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THE BOUNDED LOCAL OPERATORS IN THE BANACH SPACE OF HÖLDER FUNCTIONS

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Abstract. It is known that every locally defined operator acting between two Hölder spaces is a Nemytskii superposition operator. We show that if such an operator is bounded in the sense of the norm, then its generator is continuous.

1. Introduction

Let $I \subset \mathbb{R}$ be an arbitrary interval and by $\mathbb{R}^I$ we denote the set of all functions $\varphi : I \to \mathbb{R}$. For a given two-place function $h : I \times \mathbb{R} \to \mathbb{R}$, the mapping $K : \mathbb{R}^I \to \mathbb{R}^I$ defined by

$$K(\varphi)(x) := h(x, \varphi(x)), \quad \varphi \in \mathbb{R}^I,\ x \in I,$$

is called a Nemytskii superposition operator of the generator h.

It is known that every locally defined operator mapping the set of continuous functions $C(I, \mathbb{R})$ into itself must be a superposition operator [2]. Moreover, K maps $C(I, \mathbb{R})$ into itself if and only if its generator h is continuous. At this background it is surprising enough that there are discontinuous functions $h : I \times \mathbb{R} \to \mathbb{R}$ generating the superpositions operators K which map the space of continuously differentiable functions $C^1(I,\mathbb{R})$ into itself (cf. [1, p. 209]). In [3] it has been proved that if a locally defined operator maps the Banach space $H_\phi(I,\mathbb{R})$ of all Hölder functions $\varphi : I \to \mathbb{R}$ into $H_\psi(I,\mathbb{R})$, then it is a Nemytskii superposition operator. The purpose of this paper is to show that if, additionally, K is bounded with respect to $H_\phi(I,\mathbb{R})$–norm, then its generator must be continuous.

2. Main result

Let $\phi : (0,\infty) \to (0,\infty)$ satisfy the following condition:

(*i*) ϕ is strictly increasing, $\phi(0+) := \lim_{t\to 0+} \phi(t) = 0$ and the function

$$(0,\infty) \ni t \to \frac{\phi(t)}{t}$$

is decreasing.

Let us note the following (easy to verify)

Remark 1. If $\phi : (0,\infty) \to (0,\infty)$ satisfies condition (i), then ϕ is subadditive and continuous.

Let $I \subset \mathbb{R}$ be an interval and let $x_0 \in I$ be arbitrarily fixed. For a given $\phi : (0,\infty) \to (0,\infty)$, having the above properties, by $H_\phi(I,\mathbb{R})$ we denote the Banach space of all Hölder functions $\varphi : I \to \mathbb{R}$ equipped with the norm

$$\|\varphi\|_\phi := |\varphi(x_0)| + \sup_{x,y\in I, x\neq y} \frac{|\varphi(x)-\varphi(y)|}{\phi(|x-y|)}.$$

Clearly, $\varphi \in H_\phi(I,\mathbb{R})$ if and only if there exists a constant $c > 0$ such that

$$|\varphi(x)-\varphi(y)| \leq c\phi(|x-y|), \quad x,y \in I.$$

Let us notice that if $\phi(t) = t^\alpha$ for some $\alpha \in (0,1]$, then $H_\alpha(I,\mathbb{R}) := H_\phi(I,\mathbb{R})$ is the classical Hölder functions space and $H_1(I,\mathbb{R})$ becomes the Banach space of Lipschitz functions.

Definition. *Let $\phi,\psi : (0,\infty) \to (0,\infty)$ satisfy condition (i). An operator $K : H_\phi(I,\mathbb{R}) \to H_\psi(I,\mathbb{R})$ is said to be locally defined if for any open interval $J \subset \mathbb{R}$ and for any functions $\varphi,\psi \in H_\phi(I,\mathbb{R})$,*

$$\varphi|_{J\cap I} = \psi|_{J\cap I} \Rightarrow K(\varphi)|_{J\cap I} = K(\psi)|_{J\cap I},$$

where $\phi|_{J\cap I}$ denotes the restriction of φ to $J \cap I$.

In [3] the following result was proved:

Theorem 1. ([3], Corollary 2). *Let $I \subset \mathbb{R}$ be an interval. If a locally defined operator K maps $H_\phi(I,\mathbb{R})$ into $H_\psi(I,\mathbb{R})$, then there exists a unique function $h : I \times \mathbb{R} \to \mathbb{R}$ such that*

$$K(\varphi)(x) = h(x, \varphi(x)), \quad (x \in I),$$

for all $\varphi \in H_\phi(I,\mathbb{R})$, that is K is a Nemytskii operator of the generator h.

