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RESEARCH ARTICLE

On discounted dynamic programming with unbounded returns

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Abstract In this paper, we apply the idea of k-local contraction of Rincón-Zapatero and Rodriguez-Palmero (Econometrica 71:1519–1555, 2003; Econ Theory 33:381–391, 2007) to study discounted stochastic dynamic programming models with unbounded returns. Our main results concern the existence of a unique solution to the Bellman equation and are applied to the theory of stochastic optimal growth. Also a discussion of some subtle issues concerning k-local and global contractions is included.

Keywords Stochastic dynamic programming · Bellman functional equation · Contraction mapping · Stochastic optimal growth

JEL Classification C61 · D90 · E20

1 Introduction

The theory of stochastic dynamic programming (or Markov decision processes) with uncountable state space started with the fundamental work of Blackwell (1965). His

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ideas were extended in many directions, with a number of applications to economics, engineering, and operations research were presented. For a good survey the reader is referred to Bertsekas and Shreve (1978), Hernández-Lerma and Lasserre (1999), Puterman (2005) and other books and articles. A large part of the theory of stochastic optimal growth lies in the framework of dynamic programming. The classical paper of Brock and Mirman (1972) as well as the book by Stokey et al. (1989) are very much related to Blackwell's work and deal with infinite state space models. However, many issues considered by economists (like properties of trajectories, steady states for specific models, etc.) are not covered in the aforementioned books on stochastic dynamic programming (control processes). In many applications of decision processes to operations research or economics it is natural to use unbounded return functions. The bounded case with discounted evaluation directly leads to the Banach contraction mapping theorem, see, e.g., Bertsekas and Shreve (1978) or Stokey et al. (1989). The unbounded case, however, requires different methods (techniques): "weighted norms" in the underlying function spaces, or limits of solutions for "truncated models", see Durán (2003), Hernández-Lerma and Lasserre (1999), Stokey et al. (1989) and others. A large survey of the existing literature on various economic models with unbounded returns can be found in a recent volume edited by Dana et al. (2006). Here, we mention important works by Boyd (1990), Boyd and Becker (1997), Le Van and Morhaim (2002), Le Van and Vailakis (2005) representing different methods and levels of generality. Moreover, the papers by Rincón-Zapatero and Rodriguez-Palmero (2003); Rincón-Zapatero and Rodriguez-Palmero (2007), which are point of our departure, contain a great deal of information on this topic, including models with recursive utility.

The aim of this paper is to apply the *valuable idea* of Rincón-Zapatero and Rodriguez-Palmero (2003) to k-local contraction to study stochastic dynamic programming models with unbounded return functions. Our main results are concerned with the existence of a unique solution to the Bellman equation and are applied to the theory of stochastic optimal growth. We give two applications motivated by the work of Stokey et al. (1989). Before describing our stochastic dynamic programming model and stating the results, we discuss one of the results given in Rincón-Zapatero and Rodriguez-Palmero (2003) in detail. It turns out that Proposition 1(b) stated in Rincón-Zapatero and Rodriguez-Palmero (2003) is *false*. In Sect. 2 we give a counterexample to support our claim. Proposition 1(b) is fundamental for some parts of the further research demonstrated by Rincón-Zapatero and Rodriguez-Palmero (2003); Rincón-Zapatero and Rodriguez-Palmero (2007). For our purpose, we present in Sect. 3 a modification of their approach and state some fixed point results related to their Theorem 2.[1] The metric induced by the norm defined in Sect. 3 is basically the same as in Theorem 2 in Rincón-Zapatero and Rodriguez-Palmero (2003), but we do not consider in our approach their concept of a "bounded set".

[1] After finishing the first draft of this paper, we obtained a communication from Martins-da-Rocha and Vailakis (2008) where a different counterexample is shown and different corrections to Rincón-Zapatero and Rodriguez-Palmero (2003) are given. We would like to thank Filipe Martins-da-Rocha and Yiannis Vailakis for some useful comments on our work.

Sections 4–6 contain applications to dynamic programming with stochastic transition functions and economic growth, respectively. Our results on stochastic optimal growth theory are new and can be applied to multi-sector models. The weighted norm approaches of Boyd (1990), Boyd and Becker (1997), Durán (2003) applied in economic models as well as the one by Hernández-Lerma and Lasserre (1999) applied in the theory of Markov decision processes are of different nature and require additional assumptions (see Remark 9).

2 Local contractions: a counterexample

In an interesting paper, Rincón-Zapatero and Rodriguez-Palmero (2003) address the issue of existence and uniqueness of solutions of the Bellman equation in the unbounded case. The proposed method is based on the Banach Fixed Point Principle and on an ingenious idea of construction of a special metric space. Unfortunately, part (b) of Proposition 1 in Rincón-Zapatero and Rodriguez-Palmero (2003) is *false*. Below we give a *counterexample*. In Sect. 3, we present some modifications of the results in Rincón-Zapatero and Rodriguez-Palmero (2003) which are very useful to study Markov decision processes, in particular stochastic optimal growth models with unbounded returns.

Throughout this paper N and R denote, respectively, the set of positive integers and the set of real numbers. Rincón-Zapatero and Rodriguez-Palmero (2003) assume that X is a topological space such that $X = \bigcup_{j=1}^{\infty} K_j$ where $\{K_j\}$ is an increasing sequence of compact subsets of X. Assume that

$$X = \bigcup_{j=1}^{\infty} \mathrm{Int}(K_j).$$

Let $C(X)$ denote the set of all continuous real-valued functions on X. Define

$$d_j(\phi, \psi) := \max_{x \in K_j} |\phi(x) - \psi(x)|, \quad j \in N.$$

Then $\{d_j\}$ is a countable family of semimetrics and d defined by

$$d(\phi, \psi) := \sum_{j=1}^{\infty} 2^{-j} \frac{d_j(\phi, \psi)}{1 + d_j(\phi, \psi)} \quad \text{for all } \phi, \psi \in C(X) \tag{1}$$

is a complete metric on $C(X)$; for more details see Lemma 1, Remarks 1(a) and 2 in Sect. 3.

Following Rincón-Zapatero and Rodriguez-Palmero (2003); Rincón-Zapatero and Rodriguez-Palmero (2007), we say that an operator $T : C(X) \mapsto C(X)$ is a *0-local contraction* relative to a set $G \subset C(X)$ if

$$d_j(T\phi, T\psi) \le \beta_j d_j(\phi, \psi) \quad \text{for each } j \in N \text{ and for all } \phi, \psi \in G, \tag{2}$$

where $0 \le \beta_j < 1$ for every $j \in N$.

