DEMONSTRATIO MATHEMATICA
Vol. XLIII No 4 2010

Janusz Matkowski

GENERALIZATIONS OF LAGRANGE AND CAUCHY MEAN-VALUE THEOREMS

Abstract. Some generalizations of the Lagrange Mean-Value Theorem and Cauchy Mean-Value Theorem are proved and the extensions of the corresponding classes of means are presented.

1. Introduction

Recall that a function $M : I^2 \to I$ is called a *mean in* a nontrivial interval $I \subset \mathbb{R}$ I if it is *internal*, that is if

$$\min(x,y) \le M(x,y) \le \max(x,y) \qquad \text{for all} \quad x,y \in I.$$

The mean M is called *strict* if these inequalities are strict for all $x,y \in I$, $x \neq y$, and *symmetric* if $M(x,y) = M(y,x)$ for all $x,y \in I$.

The Lagrange Mean-Value Theorem can be formulated in the following way. *If a function $f : I \to \mathbb{R}$ is differentiable, then there exists a strict symmetric mean $L : I^2 \to I$ such that, for all $x,y \in I, x \neq y$,*

$$\frac{f(x)-f(y)}{x-y} = f'(L(x,y)). \tag{1}$$

If f' is one-to-one then, obviously, $L^{[f]} := L$ is uniquely determined and is called a *Lagrange mean* generated by f.

Note that formula (1) can be written in the form

$$\frac{1}{2}\left(\frac{f(x)-f(\frac{x+y}{2})}{x-\frac{x+y}{2}} + \frac{f(\frac{x+y}{2})-f(y)}{\frac{x+y}{2}-y}\right) = f'(L(x,y)), \qquad x,y \in I,\ x \neq y,$$

2000 *Mathematics Subject Classification*: Primary 26A24.
Key words and phrases: mean, mean-value theorem, Lagrange theorem, Cauchy theorem, generalization, Lagrange mean, Cauchy mean, logrithmic mean, Stolarsky mean.

or, setting $\mathcal{A}(x,y) := \frac{x+y}{2}$, in the form

$$\mathcal{A}\left(\frac{f(x)-f(\mathcal{A}(x,y))}{x-\mathcal{A}(x,y)}, \frac{f(\mathcal{A}(x,y))-f(y)}{\mathcal{A}(x,y)-y}\right) = f'(L(x,y)), \quad x,y \in I,\ x \neq y, \tag{2}$$

which shows a relationship of the mean-value theorem and the arithmetic mean. This equation has the following geometrical interpretation. The arithmetic mean of the slope of chord of the graph of f passing through the points $(x, f(x))$ and $\left(\frac{x+y}{2}, f\left(\frac{x+y}{2}\right)\right)$ and the slope of chord of the graph passing through the points $\left(\frac{x+y}{2}, f\left(\frac{x+y}{2}\right)\right)$ and $(y, f(y))$ is equal to the slope of tangent to the graph at a point $(L(x,y), f(L(x,y)))$ (cf. Figure 1).

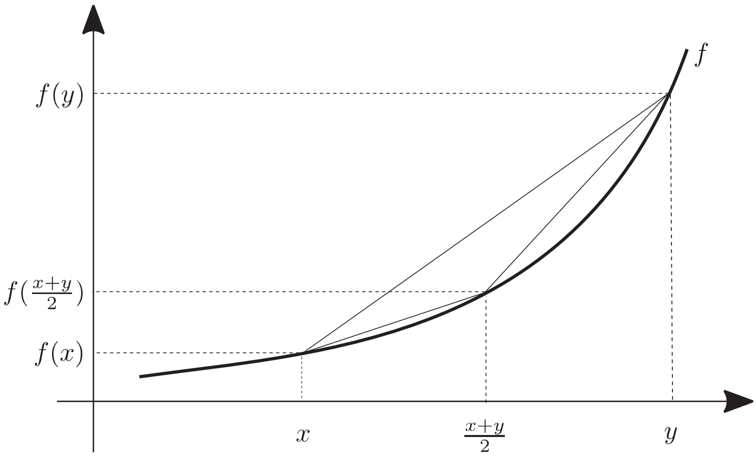

Fig. 1.

Our idea of generalization of the Lagrange Mean-Value Theorem is based on formula (2).

Let $I, J \subset \mathbb{R}$ be intervals. Assume that $M : I^2 \to I$ is a strict mean in I and $K : J^2 \to J$ is a mean in J.