We say that an operator $K : H_\phi(I,\mathbb{R}) \to H_\psi(I,\mathbb{R})$ is bounded if it maps the convergent sequences of $H_\phi(I,\mathbb{R})$ into bounded sequences in $H_\psi(I,\mathbb{R})$.

The main result reads as follows:

Theorem 2. *Let $I \subset \mathbb{R}$ be an interval. If a locally defined operator $K : H_\phi(I,\mathbb{R}) \to H_\psi(I,\mathbb{R})$ is bounded, then there exists a continuous function $h : I \times \mathbb{R} \to \mathbb{R}$ such that*

$$K(\varphi)(x) = h(x, \varphi(x)); \quad \varphi \in H_\phi(I,\mathbb{R}), \quad (x \in I).$$

Proof. By Theorem 1, there exists a function $h : I \times \mathbb{R} \to \mathbb{R}$ such that the formula of our result holds true. We shall show that h is continuous.

Without any loss of generality we can assume that $I = [a,b)$, where $0 < b \leq +\infty$, and that

$$\|\varphi\|_\phi := |\varphi(a)| + \sup_{x,y \in I, x \neq y} \frac{|\varphi(x) - \varphi(y)|}{\phi(|x-y|)}.$$

First we show that h is continuous with respect to the second variable. To this end let us fix $(x_0, y_0) \in I$ and choose arbitrarily a real sequence $(y_n)_{n\in\mathbb{N}}$ such that

$$y_n \neq y_0, \quad n \in \mathbb{N}, \quad \lim_{n\to\infty} y_n = y_0. \tag{1}$$

Let $(x_n)_{n\in\mathbb{N}}$ be a sequence such that $x_n \in I$, $n \in \mathbb{N}$, and

$$|x_n - x_0| = \phi^{-1}\left(\sqrt{|y_n - y_0|}\right), \quad n \in \mathbb{N}.$$

Hence we obtain

$$\frac{|y_n - y_0|}{\phi(|x_n - x_0|)} = \frac{|y_n - y_0|}{\phi\left(\phi^{-1}\left(\sqrt{|y_n - y_0|}\right)\right)} = \sqrt{|y_n - y_0|}, \quad n \in \mathbb{N}. \tag{2}$$

Define the functions $P_{y_n} : I \to \mathbb{R}$, $\varphi_n : I \to \mathbb{R}$, $n \in \mathbb{N}$, by the following formulas:

$$P_{y_n}(x) := y_n, \quad n \in \mathbb{N}, \tag{3}$$

$$\varphi_n(x) = \begin{cases} y_0, & \text{for } \ x \in [a, x_0], \\ \dfrac{y_n - y_0}{x_n - x_0}(x - x_0) + y_0 & \text{for } \ x \in (x_0, x_n), \ n \in \mathbb{N}, \\ y_n, & \text{for } \ x \in [x_n, b). \end{cases} \tag{4}$$

and put

$$\varphi_0(x) = y_0, \quad x \in I.$$

Of course,

$$P_{y_n}, \varphi_n \in H_\phi(I, \mathbb{R}), \quad n \in \mathbb{N}.$$

Since

$$\|P_{y_n} - \varphi_0\|_\phi = |y_n - y_0|, \quad n \in \mathbb{N},$$

applying (1) and (2), we get

$$\lim_{n\to\infty} \|P_{y_n} - \varphi_0\|_\phi = 0, \qquad \lim_{n\to\infty} \|\varphi_n - \varphi_0\|_\phi = 0. \tag{5}$$

Making use of (3), (4), the triangle inequality and by the definition of the norm, we have

$$\begin{aligned} |h(x_0, y_n) - h(x_0, y_0)| &\le |h(x_n, y_n) - h(x_0, y_n)| + |h(x_n, y_n) - h(x_0, y_0)| \\ &= |h(x_n, P_{y_n}(x_n) - h(x_0, P_{y_n}(x_0)| \\ &\quad + |h(x_n, \varphi_n(x_n)) - h(x_0, \varphi_n(x_0))| \\ &= |K(P_{y_n})(x_n) - K(P_{y_n})(x_0)| \\ &\quad + |K(\varphi_n)(x_n) - K(\varphi_n)(x_0)| \\ &= \frac{|K(P_{y_n})(x_n) - K(P_{y_n})(x_0)|}{\psi(|x_n - x_0|)} \psi(|x_n - x_0|) + \\ &\quad + \frac{|K(\varphi_n)(x_n) - K(\varphi_n)(x_0)}{\psi(|x_n - x_0|)} \psi(|x_n - x_0|) \\ &\le \|K(P_{y_n})\|_\psi \psi(|x_n - x_0|) + \|K(\varphi_n)\|_\psi \cdot \psi(|x_n - x_0|). \end{aligned}$$

Taking into account (5), the equality $\psi(0+) = 0$, the boundedness of the operator K and letting n tend to the infinity, we get the continuity of h with respect to the second variable.

