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A mean-value theorem and its applications

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ABSTRACT

For a function f defined in an interval I, satisfying the conditions ensuring the existence and uniqueness of the Lagrange mean $L^{[f]}$, we prove that there exists a unique two variable mean $M^{[f]}$ such that

$$\frac{f(x)-f(y)}{x-y}=M^{[f]}\big(f'(x),f'(y)\big)$$

for all $x,y\in I$, $x\neq y$. The mean $M^{[f]}$ is closely related $L^{[f]}$. Necessary and sufficient condition for the equality $M^{[f]}=M^{[g]}$ is given. A family of means $\{\mathcal{M}^{[t]}\colon t\in\mathbb{R}\}$ relevant to the logarithmic means is introduced. The invariance of geometric mean with respect to mean-type mappings of this type is considered. A result on convergence of the sequences of iterates of some mean-type mappings and its application in solving some functional equations is given. A counterpart of the Cauchy mean-value theorem is presented. Some relations between Stolarsky means and $\mathcal{M}^{[t]}$ means are discussed.

0. Introduction

We present some mean-value theorems, closely related to the classical results of Lagrange and Cauchy.

In the first section, under the conditions assuring the existence and uniqueness of the Lagrange mean $L^{[f]}$ for a real function f in an interval I, we prove the existence of a unique two variable mean $M^{[f]}$, accompanying to $L^{[f]}$, such that

$$\frac{f(x)-f(y)}{x-y}=M^{[f]}\big(f'(x),f'(y)\big)$$

for all $x,y\in I$, $x\neq y$ as well as we give the formula of $M^{[f]}$. We give necessary and sufficient condition for the equality $M^{[f]}=M^{[g]}$. It is analogous to that of Berrone and Moro [1] for the Lagrange means. Some examples of the accompanying Lagrange means, a mean-type mapping of such means for which the geometric mean is invariant, and application of the invariance in solving some functional equation, are presented.

It is well known that $\{\mathcal{L}^{[p]}\colon p\in\mathbb{R}\}$, the one-parameter family of logarithmic means, is, with respect to p, a continuous extension of the family of all $L^{[f]}$ means generated by the power functions $f\ (x)=x^p$ for all $p\in\mathbb{R}$ such that $p(p-1)\neq 0$. We show that the relevant family of accompanying means $\{\mathcal{M}^{[t]}\colon t\in\mathbb{R}\}$, with the parametrization $t=\frac{1}{p-1}$, is continuously extendable at the points $t=-1$ and $t=0$. It turns out that the classical means, arithmetic, harmonic, geometric and logarithmic, belong to this family. Moreover, for any $t\in\mathbb{R}$, the geometric mean $\mathcal{G}=\mathcal{M}^{[-1/2]}$ is invariant with respect to

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the mean-type mapping $(\mathcal{M}^{[t]}, \mathcal{M}^{[-(t+1)]})$, and the sequence of iterates of this mean-type mapping converges pointwise to $(\mathcal{G}, \mathcal{G})$ in $(0, \infty)^2$. We apply this fact in solving a functional equation.

In the second section, applying the main result of Section 1, we prove the relevant counterpart of the Cauchy mean-value theorem. Under the assumptions on real functions f, g defined on an interval, which guarantee the existence and uniqueness of the Cauchy mean $C^{[f,g]}$, we prove the existence of a unique mean M such that

$$\frac{f(x)-f(y)}{g(x)-g(y)} = M\left(\frac{f'(x)}{g'(x)}, \frac{f'(y)}{g'(y)}\right)$$

for all $x, y \in I$, $x \neq y$. Moreover $M = M^{[f\circ g^{-1}]}$ is the accompanying Lagrange mean of the function $f \circ g^{-1}$. Some representation formulas of the Cauchy mean with the aid of a Lagrange and its accompanying mean are proposed. A relation between Stolarsky means and the $\mathcal{M}^{[t]}$ means is presented.