Here and in the sequel $\mathbf{0}$ denotes the function ψ such that $\psi(x) = 0$ for all $x \in X$.

In Rincón-Zapatero and Rodriguez-Palmero (2003); Rincón-Zapatero and Rodriguez-Palmero (2007), a set $G \subset C(X)$ is called "bounded", if there is a sequence of positive real numbers $\{m_j\}$ such that $d_j(\phi, \mathbf{0}) \le m_j$ for each $\phi \in G$ and $j \in N$. Thus, if the set G contains an unbounded function ϕ, then the sequence $\{m_j\}$ must be unbounded as well.

A key role in some parts of Rincón-Zapatero and Rodriguez-Palmero (2003) plays the following statement (Proposition 1): If an operator $T : C(X) \mapsto C(X)$ is a 0-local contraction relative to a bounded set $G \subset C(X)$, then there exists a constant $\alpha \in [0, 1)$ such that

$$d(T\phi, T\psi) \le \alpha d(\phi, \psi) \quad \text{for all } \phi, \psi \in G. \tag{3}$$

It turns out that this proposition is false. An "a contrario" argument used in the proof (see page 1548, just before the Lebesgue dominated convergence theorem is applied) is erroneous.

Example 1 Assume that $X = (0, 1]$ and $K_j = [\frac{1}{j}, 1]$ for each $j \in N$. Let $\{m_j\}$ be an increasing sequence of positive numbers. Consider the "bounded set" $G \subset C(X)$ (in the sense of Rincón-Zapatero and Rodriguez-Palmero (2003); Rincón-Zapatero and Rodriguez-Palmero (2007)) containing functions $f_i (i \in N)$ such that $d_j(f_i, \mathbf{0}) = m_i$ for all $j \ge i$, and $d_j(f_i, \mathbf{0}) = 0$ for all $1 < j < i$. For instance take

$$f_i(x) = \begin{cases} m_i & \text{if} \quad 0 < x \le \frac{1}{i} \\ i(i-1)m_i\left(\frac{1}{i-1} - x\right) & \text{if} \quad \frac{1}{i} < x \le \frac{1}{i-1} \\ 0 & \text{if} \quad \frac{1}{i-1} < x \le 1 \end{cases}$$

for $i \in N$, $i > 1$, and $f_1 = m_1$. Assume that $\phi \in G$ if and only if there is some i such that $0 \le \phi(x) \le f_i(x)$ for all $x \in X$. Let $T\psi(x) := \beta\psi(x)$ for some $\beta \in (0, 1)$. Then $T : G \mapsto G$. Clearly, T is a 0-local contraction relative to the set G with $\beta_j = \beta$ for all $j \in N$. Take $i > 1$. Since $T\mathbf{0} = \mathbf{0}$ and $d_j(Tf_i, \mathbf{0}) = \beta m_i$ for all $j \ge i$, and $d_j(Tf_i, \mathbf{0}) = 0$ for all $j \in N$, $j < i$, we have

$$d(Tf_i, T\mathbf{0}) = \sum_{j=1}^{\infty} 2^{-j} \frac{d_j(Tf_i, \mathbf{0})}{1 + d_j(Tf_i, \mathbf{0})} = \sum_{j=i}^{\infty} 2^{-j} \frac{d_i(Tf_i, \mathbf{0})}{1 + d_i(Tf_i, \mathbf{0})}$$

$$= \frac{d_i(Tf_i, \mathbf{0})}{1 + d_i(Tf_i, \mathbf{0})} \sum_{j=i}^{\infty} 2^{-j} = 2^{-i+1} \frac{\beta m_i}{1 + \beta m_i}.$$

Suppose that there exists an $\alpha \in [0, 1)$ such that (3) holds. Taking $\phi = f_i$ and $\psi = \mathbf{0}$ in (3) we get

$$d(Tf_i, T\mathbf{0}) \leq \alpha d(f_i, \mathbf{0}) = \alpha \sum_{j=1}^{\infty} 2^{-j} \frac{d_j(f_i, \mathbf{0})}{1 + d_j(f_i, \mathbf{0})}$$
$$= \alpha \sum_{j=i}^{\infty} 2^{-j} \frac{d_i(f_i, \mathbf{0})}{1 + d_i(f_i, \mathbf{0})} = 2^{-i+1} \frac{\alpha m_i}{1 + m_i}.$$

It follows that

$$\frac{\beta m_i}{1 + \beta m_i} \leq \frac{\alpha m_i}{1 + m_i}$$

whence

$$m_i \leq \frac{\alpha - \beta}{\beta(1 - \alpha)}.$$

Since $i \in N$ is arbitrarily fixed, we have shown that the sequence $\{m_j\}$ is bounded and, consequently, the set G must be bounded in the usual sense. Note that, by the last inequality, the sequence $\{m_j\}$ can be unbounded only if $\alpha \geq 1$. Thus the unboundedness of the sequence $\{m_j\}$ excludes the contractivity of T.

The above example shows that the metric d given by (1) does not have the properties expected by Rincón-Zapatero and Rodriguez-Palmero (2003); Rincón-Zapatero and Rodriguez-Palmero (2007). This metric "kills" the contractivity of mappings on "bounded sets".[2]

3 Fixed points of local contractions

In this section, we present two fixed point results which are similar to those stated in Theorem 2 in Rincón-Zapatero and Rodriguez-Palmero (2003) and Proposition 1 in Rincón-Zapatero and Rodriguez-Palmero (2009). Let X be a nonempty set. By $\{K_j\}$ we shall denote a strictly increasing (in the sense of inclusion) sequence of subsets of X and assume that

$$X = \bigcup_{j=1}^{\infty} K_j. \tag{4}$$

Lemma 1 *Let $F(X)$ be a vector space of functions $\phi : X \mapsto R$ such that, for any $j \in N$,*

$$\|\phi\|_j := \sup_{x \in K_j} |\phi(x)| < \infty. \tag{5}$$

[2] Rincón-Zapatero and Rodriguez-Palmero (2009) recently corrected their Proposition 1 by changing the metric on $C(X)$ and under an additional assumption that $\sup_{j \in N} \beta_j < 1$.

