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A composite functional equation and invariant curves

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Abstract. The continuous solutions of a composite functional equation are characterized. An application to the problem of invariant curves is presented.

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In the theory of equations of a single variable one of the most important equations has the form

$$\psi[f(x)] = g(x, \psi(x)) \tag{1}$$

where f and g are given functions (cf. Kuczma [8]). For instance the Schröder equation

$$\psi[f(x)] = s\psi(x)$$

and the Abel equation

$$\psi[f(x)] = s + \psi(x),$$

both playing a fundamental role in iteration theory, are special cases of equation (1). This equation belongs to the so-called class of iterative functional equations mainly because the methods involving the iterative procedures are applied to find its solutions.

Note that if ψ is an invertible solution of (1) then $\varphi := \psi^{-1}$ satisfies the functional equation

$$f[\varphi(x)] = \varphi(g(\varphi(x), x)), \tag{2}$$

"conjugate" to equation (1), in which the composition of the unknown function appears.

A special case of equation (2),

$$\varphi(x\varphi(x)) = [\varphi(x)]^2 \quad \text{for } x \in [0, \infty), \tag{3}$$

which appeared in a division model of population, was considered by Dhombres [1], who characterized all continuous solutions of this equation. The continuous solutions of the equation

$$\varphi(x[\varphi(x)]^p) = [\varphi(x)]^{p+1},$$

where $p \neq 0$. are determined in [4]. The continuous solutions of the more general equation

$$\varphi(xG(\varphi(x))) = \varphi(x)G(\varphi(x)) \tag{4}$$

where G is a given continuous function, generalizing equation (3), have been established in [10].

Let us mention that increasing solutions, strictly increasing solutions and big-graph solutions of Dhombres equation (3) and its generalization were considered by Kahlig and Smital in [5–7].

The functional equations with compositions of the unknown functions, called "composite functional equations" are, in general, more difficult to examine. In particular the "iterative methods" which are useful in solving functional equation

$$F(x, \varphi(x), \varphi(f(x))) = 0$$

are not applicable.

Let $p > 0$ and $r > 0$ be fixed. In the present paper we determine all continuous functions $\varphi : (0, \infty) \to (0, \infty)$ satisfying the composite functional equation

$$\varphi[x^r G(\varphi(x))] = x^{p(r-1)}\varphi(x)[G(\varphi(x)]^p, \quad x > 0,$$

where the given function $G : (0, \infty) \to (0, \infty)$ is continuous and strictly increasing.

We apply the obtained result to the problem of invariant curves.

1 Main result

We begin with the following

Lemma 1.1. *Let $p, r \in \mathbb{R}$, $p \neq 0$, be fixed. Suppose that $G : (0, \infty) \to (0, \infty)$ is continuous and strictly increasing, and the function*

$$(0, \infty) \ni x \longmapsto x^r[G(x)]^p \tag{5}$$

is one-to-one. If $\varphi : (0, \infty) \to (0, \infty)$ satisfies the functional equation

$$\varphi[x^r G(\varphi(x))] = x^{p(r-1)}\varphi(x)[G(\varphi(x)]^p, \quad x > 0, \tag{6}$$

then the function $M : (0, \infty) \to (0, \infty)$ defined by

$$M(x) := x^r G[\varphi(x)], \quad x > 0, \tag{7}$$

is one-to-one.

