMARK 4. Equation (1) can be written in the form

$$[F(x) - F(y)][H(x) + H(y)] - (x - y)[G(x) + G(y)] = 0$$

for all $x, y \in I$, $x \neq y$. If we admit $x = y$ then this equation becomes a special case of a more general functional equation considered in [1, pp. 161–165]. However the method applied in [1] is not helpful in solving (1).

2. Some problems leading to equation (1). In this section we present some special equality problems for means which can be reduced to equation (1).

We begin by recalling the definitions of some means (cf. [3], [2])

Let $I \subset \mathbb{R}$ be an interval. A function $M : I^2 \to I$ is called a *mean in I* if

$$\min(x, y) \leq M(x, y) \leq \max(x, y), \quad x, y \in I.$$

A mean M is called *strict* if these inequalities are sharp for all $x, y \in I, x \neq y$, and *symmetric* if $M(x, y) = M(y, x)$ for all $x, y \in I$.

A mean $M : I^2 \to I$ in an interval I is called *Lagrangean* if there is a continuous and strictly monotonic function $f : I \to \mathbb{R}$, a *generator* of the mean, such that $M = L^{[f]}$, where

$$L^{[f]}(x, y) := \begin{cases} f^{-1}\left(\frac{1}{x-y}\int_x^y f(t)\,dt\right) & \text{for } x \neq y, \\ x & \text{for } x = y. \end{cases}$$

If $F : I \to \mathbb{R}$ is a primitive of f then

$$L^{[f]}(x, y) = f^{-1}\left(\frac{F(x) - F(y)}{x - y}\right), \quad x \neq y.$$

A mean $M : I^2 \to I$ is called a *Cauchy mean* if there are continuous functions $f, g : I \to \mathbb{R}$, with $g(x) \neq 0$ for all $x \in I$, and f/g one-to-one such that $M = C^{[f,g]}$ where

$$C^{[f,g]}(x, y) := \left(\frac{f}{g}\right)^{-1}\left(\frac{F(x) - F(y)}{G(x) - G(y)}\right), \quad x \neq y; \quad C^{[f,g]}(x, x) := x,$$

and F and G are primitive functions of f and g, respectively. The functions f and g are called the *generators* of the mean $C^{[f,g]}$.

Obviously, the Cauchy means are strict, symmetric, continuous, and $C^{[f,id]} = L^{[f]}$. Let us also note the following

LEMMA 1 ([6]). *Let $f, g : I \to \mathbb{R}$ be continuous, $g \neq 0$ and f/g one-to-one. Then G, a primitive of g, is invertible, and*
$$C^{[f,g]}(x,y) = G^{-1}\big(L^{[f\circ G^{-1}]}(G(x), G(y))\big), \quad x, y \in I,$$
where $L^{[f\circ G^{-1}]}$ is the Lagrangean mean of the generator $f \circ G^{-1}$.

Note that $C^{[f,g]} = C^{[g,f]}$ ([6]).

A mean $M : I^2 \to I$ is called *quasi-arithmetic* (cf. [4, Chapter II]) if there is a continuous and strictly monotonic function $\varphi : I \to \mathbb{R}$, a *generator* of the mean, such that $M = A^{[\varphi]}$, where
$$A^{[\varphi]}(x,y) = \varphi^{-1}\left(\frac{\varphi(x)+\varphi(y)}{2}\right), \quad x, y \in I.$$

A mean $M : I^2 \to I$ is called a *Bajraktarević mean* if there is a continuous and strictly monotonic $f : I \to \mathbb{R}$ in I and a continuous function $g : I \to (0, \infty)$ such that $M = B_g^{[f]}$ where
$$B_g^{[f]}(x,y) := f^{-1}\left(\frac{g(x)f(x)+g(y)f(y)}{g(x)+g(y)}\right), \quad x, y \in I.$$

In general to solve the equality problem for some classes of means requires strong regularity assumptions and leads to differential equations (cf. [5] where a general equality problem of two Cauchy means, under the assumption that the generators involved are six times differentiable, is solved by reducing it to a Riccati differential equation).