Assume that

(a) *for every $i \in N$ the set $F(K_i)$ of restrictions of all functions $\phi \in F(X)$ to K_i endowed with the norm $\|\cdot\|_i$ is a Banach space,*

(b) *if for each $i \in N$, $\varphi_i \in F(K_i)$ and $\varphi_{j+1}(x) = \varphi_j(x)$ for all $x \in K_j$, $j \in N$, then φ defined by $\varphi(x) := \varphi_j(x)$ for $x \in K_j$ belongs to $F(X)$.*

Let $c > 1$ and $m = \{m_j\}$ be an increasing unbounded sequence of positive real numbers. Let $F_m(X)$ be the set of all $\phi \in F(X)$ such that

$$\|\phi\| := \sum_{j=1}^{\infty} \frac{\|\phi\|_j}{m_j c^j} < \infty. \tag{6}$$

Then $(F_m(X), \|\cdot\|)$ is a Banach space.

Proof Take a Cauchy sequence $\{\phi_n\}$ and $\varepsilon > 0$. Thus, for some n_0,

$$\|\phi_n - \phi_k\| = \sum_{j=1}^{\infty} \frac{\|\phi_n - \phi_k\|_j}{m_j c^j} < \varepsilon \quad \text{for all } n, k \geq n_0, \tag{7}$$

whence, for any $j \in N$,

$$\|\phi_n - \phi_k\|_j < m_j c^j \varepsilon \quad \text{for all } n, k \geq n_0,$$

that is, for any $j \in N$, the sequence of restrictions $\{\phi_n |_{K_j}\}$ of $\{\phi_n\}$ to the set K_j is Cauchy. By assumption (a), for any $j \in N$, there is a function $\psi_j \in F(X)$ such that $\lim_{n\to\infty} \|\phi_n - \psi_j\|_j = 0$. Define

$$\psi(x) := \psi_j(x) \quad \text{for } x \in K_j, \ j \in N.$$

This definition is correct because $K_j \subset K_{j+1}$ for all $j \in N$. By assumption (b), $\psi \in F(X)$. Let us fix an arbitrary $J \in N$. From (7) we have

$$\sum_{j=1}^{J} \frac{\|\phi_n - \phi_k\|_j}{m_j c^j} < \varepsilon \quad \text{for all } n, k \geq n_0.$$

Letting here $n \to \infty$ we get

$$\sum_{j=1}^{J} \frac{\|\psi - \phi_k\|_j}{m_j c^j} \leq \varepsilon \quad \text{for all } k \geq n_0.$$

Fix arbitrarily $k \geq n_0$. Hence, by the triangle inequality,

$$\sum_{j=1}^{J} \frac{\|\psi\|_j}{m_j c^j} \leq \sum_{j=1}^{J} \frac{\|\psi - \phi_k\|_j}{m_j c^j} + \sum_{j=1}^{J} \frac{\|\phi_k\|_j}{m_j c^j} \leq \varepsilon + \|\phi\|.$$

whence, as $J \in N$ is arbitrary,

$$\|\psi\| = \sum_{j=1}^{\infty} \frac{\|\psi\|_j}{m_j c^j} \le \varepsilon + \|\phi\| < \infty,$$

which shows that $\psi \in F_m(X)$. Letting $k \to \infty$ in (7) we obtain

$$\|\phi_n - \psi\| = \sum_{j=1}^{\infty} \frac{\|\phi_n - \psi\|_j}{m_j c^j} \le \varepsilon \quad \text{for all} \quad n \ge n_0,$$

that is, the sequence $\{\phi_n\}$ converges to ψ in the norm $\|\cdot\|$. □

Define

$$F_{mb}(X) := \{\phi \in F(X) : \|\phi\|_j \le m_j \quad \text{for all} \quad j \in N\}.$$

Clearly, $F_{mb}(X)$ is a closed subset of $F_m(X)$.

Remark 1 In this paper, we are mainly interested in two special cases:

(a) X is a metric space, the sets K_j are compact and

$$X = \bigcup_{j=1}^{\infty} \mathrm{Int}(K_j), \tag{8}$$

$F(X)$ is the space $C(X)$ of all continuous functions on X. Let $\{\varphi_i\}$ be a sequence of continuous functions on X such that for every $j \in N$ and $x \in K_j$, $\varphi_{j+1}(x) = \varphi_j(x)$. Let $\varphi(x) := \varphi_j(x)$ for $x \in K_j$. Then $\varphi \in C(X)$. For this, take a sequence $\{x_k\}$ converging to some $x_0 \in X$. Then the set $S_0 := \{x_k : k \in N\} \cup \{x_0\}$ is compact in X. Since the sequence $\{\mathrm{Int} K_j\}$ is increasing, from (8), it follows that there is some $j_0 \in N$ such that $S_0 \subset \mathrm{Int} K_{j_0} \subset K_{j_0}$. We have $\varphi(x) = \varphi_{j_0}(x)$ for all $x \in K_{j_0}$. Hence $\varphi(x_k) = \varphi_{j_0}(x_k) \to \varphi_{j_0}(x_0) = \varphi(x_0)$ as $k \to \infty$. Thus, assumption (*b*) of Lemma 1 holds. The spaces $F_m(X)$ and $F_{mb}(X)$ will be denoted by $C_m(X)$ and $C_{mb}(X)$, respectively.

(b) (X, Σ) is a measurable space, $\{K_j\}$ is an increasing sequence of measurable sets satisfying (4), $F(X)$ is the space $M(X)$ of all measurable functions on X satisfying (5). To see that $M(X)$ satisfies assumption (*b*) of Lemma 1 consider functions $\varphi_k \in F(X)$ such that $\varphi_{j+1}(x) = \varphi_j(x)$ for each $x \in K_j$, $j \in N$, and $\varphi(x) := \varphi_j(x)$ for $x \in K_j$. Note that $\hat{\varphi}_j$ defined as $\hat{\varphi}_j(x) := \varphi_j(x)$ for $x \in K_j$ and $\hat{\varphi}_j(x) := 0$ for $x \in X \setminus K_j$ belongs to $M(X)$. Clearly, $\varphi(x) = \lim_{j\to\infty} \hat{\varphi}_j(x)$ for all $x \in X$. Hence, φ is measurable and is a member of $M(X)$. The spaces $F_m(X)$ and $F_{mb}(X)$ will be denoted by $M_m(X)$ and $M_{mb}(X)$, respectively.

Remark 2 If we drop assumption (8) in Remark 1(*a*), then continuous functions are not enough to make $F_m(X)$ complete. Consider $X = [0, 1]$, $K_j = [0, \frac{j}{j+1}] \cup \{1\}$, $j \in N$.

Then $\{\phi_n\}$ where $\phi_n(x) = x^n$ is a Cauchy sequence with respect to the norm $\|\cdot\|$ and $\phi(x) = \lim_{n\to\infty} \phi_n(x)$ is zero for $x \in [0, 1)$, $\phi(1) = 1$. Therefore, the correct definition of $F(X)$ making $F_m(X)$ complete is the set of functions which satisfy (5) and are continuous on $[0, 1)$. Note that every $F(K_j)$ (endowed with the supremum norm) is the Banach space of continuous functions on K_j, but taking continuous functions on the closed interval $[0, 1]$ is not a good choice for $F(X)$.

