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Aequationes Mathematicae

Generalized weighted quasi-arithmetic means

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Abstract. Under some natural assumptions on real functions f and g defined on a real interval I, we show that a two variable function $M_{f,g} : I^2 \to I$ defined by

$$M_{f,g}(x, y) = (f + g)^{-1}(f(x) + g(y))$$

is a generalization of the quasi-arithmetic mean. Necessary and sufficient conditions for: symmetry, quasi-arithmeticity, weighted quasi-arithmeticity, homogeneity of $M_{f,g}$, as well as equality of two such means are presented.

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1. Introduction

Let $I \subseteq \mathbb{R}$ be an interval. A function $M : I^2 \to \mathbb{R}$ is called a *mean* in I^2 if

$$\min(x, y) \le M(x, y) \le \max(x, y), \quad x, y \in I.$$

If, for all $x, y \in I, x \ne y$, these inequalities are strict, M is called *strict*; and *symmetric*, if $M(x, y) = M(y, x)$. If M is a mean in I^2 then $M(J^2) = J$ for every subinterval $J \subseteq I$; moreover M is *reflexive*, i.e.

$$M(x, x) = x, \quad x \in I.$$

Every reflexive function $M : I^2 \to I$ which is increasing with respect to each variable is a mean.

Recall that for every continuous and strictly monotonic function $h : I \to \mathbb{R}$ and $p \in (0, 1)$ the function $M_p^{[h]} : I^2 \to I$,

$$M_p^{[h]}(x, y) := h^{-1}(ph(x) + (1 - p)h(y)), \quad x, y \in I,$$

is a mean, and it is called a *quasi-arithmetic weighted mean.* The function h is called a *generator* of the mean and p its *weight.* Of course, every quasi-arithmetic weighted mean is increasing with respect to each variable and continuous.. If $p = \frac{1}{2}$ we write $M^{[h]} := M^{[h]}_{1/2}$, that is

$$M^{[h]}(x,y) := h^{-1}\left(\frac{h(x)+h(y)}{2}\right), \quad x,y \in I,$$

and $M^{[h]}$ is called a *quasi-arithmetic mean.* The family of quasi-arithmetic weighted means is one of the most important classes of means (cf. [1, Chap. 17], [2]).

In Sect. 2 of the present paper we show that under some simple and natural assumptions on the functions $f, g : I \to \mathbb{R}$, the function $M_{f,g} : I^2 \to I$ defined by

$$M_{f,g}(x,y) = (f+g)^{-1}(f(x)+g(y))$$

is a mean or a strict mean, and it is a generalization of the weighted quasi-arithmetic mean. Moreover some necessary and sufficient conditions for $M_{f,g}$ to be symmetric, quasi-arithmetic, or weighted quasi-arithmetic, are given (Theorem 1). In Sect. 3 we consider the equality problem $M_{f,g} = M_{F,G}$ (Theorem 2). The equation

$$\varphi(u+v) = \psi(u) + \gamma(v), \quad u \in J, v \in K,$$

where $\psi : J \to \mathbb{R}, \gamma : K \to \mathbb{R}$ and $\varphi : (J+K) \to \mathbb{R}$ are the unknown functions and $J, K \subset \mathbb{R}$ are some intervals, appears in the proofs. It is nonstandard Pexider functional equation, as the domains of the unknown functions are not the same. In section 4, applying the Pexider version of Cauchy functional equation, we determine all homogeneous $M_{f,g}$ means (Theorem 3). In the last section we remark that the definition $M_{f,g}$ can be easily extended to the case of k-variable means $M_{f_1,\ldots,f_k} : I^k \to I$.

2. Basic results and definitions

Lemma 1. *Let $I \subset \mathbb{R}$ be an interval. Suppose that the functions $f, g : I \to \mathbb{R}$ satisfy the following conditions: the function $f+g$ is continuous, strictly monotonic and*

$$f(I) + g(I) \subseteq (f+g)(I). \tag{1}$$

Then the function $M_{f,g} : I \times I \to \mathbb{R}$,

$$M_{f,g}(x,y) := (f+g)^{-1}(f(x)+g(y)), \tag{2}$$

is correctly defined; moreover the following conditions are equivalent:

(1) *$M_{f,g}$ is a strict mean;*
(2) *the functions f and g are continuous and strictly monotonic of the same type monotonicity as the function $f+g$.*

Proof. Inclusion (1) guarantees that the definition of $M_{f,g}$ is correctly stated. Assume that $f+g$ is strictly increasing.