Proof. Define $D : (0, \infty) \to (0, \infty)$ by

$$D(x) := \frac{\varphi(x)}{x^p}, \quad x > 0, \tag{8}$$

and note that we can write equation (6) in the form

$$D(M(x)) = D(x), \quad x > 0. \tag{9}$$

If $M(x_1) = M(x_2)$ for some $x_1, x_2 > 0$, then, obviously, $D(x_1) = D(x_2)$ and, consequently,

$$[D(x_1)]^r [M(x_1)]^p = [D(x_2)]^r [M(x_2)]^p.$$

In view of the definitions of M and D given in (7) and (8) it follows that

$$[\varphi(x_1)]^r [G(\varphi(x_1))]^p = [\varphi(x_2)]^r [G(\varphi(x_2))]^p.$$

Since the function (5) is one-to-one, we conclude that $\varphi(x_1) = \varphi(x_2)$. Now (8) and the equality $D(x_1) = D(x_2)$ imply that $x_1 = x_2$. This completes the proof. □

Theorem 1.2. *Let $p > 0$ and $r > 1$ be fixed. Suppose that $G : (0, \infty) \to (0, \infty)$ is continuous and strictly increasing. Then a continuous function $\varphi : (0, \infty) \to (0, \infty)$ satisfies equation (6):*

$$\varphi[x^r G(\varphi(x))] = x^{p(r-1)} \varphi(x) [G(\varphi(x))]^p, \quad x > 0,$$

if, and only if, there exist $a, b \in [0, \infty]$, $a \le b$ and $a \ne b$ if $a = 0$ or $b = \infty$, such that

$$\varphi(x) = \begin{cases} \frac{G^{-1}(a^{1-r})}{a^p} x^p, & 0 < x \le a, \\ G^{-1}(x^{1-r}), & a < x \le b, \\ \frac{G^{-1}(b^{1-r})}{b^p} x^p, & x > b, \end{cases} \tag{10}$$

where G^{-1} denotes the inverse function of G.

Proof. Suppose that $\varphi : (0, \infty) \to (0, \infty)$ is continuous and satisfies equation (6). Define the functions $M, D : (0, \infty) \to (0, \infty)$ by (7) and (8). Then equation (6) becomes

$$D(M(x)) = D(x), \quad x > 0.$$

The function M is continuous and, by Lemma 1, M is one-to-one. Thus M is strictly monotonic.

First consider the case when M is strictly increasing. Put

$$\mathrm{Fix}(M) := \{x > 0 : M(x) = x\} \tag{11}$$

and note that

$$\mathrm{Fix}(M) := \{x > 0 : \varphi(x) = G^{-1}(x^{1-r})\}. \tag{12}$$

We shall prove that $\mathrm{Fix}(M)$ is a nonempty, closed subinterval of $(0, \infty)$. For an indirect argument suppose first that $\mathrm{Fix}(M)$ is empty. The continuity of M implies that either $M(x) < x$ for all $x > 0$ or $M(x) > x$ for all $x > 0$. Assume that $M(x) < x$ for all $x > 0$. Then, from the definition (7) of M and the increasing monotonicity of M, we would have

$$\varphi(x) < G^{-1}(x^{1-r}), \quad x > 0. \tag{13}$$

Notice that

$$(0, \infty) = \bigcup_{n=-\infty}^{\infty} [M^{n+1}(1), M^n(1)].$$

Since M and D are continuous and M is strictly increasing, we hence get

$$D((0, \infty)) = \bigcup_{n=-\infty}^{\infty} D([M^{n+1}(1), M^n(1)]).$$

Since, from equation (9), $D \circ M = D$, we hence get

$$D((0, \infty)) = D([M(1), 1]),$$

whence, setting

$$c := \inf D((0, \infty)), \quad C := \sup D((0, \infty)),$$

we obtain

$$0 < c \le D(x) \le C < \infty \quad \text{for all } x > 0,$$

that is

$$0 < cx^p \le \varphi(x) \le Cx^p < \infty \quad \text{for all } x > 0.$$

Since $p > 0$, the graph of φ would be located between two graphs of increasing power functions. This is a contradiction. Indeed, according to inequality (13), the graph of φ is situated below the graph of the function

$$(0, \infty) \ni x \longmapsto G^{-1}(x^{1-r}),$$

that for $r > 1$ is decreasing. If $M(x) > x$ for all $x > 0$ we can argue similarly. This proves that $\mathrm{Fix}(M) \neq \emptyset$.