In the remaining part of this section we assume that properties (a) and (b) in Lemma 1 are satisfied.

Let $G \subset F_m(X)$ and $k \in \{0, 1\}$. Inspired by Rincón-Zapatero and Rodriguez-Palmero (2003), we say that a mapping $T : F_m(X) \mapsto F(X)$ is a *k-local contraction* (relative to the set G) if there is a $\beta \in [0, 1)$ such that

$$\|T\phi - T\psi\|_j \le \beta\, \|\phi - \psi\|_{j+k} \quad \text{for all } \phi,\ \psi \in G \text{ and } j \in N.$$

Note that this definition is in some sense stronger than that of Rincón-Zapatero and Rodriguez-Palmero (2003).

Proposition 1 *Let $T : F_m(X) \mapsto F(X)$ be a 0-local contraction relative to $G = F_m(X)$. Then*

$$\|T\phi - T\psi\| \le \beta\|\phi - \psi\|, \tag{9}$$

for any $\phi, \psi \in F_m(X)$. If $T\mathbf{0} \in F_m(X)$, then T maps $F_m(X)$ into itself and has a unique fixed point $\phi^* \in F_m(X)$. If, in addition,

$$\|T\mathbf{0}\|_j \le (1-\beta)m_j \quad \text{for all } j \in N,$$

then $T : F_{mb}(X) \mapsto F_{mb}(X)$ and has a unique fixed point $\phi^* \in F_{mb}(X)$.

Proof It is easy to see that (9) holds. Assume that $T\mathbf{0} \in F_m(X)$. Note that, for all $\phi \in F_m(X)$,

$$\|T\phi\| = \|T\phi - T\mathbf{0} + T\mathbf{0}\| \le \|T\phi - T\mathbf{0}\| + \|T\mathbf{0}\| \le \beta\,\|\phi\| + \|T\mathbf{0}\| < \infty.$$

Then T maps $F_m(X)$ into itself and is a contraction. Suppose now that $\phi \in F_{mb}(X)$. For each $j \in N$, we have $\|T\mathbf{0}\|_j \le (1-\beta)m_j$. Thus

$$\|T\phi\|_j \le \|T\phi - T\mathbf{0}\|_j + \|T\mathbf{0}\|_j \le \beta\|\phi\|_j + (1-\beta)m_j \le \beta m_j + (1-\beta)m_j = m_j.$$

The existence of a unique fixed point for T in $F_m(X)$ or $F_{mb}(X)$ follows from the Banach Contraction Principle. □

Remark 3 Rincón-Zapatero and Rodriguez-Palmero (2003); Rincón-Zapatero and Rodriguez-Palmero (2009) view $C_{mb}(X)$ as a closed subset of a space endowed with a *non-homogeneous metric* and study contractions on closed subsets of $C_{mb}(X)$. We allow for a larger domain $C_m(X)$ and work with a metric induced by a norm.

The following result is closely related to Theorem 2 in Rincón-Zapatero and Rodriguez-Palmero (2003).

Proposition 2 *Let $T : F_m(X) \mapsto F_m(X)$ be a 1-contraction relative to $G = F_m(X)$. If*

$$\gamma := c\beta \sup_{j \in N} \frac{m_{j+1}}{m_j} < 1,$$

then T is a contraction mapping from $F_m(X)$ into itself with the contraction coefficient γ and has a unique fixed point $\phi^* \in F_m(X)$.

Proof For $\phi, \psi \in F_m(X)$ we have

$$\begin{aligned}\|Tf - Tg\| &= \sum_{j=1}^{\infty} \frac{\|Tf - Tg\|_j}{m_j c^j} \le \sum_{j=1}^{\infty} \beta \frac{\|f - g\|_{j+1}}{m_j c^j} \\ &= \sum_{j=1}^{\infty} \left(\beta c \frac{m_{j+1}}{m_j}\right) \frac{\|f - g\|_{j+1}}{m_{j+1} c^{j+1}} \le \gamma \sum_{j=1}^{\infty} \frac{\|f - g\|_{j+1}}{m_{j+1} c^{j+1}} \\ &\le \gamma \sum_{j=1}^{\infty} \frac{\|f - g\|_j}{m_j c^j} = \gamma \|f - g\|.\end{aligned}$$

Thus T is contractive and by the Banach Contraction Principle has a unique fixed point $\phi^* \in F_m(X)$. □

Remark 4 We have shown that having a k-local contraction mapping T with $k \in \{0, 1\}$, on a subspace of $F(X)$, one can contruct a Banach space using some subset, say S, of $F(X)$ on which T is contractive. Then the unique fixed point of T in S can be obtained by taking the limit (in the norm on S) of the iterations $T^n \phi_0$ with an arbitrary fixed function $\phi_0 \in S$.

4 The model and main results

We start with some preliminaries. Let (X, Σ) be a measurable space, Y a separable metric space. A set-valued mapping A from X into the family of nonempty subsets of Y is called (weakly) *measurable* if $A^{-1}(D) := \{x \in X : A(x) \cap D \neq \emptyset\} \in \Sigma$ for every open set $D \subset Y$. Assume now that X is a metric space. Then a set-valued mapping A is called *continuous* if $A^{-1}(D)$ is closed for each closed set $D \subset Y$ and open for every open set $D \subset Y$. Clearly, a continuous set-valued mapping A is measurable if Σ is the Borel σ-algebra on X. It is well-known that any measurable mapping A having nonempty compact values $A(x)$ for all $x \in X$ admits a measurable selector, see Kuratowski and Ryll-Nardzewski (1965).

Fix a measurable *compact* set-valued mapping A and define

$$C := \{(x, a) : x \in X, a \in A(x)\}. \tag{10}$$

Then C is a measurable subset of $X \times Y$ endowed with the product σ-algebra, see Himmelberg (1975).

Lemma 2 *Let $g : C \mapsto R$ be a measurable function such that $a \mapsto g(x, a)$ is continuous on $A(x)$ for each $x \in X$. Then*

$$g^*(x) := \max_{a \in A(x)} g(x, a)$$

is measurable and there exists a measurable mapping $f^* : X \mapsto Y$ such that

$$f^*(x) \in arg \max_{a \in A(x)} g(x, a)$$

for all $x \in X$.

This fact follows from the measurable selection theorem of Kuratowski and Ryll-Nardzewski (1965) and Lemma 1.10 in Nowak (1984).