In section 2 we show that if $f : I \to \mathbb{R}$ is a differentiable function and $f'(I) \subset J$, then there exists a strict mean $L : I^2 \to I$ such that, for all $x, y \in I$, $x \neq y$,

$$K\left(\frac{f(x)-f(M(x,y))}{x-M(x,y)}, \frac{f(M(x,y))-f(y)}{M(x,y)-y}\right) = f'(L(x,y)).$$

Moreover, $L := L_{M,K}^{[f]}$ is unique if f' is one-to-one, and symmetric if M and K are symmetric. Putting

$$\mathcal{A}_w(x,y) := wx + (1-w)y, \qquad x, y \in \mathbb{R},$$

for arbitrarily fixed $w \in (0,1)$, we observe that for $M := \mathcal{A}_w$ and $K = \mathcal{A}_{1-w}$ the above result reduces to the original Lagrange theorem. In particular, in this case, L does not depend on w. Since $L^{[f]}_{A,A} = L^f$, the mean $L^{[f]}_{M,K}$ generalizes the Lagrange mean L^f. As an application we obtain a generalization of the family of logarithmic means.

The Cauchy Mean-Value Theorem can be formulated as follows. *If* $f, g : I \to R$ *are differentiable and* $g'(x) \neq 0$ *for all* $x \in I$ *then there exists a strict symmetric mean* $C : I^2 \to I$ *such that for all* $x, y \in I, x \neq y$,

$$\frac{f(x) - f(y)}{g(x) - g(y)} = \frac{f'(C(x,y))}{g'(C(x,y))}. \tag{3}$$

Moreover, if $\frac{f'}{g'}$ is one-to-one then $C^{[f,g]} := C$ is unique and it is called a *Cauchy mean* generated by f and g.

The assumptions of Cauchy's Mean-Value Theorem imply that g is continuous, strictly monotonic. It follows that the quasi-arithmetic mean $M^{[g]} : I^2 \to I$,

$$M^{[g]}(x,y) := g^{-1}\left(\frac{g(x) + g(y)}{2}\right), \qquad x, y \in I,$$

is well defined, and (3), the original Cauchy formula, can be written in the form

$$\mathcal{A}\left(\frac{f(x) - f(M^{[g]}(x,y))}{g(x) - g(M^{[g]}(x,y))}, \frac{f(M^{[g]}(x,y)) - f(y)}{g(M^{[g]}(x,y)) - g(y)}\right) = \frac{f'(C(x,y))}{g'(C(x,y))} \tag{4}$$

for all $x, y \in I$, $x \neq y$.

In Section 3 we generalize the Cauchy Mean-Value Theorem showing that if $f, g : I \to \mathbb{R}$ are differentiable, $g'(x) \neq 0$ for all $x \in I$, and $\frac{f'}{g'}(I) \subset J$, then formula (4) remains true on replacing the quasi-arithmetic mean $M^{[g]}$ by an arbitrary strict mean M and the arithmetic mean $\mathcal{A}$ by an arbitrary mean K. Moreover, $C^{[f,g]}_{M,K} := C$ is unique if $\frac{f'}{g'}$ is one-to-one, and symmetric if M and K are symmetric. In the case when $M = M^{[g]}_w$ where $M^{[g]}_w(x,y) := g^{-1}(wg(x) + (1-w)g(y))$ and $K = \mathcal{A}_{1-w}$ for some $w \in (0,1)$, this generalization reduces the original Cauchy theorem. Applying among other the Stolarsky means ([4], [5]) we obtain some new classes of means.

2. A generalization of Lagrange's Mean-Value Theorem

Theorem 1. *Let* $I, J \subset \mathbb{R}$ *be intervals. Suppose that* $M : I^2 \to I$ *is a strict mean and* $K : J^2 \to J$ *is a mean. If a function* $f : I \to \mathbb{R}$ *is differentiable* *and $f'(I) \subset J$, then there exists a strict mean $L: I^2 \to I$ such that*

$$K\left(\frac{f(x)-f(M(x,y))}{x-M(x,y)}, \frac{f(M(x,y))-f(y)}{M(x,y)-y}\right) = f'(L(x,y)), \qquad x,y \in I,\ x \neq y. \tag{5}$$

Moreover, if f' is one-to-one then $L =: L^{[f]}_{M,K}$ is unique,

(6)

$$L^{[f]}_{M,K}(x,y) = \begin{cases} (f')^{-1}\left(K\left(\frac{f(x)-f(M(x,y))}{x-M(x,y)}, \frac{f(M(x,y))-f(y)}{M(x,y)-y}\right)\right) & \text{for } x \neq y, \\ x & \text{for } x = y, \end{cases}$$

and $L^{[f]}_{M,K}$ is symmetric if M and K are symmetric.