1. An accompanying of the Lagrange theorem

We begin this section with the following:

Theorem 1. *Let $I \subseteq \mathbb{R}$ be an interval, $f : I \to \mathbb{R}$ be a differentiable function. Then $J := f'(I)$ is an interval and, if f' is one-to-one, then there exists a unique strict mean $M : J^2 \to J$ such that for any $x, y \in I$, $x \neq y$,*

$$\frac{f(x)-f(y)}{x-y} = M\big(f'(x), f'(y)\big).$$

Moreover,

$$M(u, v) = \frac{f[(f')^{-1}(u)] - f[(f')^{-1}(v)]}{(f')^{-1}(u) - (f')^{-1}(v)}, \quad u, v \in J,\ u \neq v. \tag{1}$$

Proof. The Darboux property of derivative implies that J is an interval and f' is strictly monotonic and continuous. Take any $u, v \in J$, $u \neq v$. There are unique $x, y \in I$, $x \neq y$, such that $u = f'(x)$, $v = f'(y)$. By the Lagrange mean-value theorem and the strict monotonicity of f', there is a unique $\xi \in (\min(x, y), \max(x, y))$ such that

$$\frac{f(x)-f(y)}{x-y} = f'\big(\xi(x, y)\big),$$

that is

$$\frac{f[(f')^{-1}(u)] - f[(f')^{-1}(v)]}{(f')^{-1}(u) - (f')^{-1}(v)} = f'\big(\xi\big(\big(f'\big)^{-1}(u), \big(f'\big)^{-1}(v)\big)\big). \tag{2}$$

Setting

$$M(u, v) := \frac{f[(f')^{-1}(u)] - f[(f')^{-1}(v)]}{(f')^{-1}(u) - (f')^{-1}(v)}, \tag{3}$$

and taking into account that

$$\min\big(f'(x), f'(y)\big) < f'(\xi) < \min\big(f'(x), f'(y)\big),$$

we have

$$\min(u, v) < M(u, v) < \max(u, v).$$

Putting $M(u, u) = u$ for all $u \in J$ we see that M is a strict mean.

For arbitrary $x, y \in I$, $x \neq y$, setting $u := f'(x)$, $v := f'(y)$ in (3) we obtain

$$M\big(f'(x), f'(y)\big) = \frac{f(x)-f(y)}{x-y}.$$

The uniqueness of M follows from this formula (it is enough to put here $x = (f')^{-1}(u)$ and $y = (f')^{-1}(v)$ for $u \neq v$). $\square$

Denote by $M^{[f]}$ the mean M in formula (1). Thus, under the assumptions of Theorem 1, the mean $M^{[f]} : J^2 \to J$ is defined by

$$M^{[f]}(u, v) := \begin{cases} \frac{f[(f')^{-1}(u)] - f[(f')^{-1}(v)]}{(f')^{-1}(u) - (f')^{-1}(v)} & \text{for } u \neq v, \\ u & \text{for } u = v, \end{cases} \quad u, v \in J := f'(I). \tag{4}$$

Note that $M^{[f]}$ is continuous and symmetric.

It is well known that, under the assumptions of Theorem 1, the function f generates the *Lagrange mean* defined by

$$L^{[f]}(x,y) := \begin{cases} (f')^{-1}(\frac{f(x)-f(y)}{x-y}) & \text{for } x \neq y, \\ x & \text{for } x = y, \end{cases} \quad x, y \in I.$$

As ξ in (2) coincides with $L^{[f]}$, we have the following obvious

Proposition 1. *Let $I \subseteq \mathbb{R}$ be an interval, $f : I \to \mathbb{R}$ be a differentiable function such that f' is one-to-one. Then*

$$M^{[f]}(u,v) = f'\bigl(L^{[f]}\bigl(\bigl(f'\bigr)^{-1}(u), \bigl(f'\bigr)^{-1}(v)\bigr)\bigr), \quad u, v \in J := f'(I),$$

and

$$L^{[f]}(x,y) = \bigl(f'\bigr)^{-1}\bigl(M^{[f]}\bigl(f'(x), f'(y)\bigr)\bigr), \quad x, y \in I.$$

Thus the means $L^{[f]}$ and $M^{[f]}$ are closely related. The mean $M^{[f]}$ could be referred to as *accompanying of the Lagrange mean* $L^{[f]}$.