Now we shall prove that $\mathrm{Fix}(M)$ is an interval. Assume, for the contrary, that it is not the case. Since $\mathrm{Fix}(M)$ is closed, there exist nonnegative $c, d \in \mathrm{Fix}(M)$,

$c < d$ such that $(c:d) \cap \mathrm{Fix}(M) = \emptyset$. It follows that either $M(x) < x$ for all $x \in (c:d)$ or $M(x) > x$ for all $x \in (c:d)$. In the first case, for every $x \in (c:d)$, the sequence of iterates $(M^n(x))_{n\in\mathbb{N}}$ is decreasing, and

$$\lim_{n\to\infty} M^n(x) = c, \quad x \in [c,d).$$

From equation (9) we get

$$D(x) = D(M^n(x)), \quad x \in (0,\infty),\ n \in \mathbb{N}. \tag{14}$$

Hence, by the continuity of D, letting $n \to \infty$, we obtain

$$D(x) = D(c), \quad x \in [c,d),$$

whence, again by the continuity of D,

$$D(d) = D(c),$$

which means that

$$\frac{\varphi(c)}{c^p} = \frac{\varphi(d)}{d^p}.$$

Since $c < d$, $p > 0$, and the increasing monotonicity of the power function $(0,\infty) \ni x \mapsto x^p$ implies that $c^p < d^p$, we hence get

$$\varphi(c) < \varphi(d).$$

On the other hand we have $M(c) = c$, $M(d) = d$, whence

$$G(\varphi(c)) = c^{1-r}, \quad G(\varphi(d)) = d^{1-r}.$$

The increasing monotonicity of G implies that

$$c^{1-r} = G(\varphi(c)) < G(\varphi(d)) = d^{1-r}.$$

Since, for $r > 1$, the power function $(0,\infty) \ni x \mapsto x^{1-r}$ is decreasing, we conclude that $d < c$. This contradiction proves that $\mathrm{Fix}(M)$ is an interval.

As the proof of the fact that $\mathrm{Fix}(M)$ is an interval also in the second case when $M(x) > x$ for all $x \in (c,d)$ is analogous, we omit it.

Now put

$$a := \inf \mathrm{Fix}(M), \quad b := \sup \mathrm{Fix}(M).$$

According to what we have already proved,

$$0 \le a < +\infty, \quad a \le b, \quad 0 < b \le +\infty.$$

By the continuity of M we have $\mathrm{Fix}(M) = [a, b] \cap (0, \infty)$. Hence, taking into account (12), we get

$$\varphi(x) = G^{-1}(x^{1-r}), \quad x \in [a, b] \cap (0, \infty).$$

If $b < \infty$ then we have either $M(x) < x$ for all $x > b$, or $M(x) > x$ for all $x > b$. If the first case occurs then, for all $x > b$,

$$\lim_{n\to\infty} M^n(x) = b,$$

whence, by (14) and the continuity of D,

$$D(x) = D(b), \quad x > b.$$

Assume that $M(x) > x$ for all $x > b$. Then, for every $x > b$,

$$\lim_{n\to\infty} M^{-n}(x) = b$$

and, for the same reason,

$$D(x) = D(b), \quad x > b.$$

Now the definition of D and the relation $b \in \mathrm{Fix}(M)$ imply that

$$\varphi(x) = \frac{G^{-1}(b^{1-r})}{b^p} x^p, \quad x > b.$$

If $0 < a < \infty$, in a similar way we show that

$$\varphi(x) = \frac{G^{-1}(a^{1-r})}{a^p} x^p, \quad 0 < x < a.$$

Thus we have proved that every continuous solution $\varphi : (0, \infty) \to (0, \infty)$ of equation (6) must be of the form (10). Since it is easy to verify that the function (10) satisfies equation (6), it completes the proof in the case when M is increasing.