Proof. Take $x, y \in I$, $x \neq y$. We can assume, without any loss of generality, that $x < y$. Since M is strict, we have $x < M(x,y) < y$. By the Lagrange Mean-Value Theorem there are $r, s \in I$, $r = r(x,y)$, $s = s(x,y)$,

$$x < r < M(x,y), \qquad M(x,y) < s < y, \tag{7}$$

such that

$$\frac{f(x)-f(M(x,y))}{x-M(x,y)} = f'(r), fracf(M(x,y)) - f(y)M(x,y) - yf'(s). \tag{8}$$

Since

$$\min(f'(r), f'(s)) \leq K(f'(r), f'(s)) \leq \max(f'(r), f'(s)),$$

the Darboux property of the derivative (cf. [3], p. 108, Theorem 5.12)

implies that there exists $t = t(r,s) \in I$,

$$r \leq t \leq s, \tag{9}$$

such that

$$K(f'(r), f'(s)) = f'(t). \tag{10}$$

From (7) and (9) we have

$$x = \min(x,y) < t(r(x,y), s(x,y)) < \max(x,y) = y.$$

Putting

$$L(x,y) := t(r(x,y), s(x,y)) \text{ for } x \neq y, \qquad \text{and} \qquad L(x,y) := x \text{ for } x = y,$$

we get a mean L which, according to (8) and (10), satisfies (5).

If f' is one-to-one, formula (6) is a consequence of (5). The remaining statement is obvious. ∎

Remark 1. Let a function f satisfy the conditions of Theorem 1 and let $w \in (0,1)$ be fixed. Denote by $\mathcal{A}_w$ the weighted arithmetic mean

$$\mathcal{A}_w(x,y) := wx + (1-w)y, \qquad x, y \in \mathbb{R}.$$

It easy to verify that formula (5) of Theorem 1 with $M := \mathcal{A}_w$ and $K := \mathcal{A}_{1-w}$ reduces to the classical Lagrange theorem. For $w = \frac{1}{2}$ we get relation (2). Note also that, for each $w \in (0,1)$, $\mathcal{A} = \mathcal{A}_w \otimes \mathcal{A}_{1-w}$, that $\mathcal{A}$ is a Gauss composition of the means $\mathcal{A}_w$ and $\mathcal{A}_{1-w}$ (cf. for instance [1]),

In particular we have

Corollary 1. *Suppose that $M : I^2 \to I$ is a strict mean in an interval I and $w \in (0,1)$. If $f : I \to \mathbb{R}$ is differentiable and f' is one to one, then*

$$L^{[f]}_{\mathcal{A}_w,\mathcal{A}_{1-w}} = L^{[f]}.$$

Taking $I \subset (0,\infty)$, $K = \mathcal{A}$ and $f(x) = x^{p+1}$ for some $p \in \mathbb{R}$, $-1 \neq p \neq 0$, in Theorem 1, we obtain

$$L^{[p]}_{M,\mathcal{A}}(x,y) := L^{[f]}_{M,\mathcal{A}}(x,y) = \\ = \left(\frac{1}{2(p+1)}\left(\frac{x^{p+1} - [M(x,y)]^{p+1}}{x - M(x,y)} + \frac{[M(x,y)]^{p+1} - y^{p+1}}{M(x,y) - y}\right)\right)^{1/p}$$

for all $x, y \in I$, $x \neq y$. To complete this definition it is enough to put $L^{[p]}_{M,\mathcal{A}}(x,x) := x$.