Theorem 2. *Let $I, I_1 \subseteq \mathbb{R}$ be intervals. Suppose that $f : I \to \mathbb{R}$, $g : I_1 \to \mathbb{R}$, are differentiable and such that f', g' are one-to-one and $f'(I) = g'(I_1)$. Then $M^{[f]} = M^{[g]}$ if, and only if, there are $a, b \in \mathbb{R}$, $a \neq 0$, such that*

$$\bigl(g'\bigr)^{-1}(u) = a\bigl(f'\bigr)^{-1}(u) + b, \quad u \in f'(I).$$

Proof. Suppose that $M^{[f]} = M^{[g]}$. Putting $J := f'(I)$, we have

$$\frac{f[(f')^{-1}(u)] - f[(f')^{-1}(v)]}{(f')^{-1}(u) - (f')^{-1}(v)} = \frac{g[(g')^{-1}(u)] - g[(g')^{-1}(v)]}{(g')^{-1}(u) - (g')^{-1}(v)}, \quad u, v \in J,\ u \neq v.$$

Hence, setting

$$h := \bigl(g'\bigr)^{-1} \circ f',$$

we get

$$\frac{f(x) - f(y)}{x - y} = \frac{g[h(x)] - g[h(y)]}{h(x) - h(y)}, \quad x, y \in I,\ x \neq y. \tag{5}$$

Without any loss of generality we may assume that $0 \in I$ and $0 \in I_1$. (If it is not the case we can translate these intervals and redefine in an obvious way the functions f and g.) Since for arbitrary $c, c_1 \in \mathbb{R}$, the functions $f + c$ and $g + c_1$ also satisfy this equation, i.e.

$$\frac{[f(x) + c] - [f(y) + c]}{x - y} = \frac{\{g[h(x)] + c_1\} - \{g[h(y)] + c_1\}}{h(x) - h(y)}, \quad x, y \in I,\ x \neq y,$$

we can assume that $f(0) = g(0) = 0$.

Hence, setting $y = 0$ in (5), we obtain

$$\frac{f(x)}{x} = \frac{g[h(x)]}{h(x) - h(0)}, \quad x \in I,\ x \neq 0,$$

whence

$$g\bigl[h(x)\bigr] = \frac{f(x)}{x}\bigl[h(x) - h(0)\bigr], \quad x \in I,\ x \neq 0.$$

Applying this formula in (5) we obtain

$$\frac{f(x) - f(y)}{x - y} = \frac{\frac{f(x)}{x}[h(x) - h(0)] - \frac{f(y)}{x}[h(y) - h(0)]}{[h(x) - h(0)] - [h(y) - h(0)]}, \quad x, y \in I \backslash \{0\},\ x \neq y,$$

which, after simple calculations, reduces to the equality

$$\bigl[yf(x) - xf(y)\bigr]\bigl\{x\bigl[h(y) - h(0)\bigr] - y\bigl[h(x) - h(0)\bigr]\bigr\} = 0 \tag{6}$$

for all $x, y \in I \backslash \{0\}$, $x \neq y$.

Note that the closure of the set

$$\bigl\{(x,y) \in I^2\colon\ yf(x) - xf(y) = 0\bigr\}$$

has empty interior. Indeed, in the opposite case, by the continuity of f, we would have $f(x)/x = f(y)/y$ in a nontrivial subinterval of I and, consequently, f' would be constant in this subinterval, which contradicts the assumption. Now (6) and the continuity of h imply that

$$x\big[h(y) - h(0)\big] - y\big[h(x) - h(0)\big] = 0, \quad x, y \in I,$$

whence, for some $a \in \mathbb{R}$, $a \neq 0$, and $b := h(0)$,

$$h(x) = ax + b, \quad x \in I.$$

By the definition of h,

$$\big(g'\big)^{-1}(u) = a\big(f'\big)^{-1}(u) + b, \quad u \in J.$$

Since the converse implication is easy to verify, the proof is completed. □

The relevant result for the Lagrange means reads as follows.