If in addition we assume that X is a metric space and A is continuous, then g^* is a continuous function by Berge's maximum theorem, see pp. 115–116 in Berge (1963).

A discrete-time *Markov decision process* considered in this paper is defined by the objects: X, Y, $\{A(x)\}_{x \in X}$, u, q, and β satisfying the following assumptions:

A1: X is the *state space* endowed with a σ-algebra Σ.
A2: Y is a separable metric *space of actions* of the decision maker. For any $x \in X$, $A(x)$ is a *compact* subset of Y representing the set of all actions available in state $x \in X$. It is assumed that the set-valued mapping $x \mapsto A(x)$ is measurable. Define C as in (10).
A3: $u : C \to R$ is a (product) measurable *instantaneous return function*.
A4: q is a transition probability from C to X, called the *law of motion* among states. If x_t is a state at the beginning of period t of the process and an action $a_t \in A(x_t)$ is selected, then $q(\cdot|x_t, a_t)$ is the probability distribution of the next state x_{t+1}.
A5: $\beta \in (0, 1)$ and is called the *discount factor*.

A *policy* is a sequence $\pi = \{\pi_t\}$ where π_t is a measurable mapping which associates an action $a_t \in A(x_t)$ for any admissible history of the process up to state $x_t \in X$.[3] Let Π denote the set of all policies. Note that we restrict our attention to non-randomized policies which are enough to study the discounted models. For a more formal definition of a general policy the reader is referred to Bertsekas and Shreve (1978) or Hernández-Lerma and Lasserre (1999). As usual, a *stationary policy* can be identified with a measurable mapping $\varphi : X \mapsto Y$ such that $\varphi(x) \in A(x)$ for each $x \in X$. More formally, a stationary policy is a constant sequence π with $\pi_t = \varphi$. We denote by Φ the set of all stationary policies and identify Φ with the nonempty set of measurable selectors of the mapping $x \mapsto A(x)$. Clearly, if a policy $\varphi \in \Phi$ is used, then the action selected at state x_t of the process is $a_t = \varphi(x_t)$.

[3] A history is $h_t = x_1$ for $t = 1$, $h_t = (x_1, a_1, \ldots, x_{t-1}, a_{t-1}, x_t)$ for $t \geq 2$, $a_\tau \in A(x_\tau)$, $\tau = 1, \ldots, t-1$.

For each initial state $x_1 = x$ and any policy $\pi \in \Pi$, the *expected discounted return* over an infinite future is defined as:

$$J(x,\pi) := E_x^\pi\left(\sum_{t=1}^{\infty} \beta^{t-1} u(x_t, a_t)\right), \tag{11}$$

where E_x^π denotes the expectation operator with respect to the unique conditional probability measure P_x^π defined (on the space of histories, endowed with the product σ-algebra, starting at the state x) by π and the transition probability q according to the Ionescu–Tulcea Theorem, see Proposition V.1.1 in Neveu (1965); for a detailed discussion consult Bertsekas and Shreve (1978) or Hernández-Lerma and Lasserre (1999). We shall accept conditions under which the expected returns (11) are well-defined.

We now describe some regularity assumptions on the return and transition probability functions.

C1: Let X be a metric space and $\{K_j\}$ a strictly increasing family of compact sets that satisfy (8). Let $C_c(X)$ be the space of all continuous functions on X with compact supports. Suppose that the set-valued mapping $x \mapsto A(x)$ is continuous. In addition, assume that the return function u is continuous and, for any $v \in C_c(X)$,

$$(x,a) \mapsto \int_X v(y) q(\mathrm{d}y|x,a)$$

is also continuous on the set C.

If X is not necessarily a topological space, we accept the following regularity condition:

C2: For every $x \in X$, any measurable set $D \subset X$, the functions $a \mapsto u(x,a)$ and $a \mapsto q(D|x,a)$ are continuous on $A(x)$.

Remark 5 The continuity assumptions of the above type are typical in the theory of Markov decision processes, see Schäl (1975) and Hernández-Lerma and Lasserre (1999). Using approximation by measurable step functions one can conclude from **C2** that $a \mapsto \int_X v(y)q(\mathrm{d}y|x,a)$ is continuous on $A(x)$ for any $x \in X$ and every bounded measurable function v on X.

Under **C1** or **C2** we can define

$$u_j(x) := \max_{a\in A(x)} |u(x,a)| \quad \text{if } x \in K_j \quad \text{and} \quad r_j := \sup_{x\in K_j} u_j(x). \tag{12}$$

Consider the sequences $\{m_j\}$ and $\{K_j\}$ as in Sect. 3. Assume that (4) holds. We can now describe our basic assumptions.

D1: For every $j \in N$ and $x \in K_j, a \in A(x)$, we have $q(K_j|x,a) = 1$.

D2: For every $j \in N, x \in K_j, a \in A(x)$, we have $q(K_{j+1}|x,a) = 1$. In addition, we assume that there exists $c > 1$ such that

$$\gamma := c\beta \sup_{j \in N} \frac{m_{j+1}}{m_j} < 1. \tag{13}$$

Moreover, there exists a function $h \in M_m(X)$ ($h \in C_m(X)$ when X is a metric space) such that for every $j \in N$ and $x \in K_j, |u_j(x)| \le h(x)$.

Note that (13) implies that

$$\sum_{t=1}^{\infty} (c\beta)^t m_t < \infty. \tag{14}$$

Lemma 3 *Assume* (4) *and either* **D1** *together with* $r_j \le m_j$ *for all* $j \in N$ *or* **D2**. *Then the expected returns* (11) *are finite.*

Proof Suppose that **D1** holds. Choose any $j \in N$ and $x \in K_j$. For any $t \ge 2$, we have $E_x^\pi(|u(x_t,a_t)|) \le r_j \le m_j$. Hence $|J(x,\pi)| \le \frac{m_j}{1-\beta}$. Let **D2** be satisfied. Using the norm (6), define $r := \|h\|$. Observe that $\|h\|_i \le rm_i c^i$ for all $i \in N$. Let $x \in K_j$. Then for any $t \ge 2$ we have

$$|E_x^\pi(u(x_t,a_t))| \le E_x^\pi(h(x_t)) \le rm_{j+t-1}c^{j+t-1}.$$

This and (14) imply that

$$|J(x,\pi)| \le \sum_{t=1}^{\infty} \beta^{t-1} E_x^\pi(|u(x_t,a_t)|) \le \sum_{t=1}^{\infty} r\beta^{t-1}c^{j+t-1}m_{j+t-1}$$
$$= \frac{r}{\beta^j} \sum_{t=1}^{\infty} (c\beta)^{j+t-1} m_{j+t-1} < \infty,$$

which completes the proof. □

The Bellman functional equation (BE) plays a crucial role in the theory of discounted Markov decision processes. We now describe its form. For any function $v : X \mapsto R$ which is integrable with respect to all $q(\cdot|x,a)$ where $(x,a) \in C$, define

$$Lv(x,a) := u(x,a) + \beta \int_X v(y)q(dy|x,a), \quad (x,a) \in C.$$

Using this notation we can write BE in the form

$$v^*(x) = \max_{a \in A(x)} Lv^*(x,a), \quad x \in X. \tag{15}$$

In this paper we are interested in the existence of a unique solution to (15) in the space $C_m(X)$ when X is a metric space or in $M_m(X)$ in the more general state space case.