Simple calculations show that, for all $x, y \in I$, $x \neq y$,

$$\lim_{p\to -1} L^{[p]}_{M,\mathcal{A}}(x,y) = \left(\frac{1}{2}\left(\frac{\log x - \log M(x,y)}{x - M(x,y)} + \frac{\log M(x,y) - \log y}{M(x,y) - y}\right)\right)^{-1} \\ = \mathcal{H}(\mathcal{L}(x, M(x,y)), \mathcal{L}(M(x,y), y)),$$

and

$$\lim_{p\to 0} L^{[p]}_{M,\mathcal{A}}(x,y) = e^{-1}\left(\frac{x^x}{M(x,y)^{M(x,y)}}\right)^{1/(x-M(x,y))}\left(\frac{M(x,y)^{M(x,y)}}{y^y}\right)^{1/(M(x,y)-y)} \\ = \mathcal{G}(\mathcal{J}(x, M(x,y)), \mathcal{J}(M(x,y), y)),$$

where $\mathcal{H}$, $\mathcal{L}$, $\mathcal{G}$, $\mathcal{J}$ denote, respectively, the harmonic, logarithmic, geometric and identric means defined by

$$\mathcal{H}(x,y) = \frac{2xy}{x+y}, \qquad \mathcal{L}(x,y) = \frac{x-y}{\log x - \log y},$$

$$\mathcal{G}(x,y) = \sqrt{xy}, \qquad \mathcal{J}(x,y) = \frac{1}{e}\left(\frac{x^x}{y^y}\right)^{\frac{1}{x-y}}$$

for all $x, y > 0$, $x \neq y$. Hence, denoting by $\mathcal{P}_1$ and $\mathcal{P}_2$ the projective means, we get the following

Corollary 2. *Let $I \subset (0,\infty)$ be an interval, $M : I^2 \to I$ a strict mean and $p \in \mathbb{R}$. Then the function $\mathcal{L}^{[p]}_{M,\mathcal{A}} : I^2 \to I$ defined by*

$$\mathcal{L}^{[p]}_{\mathcal{A},M} := \begin{cases} L^{[p]}_{M,\mathcal{A}} & \text{if } p \neq -1, 0, \\ \mathcal{H} \circ (\mathcal{L} \circ (\mathcal{P}_1, M), \mathcal{L} \circ (M, \mathcal{P}_2)) & \text{if } p = -1, \\ \mathcal{G} \circ (\mathcal{J} \circ (\mathcal{P}_1, M), \mathcal{J} \circ (M, \mathcal{P}_2)) & \text{if } p = 0, \end{cases}$$

is a strict mean in I. The mean $\mathcal{L}^{[p]}_{\mathcal{A},M}$ is symmetric if M is symmetric. For each $x, y \in I$ the function

$$p \to \mathcal{L}^{[p]}_{\mathcal{A},M}(x, y) \quad \text{is continuous in } \mathbb{R}.$$

Moreover, if $M = \mathcal{A}$ then

$$\mathcal{L}^{[p]}_{\mathcal{A},M} = \mathcal{L}^{[p]}, \qquad p \in \mathbb{R},$$

where $\mathcal{L}^{[p]}$ is the logarithmic mean of order p.

Thus the family of means $\{\mathcal{L}^{[p]}_{\mathcal{A},M} : p \in \mathbb{R}\}$ generalizes $\{\mathcal{L}^{[p]} : p \in \mathbb{R}\}$, a well known family of logarithmic means of order p (cf. for instance [2], p. 385).

3. A generalization of Cauchy's Mean-Value Theorem

Theorem 2. *Let $I, J \subset \mathbb{R}$ be intervals. Suppose that $M : I^2 \to I$ is a strict mean in I and $K : J^2 \to J$ is a mean. If the functions $f, g : I \to \mathbb{R}$ are differentiable, $g'(x) \neq 0$ for all $x \in I$, and $\frac{f'}{g'}(I) \subset J$, then there exists a strict mean $C : I^2 \to I$ such that, for all $x, y \in I$, $x \neq y$,*

$$K\left(\frac{f(x) - f(M(x,y))}{g(x) - g(M(x,y))}, \frac{f(M(x,y)) - f(y)}{g(M(x,y)) - g(y)}\right) = \frac{f'(C(x,y))}{g'(C(x,y))}. \tag{11}$$

Moreover, if $\frac{f'}{g'}$ is one-to-one, then $C =: C^{[f,g]}_{M,K}$ is unique,

$$C^{[f,g]}_{M,K}(x,y) = \begin{cases} \left(\frac{f'}{g'}\right)^{-1}\left(K\left(\frac{f(x)-f(M(x,y))}{g(x)-g(M(x,y))}, \frac{f(M(x,y))-f(y)}{g(M(x,y))-g(y)}\right)\right) & \text{for } x \neq y, \\ x & \text{for } x = y, \end{cases} \tag{12}$$

and $C^{[f,g]}_{M,K}$ symmetric if M and K are symmetric.