Remark 1. (See [1], cf. also [5].) Let $I \subseteq \mathbb{R}$ be an interval. Suppose that $f, g : I \to \mathbb{R}$ are differentiable and such that f', g' are one-to-one. Then $L^{[f]} = L^{[g]}$ if, and only if, there are $a, b \in \mathbb{R}$, $a \neq 0$, such that

$$g(x) = af(x) + b, \quad x \in I.$$

Considering the Lagrange means $L^{[f]}$ for the power functions $f(x) = x^p$ $(x > 0)$ where $p \in \mathbb{R}$ is such that $p(p-1) \neq 0$ one can obtain the one-parameter of means logarithmic means $\{\mathcal{L}^{[p]}\colon\ p > 0\}$, $\mathcal{L}^{[p]} : (0,\infty)^2 \to (0,\infty)$ defined by (cf. [2,3])

$$\mathcal{L}^{[p]}(x, y) := \begin{cases} (\frac{1}{p}\frac{x^p - y^p}{x - y})^{1/(p-1)} & \text{if } p(p-1) \neq 0, \\ \frac{x-y}{\log x - \log y} & \text{if } p = 0, \\ e^{-1}(\frac{x^x}{y^y})^{1/(x-y)} & \text{if } p = 1 \end{cases}$$

for $x \neq y$ and $\mathcal{L}^{[p]}(x, y) := x$ for $x = y$. Here $\mathcal{L}^{[0]}$ is the *identric mean*.

For a power function $f(x) = x^p$ $(x > 0)$ with $p(p-1) \neq 0$, by (4), we have

$$M^{[f]}(u, v) = \frac{1}{p}\frac{u^{1+1/(p-1)} - v^{1+1/(p-1)}}{u^{1/(p-1)} - v^{1/(p-1)}}, \quad u, v > 0,\ u \neq v. \tag{7}$$

For any $t \in \mathbb{R}\backslash\{0, 1\}$ there is a unique $p \in \mathbb{R}$, $p(p-1) \neq 0$ such that $t = \frac{1}{p-1}$. Setting $t = \frac{1}{p-1}$ in (7) we define $\mathcal{M}^{[t]} : (0,\infty)^2 \to (0,\infty)$ for any $t \in \mathbb{R}\backslash\{0, -1\}$ by

$$\mathcal{M}^{[t]}(u, v) := \frac{t}{1+t}\frac{u^{t+1} - v^{t+1}}{u^t - v^t}, \quad u, v > 0,\ u \neq v.$$

As it is easy to verify that, for $u, v > 0$, $u \neq v$,

$$\lim_{t \to -1} \mathcal{M}^{[t]}(u, v) = uv\frac{\log u - \log v}{u - v},$$

and

$$\lim_{t \to 0} \mathcal{M}^{[t]}(u, v) = \frac{u - v}{\log u - \log v},$$

we have the following

Remark 2. The one-parameter family of means $\{\mathcal{M}^{[t]}\colon\ t \in \mathbb{R}\}$ defined by

$$\mathcal{M}^{[t]}(u, v) := \begin{cases} \frac{t}{1+t}\frac{u^{t+1} - v^{t+1}}{u^t - v^t} & \text{if } -1 \neq t \neq 0, \\ uv\frac{\log u - \log v}{u - v} & \text{if } t = -1, \\ \frac{u-v}{\log u - \log v} & \text{if } t = 0 \end{cases} \tag{8}$$

for $u \neq v$ and $\mathcal{M}^{[t]}(u, u) := u$ for all $u > 0$, is continuous in t. For any $t \in \mathbb{R}$, the mean $\mathcal{M}^{[t]}$ is strict, symmetric and positively homogeneous. Moreover

$$\mathcal{M}^{[-2]} = \mathcal{H}, \qquad \mathcal{M}^{[-1]} = M^{[\log]}, \qquad \mathcal{M}^{[-1/2]} = \mathcal{G}, \qquad \mathcal{M}^{[0]} = M^{[\exp]}, \qquad \mathcal{M}^{[1]} = \mathcal{A},$$

where $\mathcal{H}(u,v) := \frac{2uv}{u+v}$, $\mathcal{G}(u,v) := \sqrt{uv}$, $\mathcal{A}(u,v) := \frac{u+v}{2}$ are, respectively, the harmonic, geometric and arithmetic means, $\mathcal{M}^{[0]} = M^{[\exp]}$ is the logarithmic mean, and

$$\lim_{t\to+\infty} \mathcal{M}^{[t]}(u,v) = \max(u,v); \qquad \lim_{p\to-\infty} \mathcal{M}^{[t]}(u,v) = \min(u,v).$$

