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On the existence of a convex solution of the functional equation $\varphi(x) = h(x, \varphi[f(x)])$

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Abstract. In this paper we consider the functional equation $\varphi(x) = h(x, \varphi[f(x)])$. Under some conditions on given functions f and h we obtain the existence of a convex solution $\varphi: \langle 0, a) \to R$ such that $\varphi(0) = 0$. It is assumed that $f(0) = 0$.

In the present paper we consider the problem of the existence of a convex solution of the functional equation

$$\varphi(x) = h[x, \varphi[f(x)]], \tag{1}$$

where f and h are given and φ is an unknown function.

A real function φ defined in a convex set $D \subset R^n$ [1] is convex iff for all $x, y \in D$ and $\lambda \in (0, 1)$

$$\varphi(\lambda x + (1-\lambda) y) \leqslant \lambda\varphi(x) + (1-\lambda)\varphi(y).$$

We assume that

(i) f is increasing, convex in an interval $I = \langle 0, a)$ and

$$f(0) = 0, \quad f(x) < x \quad \text{for } 0 < x < a,$$

(ii) $\Omega \subset R^2$ is a convex set such that $(0, 0) \in \Omega$; h is increasing with respect to each variable and convex in Ω, and $h(0, 0) = 0$,

(iii) for every $x \in I$, $h[f(x), \Omega_{f(x)}] \subset \Omega_x$, where $\Omega_x = \{y : (x, y) \in \Omega\}$.

Remark 1. The convexity of Ω implies that the function $\alpha(x) = \inf \Omega_x$ is convex in I and $\beta(x) = \sup \Omega_x$ is concave in I. Moreover, if for a certain $x_0 \in I$ we have $\alpha(x_0) = -\infty$, then $\alpha(x) = -\infty$ for every $x \in I$. Similarly, if for a $x_0 \in I$ we have $\beta(x_0) = +\infty$, then $\beta = +\infty$ in I.

Thus we may confine our considerations to the following two cases: $\beta < +\infty$ and $\beta = +\infty$.

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[1] Here R^n is a linear metric space with the operations and the metric ϱ defined as follows. Let $x = (x_1, \ldots, x_n)$, $y = (y_1, \ldots, y_n) \in R^n$, and let $\lambda \in R$. Then $x + y = (x_1 + y_1, \ldots, x_n + y_n)$, $\lambda x = (\lambda x_1, \ldots, \lambda x_n)$ and $\varrho(x, y) = [(x_1 - y_1)^2 + \ldots + (x_n - y_n)^2]^{1/2}$.

1. In this section we consider the simpler case: $\beta < +\infty$. We shall prove the following

THEOREM 1. *Suppose that Ω is closed and let conditions (i)–(iii) be fulfilled. If for a certain $x_0 \in I$ we have $\sup \Omega_{x_0} < +\infty$, then there exists at least one increasing and convex function $\varphi: I \to R$ such that $\varphi(0) = 0$, fulfilling equation (1) in I.*

Proof. 1° Suppose that there exists a positive number $c \leqslant a$ such that

$$\alpha(x) = \inf \Omega_x \leqslant 0, \quad x \in \langle 0, c), \tag{2}$$

and let us put

$$\varphi_0(x) = 0, \quad x \in \langle 0, c). \tag{3}$$

Next, we define the sequence φ_n by the recurrent relation

$$\varphi_n(x) = h[x, \varphi_{n-1}[f(x)]], \quad n = 1, 2, \ldots \tag{4}$$

It follows from (ii) and (iii) that $\beta(x) \geqslant 0$ for $x \in I$. Thus we have $\alpha(x) \leqslant \varphi_0(x) \leqslant \beta(x)$ for $x \in \langle 0, c)$. This together with $f(x) < x$ yields $\varphi_0[f(x)] \in \Omega_x$ for $x \in \langle 0, c)$. Suppose that for a certain $n \geqslant 1$ and for all $x \in \langle 0, c)$ we have $\varphi_{n-1}[f(x)] \in \Omega_x$. In view of (4) this means that φ_n is well defined in $\langle 0, c)$. Then $\varphi_{n-1}[f^2(x)] \in \Omega_{f(x)}$ and according to (4) and (iii) we get

$$\varphi_n[f(x)] = h[f(x), \varphi_{n-1}[f^2(x)]] \in h(f(x), \Omega_{f(x)}) \subset \Omega_x.$$