Proposition 3 *Assume* **D1**. *If* **C1** *(***C2** *and $r_j < \infty$ for each $j \in N$) is satisfied, then there exist an increasing unbounded sequence $m = \{m_j\}$ and a unique function $v^* \in C_m(X)$ ($v^* \in M_m(X)$) which satisfies the Bellman equation.*

Proof First assume **C1**. By the maximum theorem of Berge (1963), every function u_j is continuous on the compact set K_j. Therefore $r_j < \infty$ for each j. We can choose any increasing unbounded sequence $m = \{m_j\}$ such that $m_j \geq r_j$. Consider the closed subset $C_{mb}(X)$ of the Banach space $C_m(X)$. Define an operator T on $C_{mb}(X)$ by

$$Tv(x) := \max_{a \in A(x)} \left((1-\beta)u(x,a) + \beta \int_X v(y)q(dy|x,a) \right) \tag{16}$$

where $v \in C_{mb}(X)$, $x \in X$. By the maximum theorem of Berge (1963), Tv is continuous on every set K_j. From (8), it follows that Tv is continuous on X (recall Remark 1(*a*)). Under our assumption on q it is now easy to see that T maps $C_{mb}(X)$ into itself. Moreover, for any $v, w \in C_{mb}(X)$, we have

$$\|Tv - Tw\|_j \leq \beta \|v - w\|_j$$

for every $j \in N$. Thus, T is a 0-local contraction. By Proposition 1 and Remark 1(*a*), there exists a unique $w^* \in C_{mb}(X)$ such that $Tw^* = w^*$. Put $v^* = \frac{w^*}{1-\beta}$. Clearly, $v^* \in C_m(X)$ and is a solution to the Bellman equation. The proof under condition **C2** proceeds along similar lines if we apply Lemma 2, Proposition 1 and Remark 1(*b*). Clearly, in that case $v^* \in M_m(X)$. □

Remark 6 A modified form of (16) can be considered for $v \in M_m(X)$. Such situations we shall meet in the sequel.

Proposition 4 *Assume* **D2**. *If* **C1** *(***C2***) is satisfied, then there exists a unique function $v^* \in C_m(X)$ ($v^* \in M_m(X)$) which satisfies the Bellman equation.*

Proof We first assume **D2** and **C1**. In this proof we can consider a slightly modified form of the operator (16) defined as

$$Tv(x) := \max_{a \in A(x)} \left(u(x,a) + \beta \int_X v(y)q(dy|x,a) \right) \tag{17}$$

where $v \in C_m(X)$, $x \in X$. By the maximum theorem of Berge (1963), Tv is continuous. We shall show that $Tv \in C_m(X)$. Let $u^*(x) := \max_{a \in A(x)} |u(x,a)|$. Then

$\|u^*\| \le \|h\|$. Choose any $v \in C_m(X)$. Define

$$\eta(x) = \max_{a \in A(x)} \left| \int_X v(y) q(dy|x,a) \right|, \quad x \in X.$$

Clearly, η is continuous. If $x \in K_j$, then under **D2**, we have $\|\eta\|_j \le \|v\|_{j+1}$ for all $j \in N$. Consequently,

$$\|\eta\| \le \frac{1}{\beta} \sum_{j=1}^{\infty} \frac{\|v\|_{j+1}}{m_{j+1} c^{j+1}} \left(\frac{c\beta m_{j+1}}{m_j}\right) \le \frac{\gamma \|v\|}{\beta} \le \frac{\|v\|}{\beta}.$$

Thus, $\|Tv\| \le \|h\| + \|v\| < \infty$. We have shown that T maps $C_m(X)$ into itself. If $v, w \in C_m(X)$, then for any j, we have

$$\|Tv - Tw\|_j \le \beta \|v - w\|_{j+1},$$

so T is a 1-local contraction. By Proposition 2 and Remark 1(a), there exists a unique $v^* \in C_m(X)$ such that $Tv^* = v^*$. Clearly, v^* is a solution to the Bellman equation. The proof under condition **C2** makes use of Lemma 2, Proposition 2, Remark 1(b) and proceeds along similar lines. □

Remark 7 If v^* is a solution to the Bellman equation, then by Lemma 2 one can find a $\varphi^* \in \Phi$ such that $\varphi^*(x) \in \arg\max_{a \in A(x)} Lv^*(x,a)$ for each $x \in X$. Using standard iteration arguments and Lemma 3, one can prove that

$$v^*(x) = J(x, \varphi^*) = \sup_{\pi \in \Phi} J(x, \pi), \quad x \in X,$$

i.e., φ^* is a stationary optimal policy. For more details about this iteration method the reader is referred to Schäl (1975), Bertsekas and Shreve (1978) or Puterman (2005). Also one can show that v^* is the limit (in the norm $\|\cdot\|$) of the sequence $T^n\mathbf{0}$ with T defined as in (17), i.e., *value iteration* holds. Moreover, $T^n\mathbf{0}$ is the optimal expected return in the n-period model, see Bertsekas and Shreve (1978).

5 Extensions to the models with discontinuous return functions or non-compact action spaces

In some applications of Markov decision processes in operations research or economics it is desirable to allow for non-compact action spaces or discontinuous return functions.[4] We describe two possibilities for extending the results of last section.

C3: Assume in **C1** that u is upper semicontinuous and $u(x,\cdot)$ is bounded below on every compact set $A(x)$, $x \in X$.