Remark 3. The families of means $\{\mathcal{M}^{[t]}\colon t \in \mathbb{R}\}$ and $\{\mathcal{L}^{[p]}\colon p \in \mathbb{R}\}$ are different. In particular, the harmonic mean $\mathcal{M}^{[-2]}$, the geometric mean $\mathcal{M}^{[-1/2]}$ and the mean $\mathcal{M}^{[-1]}$, are not members of $\{\mathcal{L}^{[p]}\colon p > 0\}$.

Proposition 2. *For any $t \in \mathbb{R}$,*

(i) *the geometric mean $\mathcal{G}$ is invariant with respect to the mean-type mapping $(\mathcal{M}^{[t]}, \mathcal{M}^{[-(t+1)]})$, that is*

$$\mathcal{G} \circ \left(\mathcal{M}^{[t]}, \mathcal{M}^{[-(t+1)]}\right) = \mathcal{G};$$

in particular, $\mathcal{G}$ is $(M^{[\exp]}, M^{[\log]})$-invariant;

(ii) *the sequence $((\mathcal{M}^{[t]}, \mathcal{M}^{[-(t+1)]})^n)_{n\in\mathbb{N}}$ of iterates of $(\mathcal{M}^{[t]}, \mathcal{M}^{[-(t+1)]})$ converges pointwise to the mean-type mapping $(\mathcal{G}, \mathcal{G})$ in $(0,\infty)^2$.*

Proof. Let $t \neq 0$ and $t \neq -1$. Then, by (8), for all $u, v > 0, u \neq v$, we have

$$\mathcal{G}\left(\mathcal{M}^{[t]}(u,v), \mathcal{M}^{[-(t+1)]}(u,v)\right) = \sqrt{\left(\frac{t}{1+t}\frac{u^{t+1}-v^{t+1}}{u^t - v^t}\right)\left(\frac{-t-1}{-t}\frac{u^{-t}-v^{-t}}{u^{-t-1}-v^{-t-1}}\right)}$$
$$= \sqrt{uv} = \mathcal{G}(u,v).$$

If $t = 0$ or $t = -1$ then, by (8), for all $u, v > 0, u \neq v$,

$$\mathcal{G}\left(\mathcal{M}^{[t]}(u,v), \mathcal{M}^{[-(t+1)]}(u,v)\right) = \sqrt{\frac{u-v}{\log u - \log v} uv \frac{\log u - \log v}{u-v}} = \mathcal{G}(u,v).$$

For $u = v$ the equality is obvious.

The second part is a consequence of the main result of [4] (cf. also [6]). □

Corollary 1. *For any $t \in \mathbb{R}$, a function $\Phi : (0,\infty)^2 \to \mathbb{R}$, continuous on the diagonal $\{(x,x)\colon x > 0\}$, satisfies the functional equation*

$$\Phi\left(\mathcal{M}^{[t]}(u,v), \mathcal{M}^{[-(t+1)]}(u,v)\right) = \Phi(u,v), \quad u, v > 0, \tag{9}$$

if, and only if, there is a single variable continuous function $\varphi : (0,\infty) \to \mathbb{R}$ such that

$$\Phi(u,v) = \varphi(uv), \quad u, v > 0.$$

Proof. Suppose that a function $\Phi : (0,\infty)^2 \to \mathbb{R}$, continuous on the diagonal, satisfies Eq. (9). From (9), by induction, for all $u, v > 0$, we have

$$\Phi(u,v) = \Phi\left(\left(\mathcal{M}^{[t]}(u,v), \mathcal{M}^{[-(t+1)]}(u,v)\right)^n\right).$$

Hence, letting $n \to \infty$, and applying Proposition 2(ii), we obtain, for all $u, v > 0$,

$$\Phi(u,v) = \Phi\left(\mathcal{G}(u,v), \mathcal{G}(u,v)\right) = \Phi(\sqrt{uv}, \sqrt{uv}) = \varphi(uv),$$

where $\varphi(z) := \Phi(z^2, z^2)$ for $z > 0$.