Now assume that M is strictly decreasing. Then, by the definition (7) of M we would have

$$\varphi(x) = G^{-1}\left(\frac{M(x)}{x^r}\right), \quad x > 0.$$

Because $r > 1$ and G is strictly increasing, the function φ, being the superposition of increasing function G^{-1} and the decreasing function $(0, \infty) \ni x \mapsto \frac{M(x)}{x^r}$, would be decreasing in $(0, \infty)$. Then the left-hand side of (6), as the superposition of two decreasing functions φ and M, would be strictly increasing. On the other hand, the right-hand side of equation (6) is of the form

$$\varphi(x)x^{-p}[M(x)]^p, \quad x > 0.$$

Since M is strictly decreasing, the superposition of M and the power function $(0,\infty) \ni x \mapsto x^p$ with $p > 0$ is decreasing. The functions φ and the power function $(0,\infty) \ni x \mapsto x^{-p}$ are also decreasing. Thus the right-hand side of equation (6) is strictly decreasing. The obtained contradiction proves that this case cannot happen. The proof is completed. □

Remark 1.3. In the case when $a = 0$ and $b = +\infty$ the continuous solution (10) has the form

$$\varphi(x) = G^{-1}(x^{1-r}), \quad x > 0.$$

Theorem 1.4. *Let the numbers $p \in \mathbb{R}$, $p > 0$, $r \in (0,1)$ be fixed. Suppose that $G : (0,\infty) \mapsto (0,\infty)$ is continuous and strictly increasing and such that*

$$\inf\left\{\frac{G(x)}{x^{(1-r)/p}} : x > 0\right\} = 0. \tag{15}$$

$$\sup\left\{\frac{G(x)}{x^{(1-r)/p}} : x > 0\right\} = \infty. \tag{16}$$

Then a continuous function $\varphi : (0,\infty) \mapsto (0,\infty)$ satisfies equation (6):

$$\varphi[x^r G(\varphi(x))] = x^{p(r-1)}\varphi(x)[G(\varphi(x))]^p, \quad x > 0,$$

if, and only if, there exists a set $K \subset \mathbb{N}$, such that either $K = \emptyset$ or $K = \{1, 2, \ldots, n\}$ for some $n \in \mathbb{N}$ or $K = \mathbb{N}$, and a family of pairwise disjoint open intervals $\{(a_k, b_k) \subset (0,\infty) : a_k \le b_k, k \in K\}$ such that

$$\varphi(x) = \frac{G^{-1}(a_k^{1-r})}{a_k^p}x^p = \frac{G^{-1}(b_k^{1-r})}{b_k^p}x^p, \quad a_k < x < b_k, \quad k \in K. \tag{17}$$

and

$$\varphi(x) = G^{-1}(x^{1-r}), \quad x \in (0,\infty) \setminus \bigcup_{k\in K}(a_k, b_k). \tag{18}$$

Proof. Suppose that a continuous $\varphi : (0,\infty) \mapsto (0,\infty)$ satisfies equation (6). Defining the functions $M, D : (0,\infty) \mapsto (0,\infty)$ by formulas (7) and (8) we can write equation (6) in form (9). Since the function (5) is one-to-one $(0,\infty) \mapsto (0,\infty)$, Lemma 1 implies that M is one-to-one as well. By the continuity of G

and φ the function M is continuous. Thus M is strictly monotonic. Denote by $\mathrm{Fix}(M)$ the set of fixed points of M. It is easy to see that

$$\mathrm{Fix}(M) = \{x > 0 : \varphi(x) = G^{-1}(x^{1-r})\}.$$

We shall show that $\mathrm{Fix}(M) \neq \emptyset$. If M is decreasing then, clearly, $\mathrm{Fix}(M) \neq \emptyset$. Suppose that M is strictly increasing and, for the indirect argument, assume that $\mathrm{Fix}(M) = \emptyset$. The continuity and strict monotonicity of M implies that either $M(x) < x$ for all $x > 0$ or $M(x) > x$ for all $x > 0$. In the first case, from the definition of M and the monotonicity of G we would have