Hence $\varphi_n[f(x)] \in \Omega_x$ for $x \in \langle 0, c)$. We prove by induction that $\varphi_n[f(x)] \in \Omega_x$ for each n, and from (4) it follows that φ_n is well defined in $\langle 0, c)$ for each n. It follows from (i) and (ii) (induction) that φ_n is an increasing sequence of increasing and convex functions in $\langle 0, c)$. Since $\beta < +\infty$ (cf. Remark 1), $\varphi_n(x)$ is bounded for every $x \in \langle 0, c)$. Thus there exists a $\varphi(x) = \lim_{n\to\infty} \varphi_n(x)$ for $x \in \langle 0, c)$ and, evidently, φ is increasing and convex in $\langle 0, c)$. Taking into account (3), (4) and (ii), we obtain $\varphi(0) = 0$. Letting $n \to \infty$ in (4), we see that φ satisfies equation (1) in $\langle 0, c)$. Using (i), (iii) and equation (1), we can extend this solution onto the whole interval I (compare M. Kuczma (²), the proof of a theorem of Kordylewski). For simplicity we denote this extension by φ. We shall prove that φ is increasing and convex in I. Let u be the supremum of all t such that φ is increasing in $\langle 0, t)$. For the indirect proof suppose that $u < a$. Since $f(u) < u$, it follows from the continuity of f that there exists a $u_1 > u$ such that $f(x) < u$ for $x \in \langle 0, u_1)$. Thus, in view of (i) and (ii), we have for $0 \leqslant x_1 < x_2 < u_1$

$$\varphi(x_1) = h(x_1, \varphi[f(x_1)]) \leqslant h(x_2, \varphi[f(x_1)]) \leqslant h(x_2, \varphi[f(x_2)]) = \varphi(x_2).$$

(²) M. Kuczma, *Functional equations in a single variable*, Monografie Matematyczne 46, PWN, Warszawa 1968, p. 70.

i.e., φ is increasing in $\langle 0, u_1)$. This contradiction completes the proof of the monotonicity of φ in I.

Now we denote by u the supremum of all t such that φ is convex in $\langle 0, t)$ and suppose that $u < a$. Since $f(u) < u$, it follows from the continuity of f that there exists a $u_1 > u$ such that $f(x) < u$ for $x \in \langle 0, u_1)$. Now from the monotonicity of φ and from conditions (i), (ii) we have for $0 \leqslant x_k < u_1$, $\lambda_k > 0$, $\lambda_1 + \lambda_2 = 1$, $k = 1, 2$

$$\begin{aligned}\varphi(\lambda_1 x_1 + \lambda_2 x_2) &= h(\lambda_1 x_1 + \lambda_2 x_2, \varphi[f(\lambda_1 x_1 + \lambda_2 x_2)]) \\ &\leqslant h(\lambda_1 x_1 + \lambda_2 x_2, \varphi[\lambda_1 f(x_1) + \lambda_2 f(x_2)]) \\ &\leqslant h(\lambda_1 x_1 + \lambda_2 x_2, \lambda_1 \varphi[f(x_1)] + \lambda_2 \varphi[f(x_2)]) \\ &\leqslant \lambda_1 h(x_1, \varphi[f(x_1)]) + \lambda_2 h(x_2, \varphi[f(x_2)]) \\ &= \lambda_1 \varphi(x_1) + \lambda_2 \varphi(x_2).\end{aligned}$$

Thus φ is convex in $\langle 0, u_1)$. This contradiction proves that we must have $u = a$, or that φ is convex in I.