Proof. Take $x, y \in I$, $x \neq y$. Without any loss of generality we can assume that $x < y$. Since M is strict, we have $x < M(x,y) < y$. By the Cauchy Mean-Value Theorem there are $r, s \in I$, $r = r(x,y)$, $s = s(x,y)$,

$$x < r < M(x,y), \qquad M(x,y) < s < y, \tag{13}$$

such that

$$\frac{f(x) - f(M(x,y))}{g(x) - g(M(x,y))} = \frac{f'(r)}{g'(r)}, \qquad \frac{f(M(x,y)) - f(y)}{g(M(x,y)) - g(y)} = \frac{f'(s)}{g'(s)}. \tag{14}$$

Since $g'(x) \neq 0$ for all $x \in I$, the Darboux property of derivative implies that g is continuous, strictly increasing in I and, consequently, the inverse function $g^{-1} : g(I) \to I$ is differentiable in the interval $g(I)$. The relation

$$\frac{f'}{g'} = (f \circ g^{-1})' \circ g,$$

the continuity of g and the Darboux property of the derivative $\left(f \circ g^{-1}\right)'$ imply that the function $\frac{f'}{g'}$ has the Darboux property. Since

$$\min b\left(\frac{f'(r)}{g'(r)}, \frac{f'(s)}{g'(s)}\right) \leq K\left(\frac{f'(r)}{g'(r)}, \frac{f'(s)}{g'(s)}\right) \leq \max\left(\frac{f'(r)}{g'(r)}, \frac{f'(s)}{g'(s)}\right),$$

it follows that there exists $t = t(r,s) \in I$,

$$r \leq t \leq s, \tag{15}$$

such that

$$K\left(\frac{f'(r)}{g'(r)}, \frac{f'(s)}{g'(s)}\right) = \frac{f'(t)}{g'(t)}. \tag{16}$$

From (13) and (15) we have

$$x = \min(x,y) < t(r(x,y), s(x,y)) < \max(x,y) = y.$$

Thus, putting

$$C(x,y) := t(r(x,y), s(x,y)) \text{ for } x \neq y, \quad \text{and} \quad C(x,y) := x \text{ for } x = y,$$

we get a mean C which, according to (14) and (16), satisfies (11).

If $\frac{f'}{g'}$ is one-to-one then formula (12) is a consequence of (11). The remaining statement is obvious. ■

Remark 2. Let the functions f and g satisfy the conditions of Theorem 2 and let $w \in (0,1)$ be fixed. Denote by $\mathcal{A}_w$ the weighted arithmetic mean

$$\mathcal{A}_w(x,y) := wx + (1-w)y, \quad x, y \in \mathbb{R}.$$

Since g is continuous and strictly monotonic, the weighted quasi-arithmetic mean $M_w^{[g]} : I^2 \to I$ given by

$$M_w^{[g]}(x,y) := g^{-1}(wg(x) + (1-w)g(y)), \quad x, y \in I,$$

is well defined. It easy to check that for $M := M_w^{[g]}$ and $K =: \mathcal{A}_{1-w}$ formula (11) in Theorem 2 reduces to (3) that is to the classical Cauchy's theorem, and for $w = \frac{1}{2}$ we get relation (4) mentioned in the Introduction.

Thus we have the following

Corollary 3. *Suppose that $M : I^2 \to I$ is a strict mean in an interval I and $w \in (0,1)$. If $f, g : I \to \mathbb{R}$ are differentiable, $g'(x) \neq 0$ for all $x \in I$,* *and* $\frac{f'}{g'}$ *is one to one, then*

$$C^{[f,g]}_{M_w^{[g]},\mathcal{A}_{1-w}} = C^{[f,g]}.$$

Taking here $I=(0,\infty)$, $w\in(0,1)$, $f(x)=x^p$, $g(x)=x^q$ where $p,q\in\mathbb{R}$, $pq(p-q)\neq 0$, we get

$$C^{[p,q]}(x,y) := C^{[f,g]}(x,y) = \left(\frac{q}{p}\left(\frac{x^p-y^p}{x^q-y^q}\right)\right)^{1/(p-q)}, \qquad x,y>0.$$