To show that h is continuous fix $(x_0, y_0) \in I \times \mathbb{R}$, take two arbitrary sequences $x_n \in I$, $y_n \in \mathbb{R}$, $n \in \mathbb{N}$, convergent to x_0 and y_0, respectively, and define $P_{y_n} : I \to \mathbb{R}$, $n \in \mathbb{N} \cup \{0\}$, by

$$P_{y_n}(x) = y_n, \quad n \in \mathbb{N} \cup \{0\}.$$

Hence, by the triangle inequality and by the definition of the norm, we have

$$
\begin{aligned}
|h(x_n, y_n) - h(x_0, y_0)| &\le |h(x_n, y_n) - h(x_0, y_n)| + |h(x_0, y_n) - h(x_0, y_0)| \\
&= |h(x_n, P_{y_n}(x_n)) - h(x_0, P_{y_n}(x_0)| \\
&\quad + |h(x_0, y_n) - h(x_0, y_0)| \\
&= |(K(P_{y_n})(x_n) - K(P_{y_n})(x_0)| \\
&\quad + |h(x_0, y_n) - h(x_0, y_0)| \\
&= \frac{|K(P_{y_n})(x_n) - K(P_{y_n})(x_0)|}{\psi(|x_n - x_0|)} \cdot \psi(|x_n - x_0|) \\
&\quad + |h(x_0, y_n) - h(x_0, y_0)| \\
&\le \|K(P_{y_n})\|_\psi \psi(|x_n, x_0|) + |h(x_0, y_n) - h(x_0, y_0)|.
\end{aligned}
$$

Since, by the definition of P_{y_n}, $n \in \mathbb{N} \cup \{0\}$,

$$\lim_{n\to\infty} \|P_{y_n} - P_{y_0}\|_\phi = 0,$$

applying the boundedness of the operator K, the equality $\psi(0+) = 0$ and the first part of the proof, i.e. the continuity of h with respect to the second variable, letting n tend to the infinity, we get the required claim. □

Remark 2. Taking in the above theorem a compact interval $I \subset \mathbb{R}$, one gets Theorem 7.3 from [1].

To construct an example showing that the assumption of the boundedness of K is essential, we need the following

Lemma. *Let $(X, d), (Y, \rho)$ be metric spaces. Suppose $A, B \subset X$ are closed, $\operatorname{int} A \cap \operatorname{int} B = \emptyset$ and adjacent in the following sense: for any $x \in A$, $y \in B$ there exists a point $z \in \delta A \cap \delta B$ such that*

$$d(x, y) = d(x, z) + d(z, y). \tag{6}$$

If the functions $f : A \to Y$ and $g : B \to Y$ are Lipschitz continuous and

$$f(z) = g(z) \quad \text{for all} \quad z \in \delta A \cap \delta B,$$

then the function $h : (A \cup B) \to Y$ defined by

$$h(x) := \begin{cases} f(x) & \text{for } x \in A, \\ g(x) & \text{for } x \in B \end{cases}$$

is Lipschitz continuous. (Here δA stands for the boundary of A.)

Proof. Since f and g are Lipschitz continuous, there is $c \in \mathbb{R}_+$ such that

$$\rho(f(x), f(y)) \le cd(x,y) \quad \text{for } x, y \in A; \quad \rho(g(x), g(y)) \le cd(x,y) \quad \text{for } x, y \in B.$$

Take $x, y \in A \cup B$ and assume that $x \in A$ and $y \in B$. By assumption, there is $z \in \delta A \cap \delta B$ such that (6) holds. Hence, by the triangle inequality,

$$\rho(h(x), h(y)) \le \rho(h(x), h(z)) + \rho(h(z), h(y)) = \rho(f(x), f(z)) + \rho(g(z), g(y))$$

$$\le cd(x,z) + cd(z,y) = cd(x,y).$$

As the remaining two cases are obvious, the proof is complete. □

Example. Define a two-place function $h : [0,1] \times \mathbb{R} \to \mathbb{R}$ by the formula

$$h(x,y) := \begin{cases} 0 & \text{if } \ y \le 0, \\ \dfrac{y}{\sqrt{x}} & \text{if } \ 0 < y \le \sqrt{x}, \\ 1 & \text{if } \ y > \sqrt{x}. \end{cases} \tag{7}$$

Observe that h is continuous in $[0,1] \times \mathbb{R} \setminus \{(0,0)\}$ and discontinuous at the point $(0,0)$. In fact we have more, namely outside of any neighbourhood of $(0,0)$, by Lemma, the function h is Lipschitzian.