$$\varphi(x) < G^{-1}(x^{1-r}), \quad x > 0. \tag{19}$$

On the other hand, the continuity of M and D, the monotonicity of M, and equation (9), imply that

$$D((0, \infty)) = D([M(1), 1]).$$

Hence, setting

$$c := \inf D([M(1), 1]), \quad C = \sup D([M(1), 1]),$$

we get

$$0 < c \leq D(x) \leq C < \infty, \quad x > 0.$$

From the definition of D we obtain the inequality

$$cx^p \leq \varphi(x) \leq Cx^p, \quad x > 0, \tag{20}$$

whence, by (19),

$$0 < cx^p < G^{-1}(x^{1-r}), \quad x > 0.$$

Replacing here x by $(\frac{x}{c})^{1/p}$ and setting $c_1 := c^{(r-1)/p}$ we obtain

$$\frac{G(x)}{x^{(1-r)/p}} < c_1, \quad x > 0,$$

which contradicts the assumption (16).

In the case when $M(x) > x$ for all $x > 0$, from the definition of M and the strict monotonicity of G we would have

$$\varphi(x) > G^{-1}(x^{1-r}), \quad x > 0,$$

whence, by (20),

$$0 < G^{-1}(x^{1-r}) < Cx^p, \quad x > 0.$$

Replacing here x by $(\frac{x}{C})^{1/p}$ and setting $C_1 := C^{(r-1)/p}$ we obtain,

$$0 < C_1 < \frac{G(x)}{x^{(1-r)/p}}, \quad x > 0,$$

which contradicts the assumption (15). Thus $\mathrm{Fix}(M)$ is nonempty. Since $\mathrm{Fix}(M)$ is closed, the set $(0, \infty) \setminus \mathrm{Fix}(M)$ is a sum of at most countable family of pairwise disjoint open intervals. Consequently,

$$(0, \infty) \setminus \mathrm{Fix}(M) = \bigcup_{k \in K} (a_k, b_k)$$

where either $K = \emptyset$ or $K = \{1, 2, \ldots, n\}$ for some $n \in \mathbb{N}$ or $K = \mathbb{N}$. Hence

$$\mathrm{Fix}(M) = (0, \infty) \setminus \bigcup_{k \in K} (a_k, b_k),$$

and, from (12),

$$\varphi(x) = G^{-1}(x^{1-r}), \quad x \in (0, \infty) \setminus \bigcup_{k \in K} (a_k, b_k).$$

Assume that $K \neq \emptyset$. If $x \notin \mathrm{Fix}(M)$ then there exists exactly one $k \in K$ such that $x \in (a_k, b_k)$. The continuity and monotonicity of M imply that either $M(x) < x$ for all $x \in (a_k, b_k)$ or $M(x) > x$ for all $x \in (a_k, b_k)$. In the first case, making use of the continuity of D, and (9) we obtain, for all $x \in (a_k, b_k)$,

$$D(x) = D(a_k).$$

If $M(x) > x$ for all $x \in (a_k, b_k)$, then, for all $x \in (a_k, b_k)$,

$$D(x) = D(b_k).$$

The continuity of D implies that $D(a_k) = D(b_k)$. Hence, in both cases, we get

$$\varphi(x) = \frac{G^{-1}(a_k^{1-r})}{a_k^p} x^p = \frac{G^{-1}(b_k^{1-r})}{b_k^p} x^p, \quad x \in (a_k, b_k).$$

It is not difficult to show that the functions (17) and (18) satisfy equation (6).

Now a similar reasoning as in the proof of previous result shows that M cannot be decreasing. This completes the proof. □

Remark 1.5. If $K = \emptyset$ then $\mathrm{Fix}(M) = (0, \infty)$ and the continuous solution of equation (6) is of the form $\varphi(x) = G^{-1}(x^{1-r})$ for all $x \in (0, \infty)$.