The family of Stolarsky means $\left\{\mathcal{E}^{[p,q]} : p,q\in\mathbb{R}\right\}$ can be defined as follows ([3], cf. also [1], p. 385):

$$\mathcal{E}^{[p,q]} := \begin{cases} C^{[p,q]} & \text{if } pq(p-q)\neq 0,\\ \mathcal{L}\circ(\mathcal{P}_1^p,\mathcal{P}_2^p))^{1/p} & \text{if } p\neq 0,\ q=0,\\ \mathcal{L}\circ(\mathcal{P}_1^q,\mathcal{P}_2^q))^{1/q} & \text{if } p=0,\ q\neq 0,\\ \mathcal{J}\circ(\mathcal{P}_1{}^q,\mathcal{P}_2^q)^{1/q} & \text{if } p=q\neq 0,\\ \mathcal{G} & \text{if } p=q=0. \end{cases}$$

(Here $\mathcal{L}$ is the logarithmic mean, $\mathcal{J}$ is the identric mean, $\mathcal{P}_1$ and $\mathcal{P}_2$ the projective means, and $\mathcal{G}$ is the geometric mean.) Note that

$$\mathcal{E}^{[p,q]} = \mathcal{E}^{[q,p]} \qquad \text{for all } p,q\in\mathbb{R}.$$

Example 1. Taking $I\subset(0,\infty)$, $K:=\mathcal{A}$, $f(x)=x^p$, $g(x)=x^q$ where $pq(p-q)\neq 0$ in Theorem 2, for $x\neq y$, we obtain

$$\begin{aligned} C^{[p,q]}_{M,\mathcal{A}}(x,y) &:= C^{[f,g]}_{M,\mathcal{A}}(x,y)\\ &= \left(\frac{q}{2p}\left(\frac{x^p-[M(x,y)]^p}{x^q-[M(x,y)]^q}+\frac{[M(x,y)]^p-y^p}{[M(x,y)]^q-y^q}\right)\right)^{1/(p-q)} \end{aligned}$$

Since, for all $x,y\in I$, $x\neq y$,

$$\begin{aligned} \lim_{q\to 0} C^{[p,q]}_{M,\mathcal{A}}(x,y) &= \left(\frac{1}{2}\left(\frac{x^p-M(x,y)^p}{\log x^p-\log M(x,y)^p}+\frac{M(x,y)^p-y^p}{\log M(x,y)^p-\log y^p}\right)\right)^{1/p}\\ &= (\mathcal{A}(\mathcal{L}(x^p,[M(x,y)]^p),\mathcal{L}([M(x,y)]^p,y^p)))^{1/p},\\ \lim_{p\to 0} C^{[p,q]}_{M,\mathcal{A}}(x,y) &= \left(\frac{1}{2}\left(\frac{\log x^q-\log M(x,y)^q}{x^q-M(x,y)^q}+\frac{\log M(x,y)^q-\log y^q}{M(x,y)^q-y^q}\right)\right)^{-1/q}\\ &= (\mathcal{H}(\mathcal{L}(x^q,[M(x,y)]^q),\mathcal{L}([M(x,y)]^q,y^q)))^{1/q}, \end{aligned}$$

$$\lim_{p\to q} C_{M,\mathcal{A}}^{[p,q]}(x,y)$$
$$= e^{-1/q}\left[\left(\frac{x^{x^q}}{M(x,y)^{M(x,y)^q}}\right)^{1/(x^q-M(x,y)^q)}\left(\frac{M(x,y)^{M(x,y)^q}}{y^{y^q}}\right)^{1/(M(x,y)^q-y^q)}\right]^{1/2}$$
$$= \mathcal{G}[\mathcal{E}^{q,q}(x,M(x,y)),\mathcal{E}^{q,q}(M(x,y),y)],$$

and

$$\lim_{q\to 0} C_{M,\mathcal{A}}^{[q,0]}(x,y) = \lim_{q\to 0} C_{M,\mathcal{A}}^{[0,q]}(x,y)$$
$$= (xyM(x,y)^2)^{1/4} = \sqrt{\sqrt{xy}M(x,y)} = \mathcal{G}\circ(\mathcal{G},M)(x,y),$$

we obtain the following

Corollary 4. *Let $I\subset(0,\infty)$ be an interval, $M: I^2\to I$ a strict mean and $p,q\in\mathbb{R}$. Then the function $\mathcal{C}_{M,\mathcal{A}}^{[p,q]}: I^2\to I$ defined by*