If $M_{f,g}$ is a strict mean then, for all $x, y \in I$, $x<y$,

$$x < M_{f,g}(x,y) < y,$$

which, by (2) and the increasing monotonicity of $f+g$, can be written in the form

$$f(x)+g(x) < f(x)+g(y) < f(y)+g(y),$$

and, consequently,

$$g(x) < g(y) \quad \text{and} \quad f(x) < f(y),$$

which proves that the functions f and g are strictly increasing. The continuity of f and g follows from the assumed continuity of $f+g$.

We omit the similar argument in the case when $f+g$ is decreasing.

If f and g are continuous and either both strictly increasing or both strictly decreasing, then it easy to verify that $M_{f,g}$ defined by (2) is a strict mean. $\square$

In a similar way one can prove the following

Lemma 2. *Let $I \subset \mathbb{R}$ be an interval. Suppose that the functions $f, g: I \to \mathbb{R}$ satisfy the following conditions: the function $f+g$ is continuous, strictly increasing (respectively, strictly decreasing) and*

$$f(I)+g(I) \subseteq (f+g)(I).$$

Then the function $M_{f,g}: I \times I \to \mathbb{R}$,

$$M_{f,g}(x,y) := (f+g)^{-1}(f(x)+g(y)),$$

is correctly defined; moreover the following conditions are equivalent:
(1) *$M_{f,g}$ is a mean;*
(2) *the functions f and g are continuous and nondecreasing (respectively, nonincreasing).*

Remark 1. Note that condition (1) implies that $f(I)+g(I) = (f+g)(I)$, and there is no a non-trivial subinterval of I on which both f and g are constant.

Corollary 1. *Let $I \subset \mathbb{R}$ be an interval. Suppose that the functions $f, g: I \to \mathbb{R}$ are continuous and monotonic.*

(1) *If f and g are both strictly increasing or both strictly decreasing then the function $M_{f,g}$ defined by (2) is a strict mean.*
(2) *If f and g are both increasing or both decreasing, and there is no a non-trivial subinterval of a common constancy for f and g, then the function $M_{f,g}$ defined by (2) is a mean.*

Definition 1. (1) If the functions f and g satisfy the conditions of Corollary 1.(1), then $M_{f,g}$ is called a generalized weighted strict quasi-arithmetic mean in I.

(2) If f the functions and g satisfy the conditions of Corollary 1.(2), then $M_{f,g}$ is called a generalized weighted quasi-arithmetic mean in I.

The functions f and g are called generators of the mean $M_{f,g}$.

Theorem 1. *Let $I \subset \mathbb{R}$ be an interval and let $f, g : I \to \mathbb{R}$. Suppose that $M_{f,g}$ is a generalized weighted quasi-arithmetic mean in I. Then*

(1) *$M_{f,g}$ is symmetric if, and only if, there is $c \in \mathbb{R}$ such that*

$$g(x) = f(x) + c, \quad x \in I;$$

(2) *$M_{f,g}$ is quasi-arithmetic if, and only if, there is $c \in \mathbb{R}$ such that*

$$g(x) = f(x) + c, \quad x \in I;$$

(3) *$M_{f,g}$ is equal to a weighted quasi-arithmetic $M_p^{[h]}$ with a strictly monotonic and continuous generator $h : I \to \mathbb{R}$ and a weight $p \in (0, 1)$, that is,*

$$M_{f,g}(x, y) = M_p^{[h]}(x, y), \quad x, y \in I,$$

if, and only if,

$$f(x) = aph(x) + b, \quad g(x) = a(1-p)h(x) + c, \quad x \in I,$$

for some $a, b, c \in \mathbb{R}$, $a \neq 0$.

Proof. By (2), $M_{f,g}$ is symmetric iff

$$g(x) - f(x) = g(y) - f(y), \quad x, y \in I,$$

that is iff the function $g - f$ is constant. This proves part 1.