Since the converse implication is easy to verify, the proof is completed. □

2. A counterpart of Cauchy mean-value theorem

Theorem 3. *Let $I \subseteq \mathbb{R}$ be an interval, $f, g : I \to \mathbb{R}$ be differentiable functions such that $g'(x) \neq 0$ for all $x \in I$. Then $J := \frac{f'}{g'}(I)$ is an interval and, if the function $\frac{f'}{g'}$ is one-to-one, there exists a unique strict mean $M : J^2 \to J$ such that for any $x, y \in I, x \neq y$,*

$$\frac{f(x) - f(y)}{g(x) - g(y)} = M\left(\frac{f'(x)}{g'(x)}, \frac{f'(y)}{g'(y)}\right).$$

Moreover, $M = M^{[f\circ g^{-1}]}$ and

$$M^{[f\circ g^{-1}]}(u,v) = \frac{f[(\frac{f'}{g'})^{-1}(u)] - f[(\frac{f'}{g'})^{-1}(v)]}{g[(\frac{f'}{g'})^{-1}(u)] - g[(\frac{f'}{g'})^{-1}(v)]} \quad \text{for } u \neq v. \tag{10}$$

Proof. The function $f \circ g^{-1}$ is differentiable in the interval $g(I)$ and

$$\left(f \circ g^{-1}\right)' = \frac{f' \circ g^{-1}}{g' \circ g^{-1}} = \frac{f'}{g'} \circ g^{-1}.$$

Applying Theorem 1 and (4) for the function $f \circ g^{-1}$ we conclude that there is a unique strict mean $M^{[f \circ g^{-1}]}$ in the interval $J = (\frac{f'}{g'} \circ g^{-1})(g(I)) = \frac{f'}{g'}(I)$ such that, for all $r, s \in g(I)$, $r \neq s$, we have

$$\begin{aligned}\frac{f \circ g^{-1}(r) - f \circ g^{-1}(s)}{r - s} &= M^{[f \circ g^{-1}]}\left(\left(f \circ g^{-1}\right)'(r), \left(f \circ g^{-1}\right)'(s)\right) \\ &= M^{[f \circ g^{-1}]}\left(\frac{f'}{g'} \circ g^{-1}(r), \frac{f'}{g'} \circ g^{-1}(s)\right),\end{aligned}$$

whence, setting $r = g(x)$, $s = g(y)$, we obtain, for all $x, y \in I$, $x \neq y$,

$$\frac{f(x) - f(y)}{g(x) - g(y)} = M^{[f \circ g^{-1}]}\left(\frac{f'(x)}{g'(x)}, \frac{f'(y)}{g'(y)}\right).$$

Replacing here x and y respectively by $\frac{f'}{g'}(u)$ and $\frac{f'}{g'}(v)$ for $u, v \in J$, $u \neq v$, we get the second part of our result. □

Under the assumptions of Theorem 3, the functions f and g generate the *Cauchy mean* $C^{[f,g]} : I^2 \to I$ defined by

$$C^{[f,g]}(x, y) := \begin{cases} (\frac{f'}{g'})^{-1}(\frac{f(x)-f(y)}{g(x)-g(y)}) & \text{for } x \neq y, \\ x & \text{for } x = y. \end{cases}$$

We have the following

Proposition 3. *If f and g satisfy the assumptions of Theorem 3, then*

$$C^{[f,g]}(x, y) = g^{-1}\left(L^{[f \circ g^{-1}]}\left(g(x), g(y)\right)\right), \quad x, y \in I,$$

and

$$L^{[f \circ g^{-1}]}(u, v) = g\left(C^{[f,g]}\left(g^{-1}(u), g^{-1}(v)\right)\right), \quad u, v \in J,$$

where $L^{[f \circ g^{-1}]}$ is the Lagrange mean of the function $f \circ g^{-1}$.