2° Now, suppose that there is no a $c > 0$ such that (2) holds. Then according to the convexity of Ω, the function $a(x) = \inf \Omega_x$ has the following properties (cf. Remark 1):

(5) $\quad a(0) = 0, \quad a$ is increasing and convex in I.

We define

(6) $\quad \varphi_0(x) = a(x), \quad x \in I.$

Using (i)–(iii), it is easy to verify (induction) that the sequence (4) with φ_0 defined above is well defined for $x \in I$ and forms an increasing sequence of increasing and convex functions in I and such that $\varphi_n(0) = 0$. Moreover, $\varphi_n(x) \leqslant \beta[f^{-1}(x)] < \infty$ for $x \in I$. Thus, the function $\varphi(x) = \lim_{n\to\infty} \varphi_n(x)$ for $x \in I$ is increasing, convex, fulfils equation (1) in I and condition $\varphi(0) = 0$. This completes the proof.

2. In this section we assume that

(iv) for every $x \in I$, $\sup \Omega_x = +\infty$ and there exists a $\delta > 0$ such that $\inf \Omega_x \leqslant 0$ for $x \in (0, \delta)$.

It follows from (ii) and (iv) that there exist partial derivatives:

$$h'_x(0+, 0) = \lim_{x\to 0+} \frac{h(x, 0)}{x}, \quad h'_y(0, 0+) = \lim_{y\to 0+} \frac{h(0, y)}{y}.$$

By (i) we have

$$f'(0+) = \lim_{x\to 0+} \frac{f(x)}{x}.$$

We shall prove the following result.

THEOREM 2. *Let conditions* (i)–(iv) *be fulfilled. If*

(7) $$f'(0+)h'_y(0, 0+) < 1,$$

then there exists an increasing and convex function $\varphi: I \to R$, *fulfilling equation* (1) *in* I *and condition* $\varphi(0) = 0$.

Proof. For an $\varepsilon > 0$ we denote

$$k = h'_x(0+, 0) + \varepsilon, \quad l = h'_y(0, 0+) + \varepsilon, \quad s = f'(0+) + \varepsilon.$$

In view of (7) we can choose the $\varepsilon > 0$ so small that

(8) $$sl < 1.$$

It follows from (i) and (ii) that there exists a b, $0 < b < \delta$, such that

(9) $$h(x, y) \leqslant kx + ly, \quad x, y \in \langle 0, b\rangle$$

and

(10) $$f(x) \leqslant sx, \quad x \in \langle 0, b\rangle.$$

Let us put

(11) $$m = k(1 - sl)^{-1},$$

(12) $$c = \min(b, bm^{-1})$$

and denote by D the set

$$D = \{(x, y): 0 \leqslant x \leqslant c,\ 0 \leqslant y \leqslant mx\}.$$

It follows from (12) that $D \subset \Omega$. Let $D_x = [y: (x, y) \in D]$. Evidently, $D_x = \langle 0, mx\rangle$. We shall show that

(13) $$h[f(x), D_{f(x)}] \subset D_x, \quad x \in \langle 0, c\rangle.$$

Take $y \in D_{f(x)} = \langle 0, mf(x)\rangle$. Then by (ii), (9), (i), (10) and (11) we obtain

$$0 \leqslant h(f(x), y) \leqslant kf(x) + ly \leqslant kx + lmf(x) \leqslant (k + slm)x = mx$$

and (13) has been proved. Evidently, D is closed and convex. If we put $\Omega = D$, then all the assumptions of Theorem 1 will be fulfilled. Thus there exists an increasing and convex function $\varphi: \langle 0, c\rangle \to R$, fulfilling equation (1) in $\langle 0, c\rangle$ and condition $\varphi(0) = 0$. This solution has a unique extension onto the whole interval I, which may easily be obtained by using (iii) and equation (1) (compare M. Kuczma [2]). A similar argument as in Theorem 1 proves that this extension is increasing and convex in I. This completes the proof.

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[2] M. Kuczma, cf. [1], p. 70, Theorem 3.2.

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