To prove part 2 assume that $M_{f,g}$ is quasi-arithmetic. Since any quasi-arithmetic mean is symmetric, in view of part 1, there is a constant c such that $g(x) = f(x) + c$ for all $x \in I$. Conversely, if $g(x) = f(x) + c$ for all $x \in I$, then $(f + g)(x) = 2f(x) + c$ for $x \in I$. Hence $(f + g)^{-1}(u) = f^{-1}(\frac{u-c}{2})$ for all $u \in (f + g)(I)$, whence, for all $x, y \in I$,

$$\begin{aligned} M_{f,g}(x, y) &= (f + g)^{-1}(f(x) + g(y)) \\ &= f^{-1}\left(\frac{f(x) + (f(y) + c) - c}{2}\right) = M_f(x, y), \end{aligned}$$

so $M_{f,g}$ is a quasi-arithmetic mean of a generator f.

To prove part 3, suppose that $M_{f,g}$ is a weighted quasi-arithmetic mean. Thus there is a continuous and strictly monotonic function $h : I \to \mathbb{R}$ and a number $p \in (0, 1)$ such that

$$M_{f,g}(x, y) = h^{-1}\left(ph(x) + (1-p)h(y)\right), \quad x, y \in I,$$

that is,

$$(f+g)^{-1}(f(x)+g(y)) = h^{-1}(ph(x)+(1-p)h(y)), \quad x,y \in I.$$

Setting $\alpha := (f+g)\circ h^{-1}$ and $u = h(x), v = h(y)$ we can write this equation in the form

$$\alpha(pu+(1-p)v) = f(h^{-1}(u)) + g(h^{-1}(v)), \quad u,v \in h(I).$$

Take an arbitrary $u_0 \in \operatorname{int} h(I)$ and define $\beta : (h(I)-u_0) \to \mathbb{R}$ by

$$\beta(u) := \alpha(u+u_0) - \alpha(u_0), \quad u \in (h(I)-u_0).$$

Hence, for all $u, v \in (h(I)-u_0)$, we have

$$\begin{aligned}\beta(pu+(1-p)v) &= \alpha(pu+(1-p)v+u_0) - \alpha(u_0)\\ &= \alpha(p(u+u_0)+(1-p)(v+u_0)) - \alpha(u_0)\\ &= f(h^{-1}(u+u_0)) + g(h^{-1}(v+u_0)) - \alpha(u_0),\end{aligned}$$

that is, for all $u, v \in (h(I)-u_0)$,

$$\beta(pu+(1-p)v) = f(h^{-1}(u+u_0)) + g(h^{-1}(v+u_0)) - \alpha(u_0), \tag{3}$$

Since $0 \in (h(I)-u_0)$, and $\beta(0)=0$, taking here first $v=0$ and then $u=0$, we obtain

$$f(h^{-1}(u+u_0)) = \beta(pu) - g((h^{-1}u_0)) + \alpha(u_0), \quad u \in (h(I)-u_0), \tag{4}$$

and

$$g(h^{-1}(v+u_0)) = \beta((1-p)v) - f(h^{-1}(u_0)) + \alpha(u_0), \quad v \in (h(I)-u_0). \tag{5}$$

Setting these functions into (3), and taking into account that $\beta(0)=0$ implies the equality

$$\alpha(u_0) - f(h^{-1}(u_0)) - g(h^{-1}(u_0)) = 0, \tag{6}$$

we get

$$\beta(pu+(1-p)v) = \beta(pu) + \beta((1-p)v), \quad u,v \in (h(I)-u_0),$$

or, equivalently,

$$\beta(u+v) = \beta(u) + \beta(v), \quad u \in p(h(I)-u_0), \quad v \in (1-p)(h(I)-u_0).$$

Since $0 \in [p(h(I)-u_0)] \cap [(1-p)(h(I)-u_0)]$, the function β has a unique extension to an additive function on $\mathbb{R}$. Denote this extension by β. By the assumptions of f, g, h, the function α is strictly monotonic, and consequently, so is β. It follows that, for some $\beta(1) \neq 0$, we have

$$\beta(u) = \beta(1)u, \quad u \in (h(I)-u_0).$$

Obviously, the form of β does not depend on choice of u_0. Now, from (4), (5) and (6), we obtain

$$\begin{aligned}f(h^{-1}(u)) &= \beta(1)pu + f((h^{-1}u_0)), \quad u \in h(I),\\ g(h^{-1}(u)) &= \beta(1)(1-p)u + g(h^{-1}(u_0)), \quad u \in h(I).\end{aligned}$$

Consequently,

$$f(x) = aph(x) + b, \quad g(x) = a(1-p)h(x) + c, \quad x \in I,$$

which completes the "only if" part of the proof.