Proof. For $u, v \in J = (\frac{f'}{g'})(I)$, $u \neq v$, we have

$$\begin{aligned}L^{[f \circ g^{-1}]}(u, v) &= \left(\left(f \circ g^{-1}\right)'\right)^{-1}\left(\frac{f \circ g^{-1}(u) - f \circ g^{-1}(v)}{u - v}\right) \\ &= \left(\frac{f'}{g'} \circ g^{-1}\right)^{-1}\left(\frac{f \circ g^{-1}(u) - f \circ g^{-1}(v)}{u - v}\right) \\ &= g \circ \left(\frac{f'}{g'}\right)^{-1}\left(\frac{f \circ g^{-1}(u) - f \circ g^{-1}(v)}{u - v}\right).\end{aligned}$$

Taking here $u = g(x)$, $v = g(y)$ where $x, y \in I$, $x \neq y$ are arbitrary, we hence get

$$L^{[f \circ g^{-1}]}\left(g(x), g(y)\right) = g \circ \left(\frac{f'}{g'}\right)^{-1}\left(\frac{f(x) - f(y)}{g(x) - g(y)}\right) = g\left(C^{[f,g]}(x, y)\right),$$

whence $C^{[f,g]}(x, y) = g^{-1}(L^{[f \circ g^{-1}]}(g(x), g(y)))$. □

The first equality is a *representation formula* of the Cauchy mean with the aid of Lagrange mean.

Notation 1. If $M : J^2 \to J$ is a mean in an interval J and $\varphi : I \to J$ is continuous, one-to-one and onto, then, obviously, the function $M_\varphi : I^2 \to I$ defined by

$$M_\varphi(x, y) := \varphi^{-1}\left(M\left(\varphi(x), \varphi(y)\right)\right), \quad x, y \in I,$$

is a mean in I. In the sequel, if φ is a nonconstant power function, that is if $\varphi(t) = t^r$ for some $r \in \mathbb{R}$, $r \neq 0$, we write M_r instead of M_φ.

With this notation, the representation formula can be written in the following way.

Remark 4. Under the assumption of Theorem 3,

$$C^{[f,g]} = L_g^{[f\circ g^{-1}]} \quad \text{and} \quad L^{[f\circ g^{-1}]} = C_{g^{-1}}^{[f,g]}.$$

Note that the Cauchy mean can be also represented with the aid of the accompanying Lagrange mean. Namely, we have the following

Proposition 4. *Under the assumptions of Theorem* 3, *we have*

$$C^{[f,g]} = M_{(f'/g')}^{[f\circ g^{-1}]}, \qquad M^{[f\circ g^{-1}]} = C_{(f'/g')^{-1}}^{[f,g]},$$

and

$$M_{(f'/g')}^{[f\circ g^{-1}]} = L_g^{[f\circ g^{-1}]}, \qquad L^{[f\circ g^{-1}]} = \left(M_{(f'/g')}^{[f\circ g^{-1}]}\right)_{g^{-1}}.$$

Proof. Indeed, by the definitions of $C^{[f,g]}$ and $M^{[f\circ g^{-1}]}$, for all $x, y \in I$, we have

$$C^{[f,g]}(x,y) = \left(\frac{f'}{g'}\right)^{-1}\left(M^{[f\circ g^{-1}]}\left(\frac{f'(x)}{g'(x)}, \frac{f'(y)}{g'(y)}\right)\right),$$

and, for all $u, v \in J := (\frac{f'}{g'})(I)$,

$$M^{[f\circ g^{-1}]}(u,v) = \left(\frac{f'}{g'}\right)\left(C^{[f,g]}\left(\left(\frac{f'}{g'}\right)^{-1}(u), \left(\frac{f'}{g'}\right)^{-1}(v)\right)\right).$$