Proposition 5 *Assume* **C3** *and either* **D1** *together with the condition that* $\sup_{x\in K_j} \sup_{a\in A(x)} |u(x,a)| < \infty$ *for every* $j \in N$ *or* **D2**. *Then the Bellman equation has a unique upper semicontinuous solution.*

Proof Denote by $S(X)$ the set of all *upper semicontinuous functions* in $M(X)$. Put $S_m(X) := S(X) \cap M_m(X)$ and $S_{mb}(X) := S(X) \cap M_{mb}(X)$. Propositions 1 and 2 can be formulated for operators $T : S_{mb}(X) \mapsto S_{mb}(X)$ or $T : S_m(X) \mapsto S_m(X)$, because the indicated subsets are closed in the Banach space $F_m(X)$. By Proposition 7.31 in Bertsekas and Shreve (1978), under assumption **C3**, for any $v \in S_m(X)$, the function $v(x,a) := \int_X v(y)q(dy|x,a)$ is upper semicontinuous on every set $\{(x,a) : x \in K_j, a \in A(x)\}$, $j \in N$. From the maximum theorem of Berge (1963), it follows that Tv defined by (16) is upper semicontinuous on K_j. Assume now **D1** and that $\sup_{x\in K_j} \sup_{a\in A(x)} |u(x,a)| < \infty$ for every $j \in N$. Using our assumption (8), we infer that $Tv \in S(X)$. Now we can easily see that T maps $S_{mb}(X)$ into itself. The remaining part of the proof is an adaptation of the arguments used in proving Proposition 3. Under assumption **C3** and **D2**, consider T defined by (17). Adapting the arguments used in the proof of Proposition 4, observe that T maps the space $S_m(X)$ into itself. An application of the modified version of Proposition 2 mentioned above finishes the proof. □

C4: Let X, Y be Borel (subsets of complete separable metric) spaces. Assume that $C \subset X \times Y$ is a Borel set and for each $x \in X$, $A(x)$ is σ-*compact*, that is, $A(x)$ is the countable union of compact sets. Suppose that the sets K_j satisfying (4) are Borel and the assumption on q in **C2** holds, $u : C \mapsto R$ is Borel measurable, and for each $x \in X$, $a \mapsto u(x,a)$ is upper semicontinuous and bounded below on $A(x)$.

In this context, $M(X)$ and $M_m(X)$ consist of Borel measurable functions.

Proposition 6 *Assume* **C4**. *If* **D1** *holds and* $\sup_{x\in K_j} \sup_{a\in A(x)} |u(x,a)| < \infty$ *for all* $j \in N$ *or* **D2** *with* $h \in M_m(X)$ *is satisfied, then the Bellman equation*

$$v(x) = \sup_{a\in A(x)} Lv(x,a), \quad x \in X,$$

[4] As noted by Dutta and Mitra (1989), standard continuity assumptions are quite restrictive in intertemporal allocation models. There are more arguments to study dynamic programming problems under some discontinuity assumptions. Very often Nash equilibria in stochastic dynamic games are semicontinuous (or more generally measurable) functions of the state variable. Studying the best responses of any player to discontinuous strategies of his/her partners leads to dynamic programming under conditions similar to our assumptions in this section. We would like to emphasize that this happens even if we assume that the instantaneous utility functions and transition probabilities are jointly continuous with respect to the state and action variables. The reason is that the class of continuous strategies of the players is too narrow to prove equilibrium theorems for games, especially in the class of general strategy profiles. For a further discussion of these issues the reader is referred to Dutta and Sundaram (1992) and Nowak and Raghavan (1992).

has a unique solution $v^* \in M_m(X)$.

Proof Consider first **C4** and **D2**. It is sufficient to show that $Tv(x) := \sup_{a \in A(x)} Lv(x,a)$ maps $M_m(X)$ into $M(X)$. Let $v \in M_m(X)$. Then the function $v(x,a) := \int_X v(y)q(dy|x,a)$ is Borel measurable on C and $a \mapsto v(x,a)$ is continuous on $A(x)$ for each $x \in X$. Therefore Lv is Borel on C and $a \mapsto Lv(x,a)$ is upper semicontinuous on $A(x)$ for each $x \in X$. The fact that the function Tv belongs to $M(X)$ now follows from Corollary 1 in Brown and Purves (1973). A simple adaptation of the proof of Proposition 4 yields that if $v \in M_m(X)$, then $Tv \in M_m(X)$ and T is a 1-local contraction. The assertion now follows from Proposition 2. If **C4** and **D1** hold and $\sup_{x \in K_j} \sup_{a \in A(x)} |u(x,a)| < \infty$ for all $j \in N$, then the proof follows along similar lines to that of Proposition 3 with T defined by

$$Tv(x) := \sup_{a \in A(x)} \left((1-\beta)u(x,a) + \beta \int_X v(y)q(dy|x,a) \right), \quad v \in M_{mb}(X).$$

□

This result, Corollary 1 in Brown and Purves (1973), and standard iteration arguments in dynamic programming, see Blackwell (1965), lead to the following conclusion.

Corollary 1 *Under assumptions of Proposition 6, for any* $\epsilon > 0$ *there exists some* $\varphi^* \in \Phi$ *such that*

$$Lv^*(x, \varphi^*(x)) + \epsilon(1-\beta) \geq \sup_{a \in A(x)} Lv^*(x,a), \quad x \in X,$$

which implies that

$$\epsilon + J(x, \varphi^*) \geq \sup_{\pi \in \Pi} J(x, \pi), \quad x \in X.$$

Remark 8 The regularity assumptions **C1–C4** can be considerably weakened if the state and action spaces are Borel. One can assume that u is a Borel measurable function. Using universally measurable policies, it is possible to obtain (under similar assumptions to **D1** or **D2**) that there is an upper semi-analytic solution to the Bellman equation and (for any $\epsilon > 0$) there exists an ϵ-optimal universally measurable policy. For a background material for this modification consult Bertsekas and Shreve (1978). Finally, we would like to point out that our results can also be applied to discounted stochastic games with unbounded payoffs studied in Nowak (1984, 1985), Nowak and Raghavan (1992) and related articles under a boundedness assumption.

6 Applications to one-sector models of stochastic optimal growth

The results of Sect. 3 may have many applications to various models in operations research as studied in Hernández-Lerma and Lasserre (1999) or Puterman (2005) and in economics. We now show two applications of Propositions 3 and 4 to the theory of stochastic optimal growth. We have in mind classical models studied in Brock and Mirman (1972) and Stokey et al. (1989). However, within our framework we allow for *unbounded utility* (return) functions. Let $X = [0, \infty)$ be the set of all *capital stocks*. If x_t is a capital stock at the beginning of period t, then consumption a_t in this period belongs to $A(x_t) := [0, x_t]$. The utility of consumption a_t is $U(a_t)$ where $U : X \mapsto R$ is a fixed function. The evolution of the state process is described by some function f of the investment for the next period $y_t := x_t - a_t$ and some random variable ξ_t. In the literature, f is called *production technology*, see Stokey et al. (1989). We shall view this model as a Markov decision process with $X = [0, \infty)$, $A(x) = [0, x]$, and $u(x, a) = U(a), x \in X, a \in A(x)$. The transition probability will be specified in two different cases. Assume that $\{\xi_t\}$ are independent and have a common probability distribution μ with support included in $[0, z]$ for some $z > 1$.