Since the reverse implication is easy to verify, the proof is complete. $\square$

3. Equality of generalized weighted quasi-arithmetic means

Theorem 2. *Let $I \subset \mathbb{R}$ be an interval and let $f, g, F, G : I \to \mathbb{R}$. Suppose that $M_{f,g}$ and $M_{F,G}$ are generalized strict quasi-arithmetic means in I. Then $M_{F,G} = M_{f,g}$ if, and only if, there exist $a, b, c \in \mathbb{R}$, $a \neq 0$, such that*

$$F(x) = af(x) + b, \quad G(x) = ag(x) + c, \quad x \in I.$$

Proof. Note that without any loss of generality we can assume that $0 \in \operatorname{int}(I)$. To show it choose a point $x_0 \in \operatorname{int}(I)$ and define the functions $\hat{f}, \hat{g}, \hat{F}, \hat{G} : (I - x_0) \to \mathbb{R}$ by

$$\hat{f}(x) := f(x + x_0), \quad \hat{g}(x) := g(x + x_0),$$
$$\hat{F}(x) := F(x + x_0), \quad \hat{G}(x) := G(x + x_0).$$

Now equality $M_{F,G} = M_{f,g}$ in $I \times I$ implies that $M_{\hat{F},\hat{G}} = M_{\hat{f},\hat{g}}$ in $(I - x_0) \times (I - x_0)$ and, obviously $0 \in I - x_0$.

Suppose that $M_{F,G} = M_{f,g}$ in $I \times I$, that is

$$(F+G)^{-1}(F(x) + G(y)) = (f+g)^{-1}(f(x) + g(y)), \quad x, y \in I. \tag{7}$$

It is easy to verify that the functions $f+k, g+l, F+m, G+n$, where k, l, m, n are arbitrary fixed real numbers, also satisfy this equation. Taking $k = -f(0), l = -g(0), m = -F(0), n := -G(0)$ we can replace f, g, F, G by the functions which satisfy the considered equality and all have value 0 at 0. Thus, in the sequel, we may assume that

$$f(0) = g(0) = F(0) = G(0) = 0.$$

Putting

$$\varphi := (f+g) \circ (F+G)^{-1}, \quad \psi := F \circ f^{-1}, \quad \gamma := G \circ g^{-1}, \tag{8}$$

we can write Eq. (7) in the form

$$\varphi(u+v) = \psi(u) + \gamma(v), \quad u \in f(I), v \in g(I),$$

where $\varphi(0) = \psi(0) = \gamma(0)$ and $0 \in \operatorname{Int} f(I) \cap g(I)$. Setting here $v = 0$, we get $\varphi(u) = \psi(u)$ for all $u \in f(I)$ and, setting $v = 0$, we get $\varphi(v) = \gamma(v)$ for all $v \in g(I)$. It follows that

$$\varphi(u+v) = \varphi(u) + \varphi(v), \quad u \in f(I), \quad v \in g(I).$$

Since φ is additive in a neighborhood of 0, it has a unique additive extension on $\mathbb{R}$. We can denote this extension by φ. The continuity of φ (at least at 0) implies that there is an $a \in \mathbb{R}$, $a \neq 0$, (cf. [1, p.15, Corollary 5], [3, p. 121]) such that $\varphi(u) = au$ for all $u \in \mathbb{R}$. It follows that

$$\varphi(u) = au, \quad u \in f(I) + g(I); \quad \psi(u) = au, \quad u \in f(I); \quad \gamma(u) = au, \quad u \in g(I).$$

Hence, taking into account the above mentioned invariance of the generalized quasi-arithmetic mean with respect to additive constants of their generators, and the second and third definitions of (8), we infer that

$$F(x) = af(x) + b, \quad G(x) = ag(x) + c, \quad x \in I,$$

for some $b, c \in \mathbb{R}$. This completes the "only if" part of the proof. The converse implication is obvious. $\square$