Now two remaining equalities follow from the previous remark. □

Remark 5. (Cf. [7].) Assuming additionally that also $f'(x) \neq 0$ for all $x \in I$ in the definition of $C^{[f,g]}$, we have

$$C^{[f,g]} = C^{[g,f]}.$$

Stolarsky [8] (cf. also [2,3]), considering the power functions in the definition of Cauchy mean $C^{[f,g]}$, introduced a two parameter class of means $\mathcal{E}^{[p,q]} : (0,\infty)^2 \to (0,\infty)$, pointwise continuous with respect to the parameters p and q, which, for all $x, y > 0$, $x \neq y$, can written as follows

$$\mathcal{E}^{[p,q]}(x,y) := \begin{cases} (\frac{q}{p}\frac{x^p - y^p}{x^q - y^q})^{1/(p-q)} & \text{if } pq(p-q) \neq 0, \\ (\frac{1}{p}\frac{x^p - y^p}{\log x - \log y})^{1/p} & \text{if } p \neq 0 = q, \\ e^{-1/p}(\frac{x^{px^p}}{y^{py^p}})^{1/(px-py)} & \text{if } q = p \neq 0, \\ \sqrt{xy} & \text{if } p = q = 0. \end{cases}$$

By Remark 6 we have $\mathcal{E}^{[p,q]} = \mathcal{E}^{[q,p]}$; in particular $\mathcal{E}^{[p,0]} = \mathcal{E}^{[0,p]}$ for $p \neq 0$. Note also that $\mathcal{E}^{[p,1]} = \mathcal{L}^{[p]}$ for all $p \in \mathbb{R}$ and $\mathcal{E}^{[1,1]}$ is the identric mean.

Take $f(x) = x^p$, $g(x) = x^q$ $(x > 0)$ where $pq(p-q) \neq 0$. By (10), for all $u, v > 0$, $u \neq v$ we have

$$M^{[f\circ g^{-1}]}(u,v) = \frac{q}{p}\frac{u^{p/(p-q)} - v^{p/(p-q)}}{u^{q/(p-q)} - v^{q/(p-q)}} = \frac{\frac{q}{p-q}}{1 + \frac{q}{p-q}}\frac{u^{1+\frac{q}{p-q}} - v^{1+\frac{q}{p-q}}}{u^{\frac{q}{p-q}} - v^{\frac{q}{p-q}}},$$

whence, setting $t := \frac{q}{p-q}$ in the formula (8) for $\mathcal{M}^{[t]}$, we obtain

$$M^{[f\circ g^{-1}]} = \mathcal{M}^{[\frac{q}{p-q}]}.$$

Since

$$f \circ g^{-1}(u) = u^{p/q}, \qquad \frac{f'}{g'}(x) = \frac{p}{q}x^{p-q}, \qquad \left(\frac{f'}{g'}\right)^{-1}(u) = \left(\frac{q}{p}u\right)^{1/(p-q)},$$

applying the first equality of Proposition 1, the positive homogeneity of $\mathcal{M}^{[\frac{q}{p-q}]}$, and the assumed notation, for all $x, y > 0$, we get

$$\begin{aligned}\mathcal{E}^{[p,q]}(x,y) &= C^{[f,g]}(x,y) = \left(\frac{f'}{g'}\right)^{-1}\left(M^{[f\circ g^{-1}]}\left(\frac{f'}{g'}(x), \frac{f'}{g'}(y)\right)\right) \\ &= \left(\frac{q}{p}\mathcal{M}^{[\frac{q}{p-q}]}\left(\frac{p}{q}x^{p-q}, \frac{p}{q}y^{p-q}\right)\right)^{1/(p-q)} = \left(\mathcal{M}^{[\frac{q}{p-q}]}\left(x^{p-q}, y^{p-q}\right)\right)^{1/(p-q)} \\ &= \mathcal{M}_{p-q}^{[\frac{q}{p-q}]}(x,y).\end{aligned}$$

Thus we have shown the following

Remark 6. For all $p, q \in \mathbb{R}$, $pq(p-q) \neq 0$, we have

$$\mathcal{E}^{[p,q]} = \mathcal{M}_{p-q}^{[\frac{q}{p-q}]}.$$

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