Example 2 (*A model with multiplicative shocks*) Assume that

$$x_{t+1} = f(x_t - a_t)\xi_t, \quad t \in N, \tag{18}$$

where $f : X \mapsto R$ is a continuous and increasing function, $f(0) = 0$,

$$(0, \infty) \ni y \to \frac{f(y)}{y} \text{ is strictly decreasing;} \tag{19}$$

$$\lim_{y \to 0+} \frac{f(y)}{y} > 1 \tag{20}$$

and

$$\lim_{y \to \infty} \frac{f(y)}{y} = 0. \tag{21}$$

Conditions (19)–(21) imply that there exists $y_0 > 0$ such that

$$f(y) > y \quad \text{for all} \quad y \in (0, y_0) \quad \text{and} \quad f(y) < y \quad \text{for all} \quad y > y_0. \tag{22}$$

We shall consider the more interesting case when f is unbounded. Observe that the transition probability q is of the form: for any Borel set $B \subset X, x \in X, a \in A(x)$, we have

$$q(B|x, a) = \int_0^z 1_B(f(x - a)\xi)\mu(d\xi),$$

where 1_B is the indicator function of the set B. If $v \in C_c(X)$, then the integral

$$\int_X v(y)q(dy|x, a) = \int_0^z v(f(x - a)\xi)\mu(d\xi)$$

depends continuously on (x, a). From (22) and our additional assumptions on f, it follows that for any $j \in N$, there exists $y_j > y_0$ such that $f(y_j)z^j = y_j$. The sequence $\{y_j\}$ is increasing. Define $K_j := [0, y_j]$ for each $j \in N$. Note that if $y = x - a \in K_j$, then for any $\xi \in [0, z]$, we have $\xi f(y) \le z f(y_j) < f(y_j)z^j = y_j$. From (18) we conclude that $q(K_j|x, a) = 1$ for every $x \in K_j$, $a \in A(x)$. We have shown that assumptions of Proposition 3 are satisfied. Therefore, for *arbitrary unbounded* continuous utility function U the Bellman equation has a unique continuous solution.

Note that Stokey et al. (1989) (see pp. 104, 288) assume the following stronger conditions: $f : X \mapsto R$ is a *bounded* strictly concave continuously differentiable increasing function such that $f(0) = 0$ and (22) holds.

Example 3 (A model with additive shocks) Assume that

$$x_{t+1} = (1+\rho)(x_t - a_t) + \xi_t, \quad t \in N. \tag{23}$$

Here $\rho > 0$ is a constant *rate of growth* and ξ_t an additional *random income* received in period t. The transition probability q is of the form

$$q(B|x, a) = \int_0^z 1_B((1+\rho)(x-a) + \xi)\mu(d\xi),$$

where $B \subset X$ is a Borel set. If $v \in C_c(X)$, then the integral

$$\int_X v(y) q(dy|x, a) = \int_0^z v((1+\rho)(x-a) + \xi)\mu(d\xi)$$

is continuous in (x, a). Fix a number $d > 0$. Define $k_1 := d$ and then recursively $k_{j+1} := (1+\rho)k_j + z$ where

$$k_j = (1+\rho)^{j-1} d + \frac{z}{\rho}\left[(1+\rho)^{j-1} - 1\right], \quad j \in N.$$

Put $K_j := [0, k_j]$, $j \in N$. Assume that $U(a) := a^\sigma, \sigma \in (0, 1)$ is fixed and put $m_j := \max_{a \in K_j} U(a)$. The sequence $\{m_j\}$ is increasing, unbounded and, as

$$\frac{m_{j+1}}{m_j} = \left(\frac{\rho(1+\rho)^j d + z\left[(1+\rho)^j - 1\right]}{\rho(1+\rho)^{j-1} d + z\left[(1+\rho)^{j-1} - 1\right]}\right)^\sigma, \quad j \in N,$$

it is easy to check that the sequence $\{\frac{m_{j+1}}{m_j}\}$ is decreasing and thus

$$\sup_{j \in N} \frac{m_{j+1}}{m_j} = \frac{m_2}{m_1} = \left(1 + \rho + \frac{z}{d}\right)^\sigma.$$

Therefore γ defined in (13) satisfies

$$\gamma = c\beta\left(1+\rho+\frac{z}{d}\right)^{\sigma} < 1$$

only for some $c > 1$ and $\beta < 1$. Note that d can be arbitrarily large. For example, we can take d such that $z/d < \rho$. Then $\gamma < 1$ if $c\beta(1+2\rho)^{\sigma} < 1$. If ρ is small, then we can consider discount factors very close to one. From (23), it is easy to see that $q(K_{j+1}|x, a) = 1$ for each $x \in K_j, a \in A(x)$. Assumptions of Proposition 4 are thus satisfied. Therefore for this model the Bellman equation has a unique continuous solution.

Remark 9 The model based on assumption **D1** discussed in Proposition 3 and Example 2 can also be analyzed using the weighted norm approach (see Hernández-Lerma and Lasserre (1999) for more details on this idea). Let $\omega : X \mapsto [1, \infty)$ be a measurable *weight function*. The weighted norm of a function $\psi : X \mapsto R$ is $\|\psi\|_{\omega} := \sup_{x\in X} |\psi(x)|/\omega(x)$ if it is finite. Using this weight it is assumed that

$$\beta \sup_{(x,a)\in C} \frac{\int_X \omega(y)q(dy|x,a)}{\omega(x)} < 1. \tag{24}$$

More details can be found in Hernández-Lerma and Lasserre (1999) and related analysis in Durán (2003). This inequality does not follow from **D1** and is an additional restriction on q. Suppose that (24) holds. Then it is required that there is a constant $l > 0$ such that $|u(x, a)| \le l\omega(x)$ for all $(x, a) \in C$ (see Hernández-Lerma and Lasserre (1999)). This is a restriction on the utility functions which does not take place in Proposition 3 or Example 2. Having fixed the transition probability q as in Example 2, we can consider *arbitrary* unbounded continuous utility function $u(x, a) = U(a)$. The sequence $\{m_j\}$ is determined by u and not conversely (recall the proof of Proposition 3).

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