4. Homogeneity

Theorem 3. *Let $f, g : (0, \infty) \to \mathbb{R}$ be such that $M_{f,g}$ is a generalized weighted quasi-arithmetic mean in $(0, \infty)$. Then $M_{f,g}$ is homogeneous if, and only if, either, for some $a, b, c, d \in \mathbb{R}$, $ac > 0$,*

$$f(x) = a \log(x) + b, \quad g(x) = c \log(x) + d, \quad x > 0;$$

or, for some $p, a, b, c \in \mathbb{R}$, $p \neq 0$, $ac > 0$,

$$f(x) = ax^p + b, \quad g(x) = cx^p + d, \quad x > 0;$$

Moreover, in the first case,

$$M_{f,g}(x, y) = x^{\frac{a}{a+c}} y^{\frac{c}{a+c}}, \quad x, y > 0;$$

and, in the second case,

$$M_{f,g}(x, y) = \left(\frac{a}{a+c} x^p + \frac{c}{a+c} y^p \right)^{1/p}, \quad x, y > 0,$$

thus $M_{f,g}$ is a weighted Hölder or power mean.

Proof. Suppose that $M_{f,g}$ is homogeneous. Thus

$$(f+g)^{-1}(f(tx) + g(ty)) = t(f+g)^{-1}(f(x) + g(y)), \quad t, x, y > 0.$$

Putting, for every $t > 0$,

$$\varphi_t := (f+g) \circ \left[t(f+g)^{-1} \right], \quad \psi_t := f \circ (tf^{-1}), \quad \gamma_t := g \circ (tg^{-1}), \tag{9}$$

we can write this equation in the form

$$\varphi_t(u + v) = \psi_t(u) + \gamma_t(v), \quad u \in f((0, \infty)), \quad v \in g((0, \infty)), \quad t > 0.$$

A similar reasoning as that applied in the proof of previous result shows that for every $t > 0$ there are real numbers $k(t)$, $m(t)$, $n(t)$ such that

$$\psi_t(u) = k(t)u + m(t), \quad u \in f((0, \infty)); \quad \gamma_t(v) = k(t)v + n(t), \quad v \in g((0, \infty)).$$

For every $t > 0$ the functions $t(f+g)^{-1}$, tf^{-1} and tg^{-1} are, respectively, of the same type monotonicity as $(f+g)^{-1}, f^{-1}$ and g^{-1}. Since the composition of two strictly monotonic functions of the same type monotonicity is strictly increasing, for every $t > 0$, the functions ψ_t and γ_t are strictly increasing, we have

$$k(t) > 0, \quad t > 0,$$

and, by (9),

$$f(tx) = k(t)f(x) + m(t), \quad g(tx) = k(t)g(x) + n(t), \quad t, x > 0.$$

Applying a multiplicative version of Corollary 2 in [1], p. 242, we infer that either

$$k(t) = 1, \quad t > 0,$$

and

$$f(x) = a\log(x) + b, \quad g(x) = c\log(x) + d, \quad x > 0,$$

or there is a $p \in \mathbb{R}$, $p \neq 0$, such that

$$k(t) = t^p, \quad t > 0,$$

and

$$f(x) = ax^p + b, \quad g(x) = cx^p + d, \quad x > 0.$$

As the generators of the generalized weighted quasi-arithmetic mean must be of the same type of monotonicity, we have

$$ac > 0.$$

This completes the proof of the "only if" part of the theorem. The remaining statements are easy to verify. □

5. Remarks on generalized k-variable weighted quasi-arithmetic means $M_{f_1,\dots,f_k}$

Let us note that the k-dimensional counterparts of Lemmas 1 and 2 are valid.

Lemma 3. *Let $I \subset \mathbb{R}$ be an interval and $k \in \mathbb{N}$, $k \geq 2$, fixed. Suppose that the functions $f_1, \dots, f_k : I \to \mathbb{R}$ satisfy the following conditions: the function $f_1 + \dots + f_k$ is continuous, strictly monotonic and*

$$f_1(I) + \dots + f_k(I) \subseteq (f_1 + \dots + f_k)(I). \tag{10}$$

Then the function $M_{f_1,\dots,f_k} : I^k \to \mathbb{R}$,

$$M_{f_1,\dots,f_k}(x_1,\dots,x_k) := (f_1 + \dots + f_k)^{-1}(f_1(x_1) + \dots + f_k(x_k)), \tag{11}$$

is correctly defined; moreover the following conditions are equivalent:

(1) *$M_{f_1,\dots,f_k}$ is a strict mean;*

(2) *the functions* $f_1, \dots, f_k$ *are continuous, strictly monotonic and of the same type monotonicity as the function* $f_1 + \dots + f_k$.

Proof. Inclusion (10) implies the correctness of the definition of $M_{f_1,\dots,f_k}$. Assume that $f_1 + \dots + f_k$ is strictly increasing.