Remark 1.6. Note that composite functional equation (6) is "conjugate" to iterative functional equation (1) in the sense discussed in the Introduction, only in the case when $p = r = 1$. Then equation (6) becomes equation (4):

$$\varphi(x)G(\varphi(x)) = \varphi(xG(\varphi(x)), \quad x > 0.$$

examined in [10].

2 Application to invariant curves

We begin this section with the following

Definition 2.1. Let $D \subset \mathbb{R}^2$ be an open and connected set and $T : D \to D$, $T = (f, g)$ be a continuous selfmapping of D. Assume that $I \subset \mathbb{R}$ is an interval and $\varphi : I \to \mathbb{R}$ a continuous function. The graph $W_\varphi := \{(x, \varphi(x)) : x \in I\}$ of the function φ is called an invariant curve under the mapping T if

- $(x, \varphi(x)) \in D$ for all $x \in I$;
- for every $x \in I$ there is $x' \in I$ such that $T((x, \varphi(x))) = (x', \varphi(x'))$.

Remark 2.2. The graph of the function $\varphi : I \to \mathbb{R}$ is an invariant curve under the transform $T = (f, g)$, if the function φ satisfies the composite functional equation

$$\varphi(f(x, \varphi(x)) = g(x, \varphi(x)), \quad x \in I.$$

Let us note the following easy to verify assertion.

Remark 2.3. Let $p, r \in \mathbb{R}$ be fixed. For a given continuous function $G : (0, \infty) \mapsto (0, \infty)$ define the functions $f, g : (0, \infty)^2 \to (0, \infty)$ by

$$f(x, y) := x^r G(y), \quad g(x, y) := x^{p(r-1)}[G(y)]^p.$$

Then the graph of the function $\varphi : (0, \infty) \to (0, \infty)$ is an invariant curve under the mapping $T = (f, g)$ iff φ satisfies functional equation (6).

Hence, applying Theorems 1 and 2 of the previous section we obtain the following:

Corollary 2.4. *Let $p, r \in \mathbb{R}$, $p > 0$, $r > 1$, be fixed. Suppose that a function $G : (0, \infty) \mapsto (0, \infty)$ is continuous and strictly increasing. Then the graph* *of a continuous function $\varphi : (0,\infty) \mapsto (0,\infty)$ is an invariant curve under the transform $T : (0,\infty)^2 \to (0,\infty)^2$ given by*

$$T(x,y) := (x^r G(y), x^{p(r-1)}[G(y)]^p), \quad x,y > 0, \tag{21}$$

if, and only if, there exist $a, b \in [0,\infty]$, $a \le b$ and $a \ne b$ if $a = 0$ or $b = \infty$, such that

$$\varphi(x) = \begin{cases} \frac{G^{-1}(a^{1-r})}{a^p} x^p, & 0 < x \le a, \\ G^{-1}(x^{1-r}), & a < x \le b, \\ \frac{G^{-1}(b^{1-r})}{b^p} x^p, & x > b, \end{cases}$$

where G^{-1} denotes the inverse function of G.

Corollary 2.5. *Let $p, r \in \mathbb{R}$, $p > 0$, $r \in (0,1)$, be fixed. Suppose that a function $G : (0,\infty) \mapsto (0,\infty)$ is continuous, strictly increasing and such that*

$$\inf\left\{\frac{G(x)}{x^{(1-r)/p}} : x > 0\right\} = 0, \quad \sup\left\{\frac{G(x)}{x^{(1-r)/p}} : x > 0\right\} = \infty.$$