The problems presented below are special cases of more general ones.

PROBLEM 1. *Determine all generators f of the Lagrange means for which $L^{[f]}$ are quasi-arithmetic.*

Assume that $L^{[f]} = A^{[\varphi]}$, that is,
$$f^{-1}\left(\frac{F(x)-F(y)}{x-y}\right) = \varphi^{-1}\left(\frac{\varphi(x)+(y)}{2}\right), \quad x, y \in I,\ x \neq y,$$
and that $f, \varphi : I \to \mathbb{R}$ are strictly monotonic, differentiable and such that $f'\varphi' \neq 0$ in I.

Differentiating both sides, first with respect to x and then with respect to y, we get
$$\frac{1}{f'(L_f(x,y))}\,\frac{f(x)(x-y)-F(x)+F(y)}{(x-y)^2} = \frac{1}{\varphi'(A^{[\varphi]}(x,y))}\,\frac{\varphi'(x)}{2},$$
$$\frac{1}{f'(L_f(x,y))}\,\frac{f(y)(y-x)-F(y)+F(x)}{(x-y)^2} = \frac{1}{\varphi'(A^{[\varphi]}(x,y))}\,\frac{\varphi'(y)}{2},$$
for all $x, y \in I$, $x \neq y$. Dividing the first equation by the second we obtain
$$\frac{f(x)(x-y)-F(x)+F(y)}{f(y)(y-x)-F(y)+F(x)} = \frac{\varphi'(x)}{\varphi'(y)}, \quad x, y \in I, x \neq y,$$

which can be written in the form

$$\frac{F(x)-F(y)}{x-y}=\frac{\frac{f(x)}{\varphi'(x)}+\frac{f(y)}{\varphi'(y)}}{\frac{1}{\varphi'(x)}+\frac{1}{\varphi'(y)}}, \quad x,y\in I, x\neq y.$$

Putting $H := 1/\varphi'$ and $G := f/\varphi'$ we get equation (1) and, applying Theorem 1, we can solve Problem 1 (under the above regularity assumption on f and φ).

REMARK 5. From the last equality we get

$$f^{-1}\left(\frac{F(x)-F(y)}{x-y}\right)=f^{-1}\left(\frac{\frac{f(x)}{\varphi'(x)}+\frac{f(y)}{\varphi'(y)}}{\frac{1}{\varphi'(x)}+\frac{1}{\varphi'(y)}}\right), \quad x,y\in I, x\neq y,$$

which proves that $L^{[f]}=B^{[f]}_{1/\varphi'}$.

PROBLEM 2. *Determine all generators f,g of the Cauchy means for which $C^{[f,g]}$ are quasi-arithmetic.*

If $C^{[f,g]}$ is quasi-arithmetic then there exists a strictly monotonic and continuous $\varphi : I \to \mathbb{R}$ such that

$$C^{[f,g]}(x,y)=\varphi^{-1}\left(\frac{\varphi(x)+\varphi(y)}{2}\right), \quad x,y\in I.$$

By Lemma 1 we have

$$G^{-1}(L^{[f\circ G^{-1}]}(G(x),G(y)))=\varphi^{-1}\left(\frac{\varphi(x)+\varphi(y)}{2}\right), \quad x,y\in I,$$

whence

$$L^{[f\circ G^{-1}]}(x,y)=G\circ\varphi^{-1}\left(\frac{\varphi\circ G^{-1}(x)+\varphi\circ G^{-1}(y)}{2}\right), \quad x,y\in G(I),$$

which proves the following

REMARK 6. The Cauchy mean $C^{[f,g]}$ is quasi-arithmetic with generator φ if, and only if, the Lagrangean mean $L^{[f\circ G^{-1}]}$ is quasi-arithmetic with generator $\varphi\circ G^{-1}$.

This remark reduces the second problem to the previous one.

REMARK. Let us mention that the problem of equality of the Cauchy means was considered in [5].

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