$$\mathcal{C}_{\mathcal{A},M}^{[p,q]} := \begin{cases} C_{M,\mathcal{A}}^{[p,q]} & \text{if } pq(p-q)\neq 0,\\ (\mathcal{A}\circ(\mathcal{L}\circ(\mathcal{P}_1^p,M^p),\mathcal{L}\circ(M^p,\mathcal{P}_2^p)))^{1/p} & \text{if } p\neq 0,\ q=0,\\ (\mathcal{H}\circ(\mathcal{L}\circ(\mathcal{P}_1^q,M^q),\mathcal{L}\circ(M^q,\mathcal{P}_2^q)))^{1/q} & \text{if } p=0,\ q\neq 0,\\ \mathcal{G}\circ(\mathcal{E}^{q,q}\circ(\mathcal{P}_1,M),\mathcal{E}^{q,q}\circ(M,\mathcal{P}_2)) & \text{if } p=q\neq 0,\\ \mathcal{G}\circ(\mathcal{G},M) & \text{if } p=q=0,\end{cases}$$

is a strict mean in I and $\mathcal{C}_{\mathcal{A},M}^{[p,q]}$ is symmetric if M is symmetric. Moreover, for all $p,q\in\mathbb{R}$, $p\neq q$,

$$\mathcal{C}_{\mathcal{A},M}^{[p,q]}\neq\mathcal{C}_{\mathcal{A},M}^{[q,p]}.$$

Only the last statement needs an argument. To show it suppose, for contrary, that $\mathcal{C}_{\mathcal{A},M}^{[p,q]}=\mathcal{C}_{\mathcal{A},M}^{[q,p]}$ for a strict mean M and some $p\neq q$. First consider the case $pq\neq 0$. From the definition of $\mathcal{C}_{\mathcal{A},M}^{[p,q]}$, after obvious calculations, we hence get, for all $x,y\in I$, $x\neq y$,

$$\mathcal{A}\left(\frac{x^p-M^p}{x^q-M^q},\frac{M^p-y^p}{M^q-y^q}\right)=\mathcal{H}\left(\frac{x^p-M^p}{x^q-M^q},\frac{M^p-y^p}{M^q-y^q}\right)\quad\text{where}\quad M=M(x,y).$$

Since the values of the arithmetic and harmonic means are equal only on the diagonal, we hence get the equation

$$\frac{x^p-M^p}{x^q-M^q}=\frac{M^p-y^p}{M^q-y^q},\qquad x,y\in I,\ x\neq y,$$

which implies that

$$(x^p-M^p)(M^q-y^q)-(x^q-M^q)(M^p-y^p)=0,\qquad x,y\in I.$$

Let us fix $x,y\in I$. Without any loss of generality we can assume that $x<y$.

Define a function $\Phi : [x, y] \to \mathbb{R}$ by the formula

$$\Phi(m) := (x^p - m^p)(m^q - y^q) - (x^q - m^q)(m^p - y^p), \qquad m \in [x, y].$$

Note that $\Phi(x) = \Phi(y) = 0$. Since $\Phi'(m) = 0$ iff

$$m = \mathcal{E}^{[p,q]}(x, y),$$

there is no $M \in (x, y)$ satisfying the above equation. In the case when $pq = 0$ we can argue similarly.

References

[1] J. M. Borwein, P. B. Borwein, *Pi and the AGM, a Study in Analytic Number Theory and Computational Complexity*, John Wiley & Sons Inc., New York, 1987.

[2] P. S. Bullen, *Handbook of Means and their Inequalities*, Mathematics and Its Applications, Vol. 560, Kluwer Academic Publishers, Dordrecht-Boston-London, 2003.

[3] W. Rudin, *Principles of Mathematical Analysis*, Third Edition, McGraw-Hill, Inc., New York, 1976.

[4] K. B. Stolarsky, *Generalizations of the logarithmic mean*, Math. Mag. 48 (1975), 87–92.

[5] K. B. Stolarsky, *The power and generalized logarithmic mean*, Amer. Math. Monthly 87 (1980), 545–548.

FACULTY OF MATHEMATICS COMPUTER SCIENCE AND ECONOMETRY
UNIVERSITY OF ZIELONA GÓRA
Podgórna 50
65-246 ZIELONA GÓRA, POLAND
and
INSTITUTE OF MATHEMATICS
SILESIAN UNIVERSITY
Bankowa 14
42-007 KATOWICE, POLAND
E-mail: J.Matkowski@wmie.uz.zgora.pl

Received