Suppose that $M_{f_1,\dots,f_k}$ is a strict mean. Take $i \in \{1, \dots, k\}$ and $x, y \in I, x < y$, and put $x_j = x$ for all $j \in \{1, \dots, k\}, j \neq i$, and $x_i = y$. From the definition of the mean we have

$$x = \min(x_1, \dots, x_k) < M_{f_1,\dots,f_k}(x_1, \dots, x_k),$$

whence, by (11) and the increasing monotonicity of $f_1 + \dots + f_k$,

$$f_i(x) < f_i(y),$$

which proves that, for each $i \in \{1, \dots, k\}$, the function f_i is strictly increasing. The continuity of each f_i follows from the assumed continuity of $f_1 + \dots + f_k$.

We omit similar argument in the case when $f_1 + \dots + f_k$ is decreasing.

If $f_1, \dots, f_k$ are continuous and all either strictly increasing or strictly decreasing, then it easy to verify that $M_{f_1,\dots,f_k}$ defined by (11) is a strict mean. □

In a similar way as Lemma 2 we can prove the following

Lemma 4. *Let* $I \subset \mathbb{R}$ *be an interval and* $k \in \mathbb{N}$, $k \geq 2$, *fixed. Suppose that the functions* $f_1, \dots, f_k : I \to \mathbb{R}$ *satisfy the following conditions: the function* $f_1 + \dots + f_k$ *is continuous, strictly increasing (respectively, strictly decreasing) and*

$$f_1(I) + \dots + f_k(I) \subseteq (f_1 + \dots + f_k)(I).$$

Then the function $M_{f_1,\dots,f_k} : I^k \to \mathbb{R}$,

$$M_{f_1,\dots,f_k}(x_1, \dots, x_k) := (f_1 + \dots + f_k)^{-1}(f_1(x_1) + \dots + f_k(x_k)),$$

is correctly defined; moreover the following conditions are equivalent:

(1) $M_{f_1,\dots,f_k}$ *is a mean;*

(2) *the functions* $f_1, \dots, f_k$ *are continuous, of the same type (weak) monotonicity as the function* $f_1 + \dots + f_k$.

Remark 2. Note that condition (10) implies that

$$f_1(I) + \dots + f_k(I) = (f_1 + \dots + f_k)(I),$$

and there is no a non-trivial subinterval of I on which all functions $f_1, \dots, f_k$ are constant.

Corollary 2. *Let* $I \subset \mathbb{R}$ *be an interval. Suppose that the functions* $f_1, \dots, f_k : I \to \mathbb{R}$ *are continuous and monotonic.*

(1) *If* $f_1, \dots, f_k$ *are strictly increasing (strictly decreasing) then* $M_{f_1,\dots,f_k}$ *defined by* (11) *is a strict mean.*

(2) *If $f_1, \ldots, f_k$ are increasing (decreasing) and there is no a nontrivial subinterval of I on which all functions $f_1, \ldots, f_k$ are constant, then $M_{f_1,\ldots,f_k}$ defined by* (11) *is a mean.*

Definition 2. (1) If the functions $f_1, \ldots, f_k : I \to \mathbb{R}$ satisfy the conditions of Corollary 2.(1), then $M_{f_1,\ldots,f_k}$ defined by (11) is called a generalized strict weighted quasi-arithmetic mean in I.

(2) If the functions $f_1, \ldots, f_k : I \to \mathbb{R}$ satisfy the conditions of Corollary 2.(2), then $M_{f_1,\ldots,f_k}$ defined by (11) is called a generalized weighted quasi-arithmetic mean, in I.

The functions $f_1, \ldots, f_k$ are called generators of the mean $M_{f_1,\ldots,f_k}$.

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References

[1] Aczél, J., Dhombres, J.: Functional equations in several variables. Encyclopedia of Mathematics and its Applications, vol. 31. Cambridge University Press, Cambridge (1989)

[2] Bullen, P.S., Mitrinović, D.S., Vasić, P.M.: Means and their inequalities. Mathematics and its Applications. D. Reidel Publishing Company, Dodrecht (1988)

[3] Kuczma, M.: An Introduction to the Theory of Functional Equations and Inequalities. Cauchy's Equations and Jensen Inequality. Uniwersytet Ślaski-PWN, Warszawa (1985)

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