Denote by $\mathcal{F}[0,1]$ the set of all functions $\varphi : [0,1] \to \mathbb{R}$. Let $K : \mathcal{F}[0,1] \to \mathcal{F}[0,1]$ be the Nemytskii composition (so locally defined) operator generated by h, i.e.

$$K(\varphi)(x) := h(x, \varphi(x)), \quad x \in [0,1].$$

We shall show that K maps the space $H_1([0,1], \mathbb{R})$ of all Lipschitz continuous functions $\varphi : [0,1] \to \mathbb{R}$ into itself.

Take $\varphi \in H_1([0,1], \mathbb{R})$. If $\varphi(0) \ne 0$, then as h is Lipschitz continuous outside any neighbourhood of $(0,0)$, the function $K(\varphi)$, as composition of Lipschitz continuous functions, is Lipschitz continuous in $[0,1]$, so $K(\varphi) \in H_1([0,1], \mathbb{R})$. If $\varphi(0) = 0$, then $K(\varphi)\big|_{[\varepsilon,1]}$ is Lipschitz continuous for any $\varepsilon \in (0,1]$. In view of Lemma, it is enough to show that $K(\varphi)\big|_{[0,\varepsilon]}$ is Lipschitz continuous. To this end assume that φ satisfies the Lipschitz condition with a constant c, that is

$$|\varphi(x) - \varphi(\overline{x})| \le c|x - \overline{x}|, \quad x, \overline{x} \in [0,1].$$

Setting $\overline{x} = 0$, we hence get

$$|\varphi(x)| \le cx, \quad x \in [0,1],$$

so the graph of the function φ is contained in the triangle set

$$D := \{(x, y) : x \in [0, 1],\ |y| \leq cx\}.$$

If φ is nonpositive on any subinterval of $I \subset [0, 1]$, then, by the definition of h, we have $K(\varphi)\big|_I = 0$ and, obiously, $K(\varphi)$ is Lipschitz continiuous on I with zero Lipschitz constant. Therefore, it is enough to confine our considerations to the case when the graph of $\varphi\big|_{[0,\varepsilon]}$ is contained in the set

$$D_\varepsilon := \{(x, y) : x \in [0, \varepsilon],\ 0 \leq y \leq cx\}.$$

Let us choose $\varepsilon > 0$ such that $c < \frac{1}{\sqrt{\varepsilon}}$. Then, clearly $cx < \sqrt{x}$ for all $x \in (0, \varepsilon]$. Since for all $(x, y) \in D_\varepsilon$ we have

$$\left|\frac{\partial}{\partial x} h(x, y)\right| = \left|-\frac{y^2}{2x\sqrt{x}}\right| \leq \frac{(cx)^2}{2x\sqrt{x}} \leq \frac{c^2\sqrt{\varepsilon}}{2}$$

and

$$\left|\frac{\partial}{\partial y} h(x, y)\right| = \frac{2y}{\sqrt{x}} \leq \frac{2cx}{\sqrt{x}} \leq 2c\sqrt{\varepsilon},$$

we infer that $h\big|_{D\varepsilon}$ is Lipschitz continuous. It follows that $K(\varphi)\big|_{[0,\varepsilon]}$, as a composition of Lipchitz functions, is Lipschitz continuous.

We claim that K is unbounded. To see this take a sequence of constant functions convergent to zero, $\varphi_k : [0, 1] \to \mathbb{R}$, $k \in \mathbb{N}$, defined by $\varphi_k(x) = \frac{1}{\sqrt{k}}$. According to (7), we get

$$K(\varphi_k)(x) = \begin{cases} 1 & \text{for } 0 \leq x < \dfrac{1}{k} \\ \dfrac{1}{\sqrt{kx}} & \text{for } \dfrac{1}{k} \leq x \leq 1 \end{cases} \qquad k \in \mathbb{N}.$$

Since

$$\|K(\varphi_k)\|_\psi \geq \left|\frac{\varphi_k(x) - \varphi_k(\overline{x})}{x - \overline{x}}\right|, \quad x, \overline{x} \in [0, 1], \quad x \neq y,$$

setting $x = \frac{4}{k}$, $\overline{x} = 0$, for all $k \geq 4$, we get

$$\|K(\varphi_k)\|_\psi \geq \frac{k}{8}, \quad k \geq 4,$$

which shows that K is not bounded. $\square$

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