Then the graph of a continuous function $\varphi : (0,\infty) \mapsto (0,\infty)$ is an invariant curve under the transform $T : (0,\infty)^2 \to (0,\infty)^2$ given by (21) if, and only if, there exists a set $K \subset \mathbb{N}$, such that either $K = \emptyset$ or $K = \{1, 2, \dots, n\}$ for some $n \in \mathbb{N}$ or $K = \mathbb{N}$, and a family of pairwise disjoint open intervals $\{(a_k, b_k) \subset (0,\infty) : a_k \le b_k, k \in K\}$ such that

$$\varphi(x) = \frac{G^{-1}(a_k^{1-r})}{a_k^p} x^p = \frac{G^{-1}(b_k^{1-r})}{b_k^p} x^p, \quad a_k < x < b_k, \ k \in K,$$

and

$$\varphi(x) = G^{-1}(x^{1-r}), \quad x \in (0,\infty) \setminus \bigcup_{k\in K} (a_k, b_k).$$

Applying Remark 5 and the main result of [10] where the case $p = r = 1$ is considered we obtain:

Corollary 2.6. *Suppose that a function $G : (0,\infty) \mapsto (0,\infty)$ is continuous, strictly increasing and such that $1 \in G((0,\infty))$. Then the graph of a continuous function $\varphi : (0,\infty) \mapsto (0,\infty)$ is an invariant curve under the transform $T : (0,\infty)^2 \to (0,\infty)^2$ given by*

$$T(x,y) := (xG(y), G(y)), \quad x,y > 0.$$

if, and only if, there exist $a, b \in [0, \infty]$, $a \le b$ *and* $a \ne b$ *if* $a = 0$ *or* $b = \infty$, *such that*

$$\varphi(x) = \begin{cases} \frac{G^{-1}(1)}{a}x, & 0 < x \le a, \\ G^{-1}(1), & a < x \le b, \\ \frac{G^{-1}(1)}{b}x, & x > b, \end{cases}$$

where G^{-1} *denotes the inverse function of* G.

Recall that, under some additional regularity assumptions, the existence and uniqueness of a local invariant curve in a neighbourhood of a fixed point of a map was proved by Hadamard [2], Lattés [9], Montel [11] (cf. also Kuczma [8] for some other references).

The continuous solutions of the functional equation $\varphi(x + \varphi(x)) = P(\varphi(x))$, coming from a global-type problem of invariant curves, were considered by Jarczyk [3].

Bibliography

[1] J. Dhombres, *Some Aspects of Functional Equations*, Chulalongkorn University, Bangkok, 1979.

[2] J. Hadamard, *Sur l'itération et solutions asymptotiques des équations differentielles*, Bull. Soc. Math. France **29** (1901), 224–228.

[3] W. Jarczyk, *On continuous solutions of the equation of invariant curves*, C. Carathéodory: An International Tribute, Vol. I, edited by Th. M. Rassias, World Scientific Publ. Co. (1991), 527–542.

[4] P. Kahlig, A. Matkowska and J. Matkowski, *On a class of composite functional equations in a single variable*, Aequationes Math. **52** (1996), no. 3, 260–283.

[5] P. Kahlig and J. Smítal, *On the solutions of a functional equation of Dhombres*, Results Math. **27** (1995), no. 3-4, 362–367.

[6] P. Kahlig and J. Smítal, *On a generalized Dhombres functional equation*, Aequationes Math. **62** (2001), no. 18–29,

[7] P. Kahlig and J. Smítal, *On a generalized Dhombres functional equation. II*, Math. Bohem. **127** (2002), no. 4, 547–555.

[8] M. Kuczma. In Kahlig, Smítal, J. Results Math. 27 (1995), no. 3-4, 362–367

[9] S. Lattés, *Sur les équations fonctionnelles qui définissent une courbe ou une surface invariante par une transformation*, Ann. di Mat. (3) **25** (1906), 1–137.

[10] J. Matkowski and J. Okrzesik, *On a composite functional equation*, Demonstratio Math. **36** (2003), 653–658.

[11] P. Montel, *Leçons sur les récurrences et leursapplications*, Gauthier-Villars, Paris, 1957.

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