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Uniformly continuous superposition operators in the space of bounded variation functions

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Let $I, J \subset \mathbb{R}$ be intervals. The main result says that if a superposition operator H generated by a function of two variables $h : I \times J \to \mathbb{R}$,

$$H(\varphi)(x) := h(x, \varphi(x)),$$

maps the set $BV(I, J)$ of all bounded variation functions $\varphi : I \to J$ into the Banach space $BV(I, \mathbb{R})$ and is uniformly continuous with respect to the BV-norm, then

$$h(x, y) = a(x)y + b(x), \quad x \in I, \quad y \in J,$$

for some $a, b \in BV(I, \mathbb{R})$.

1 Introduction

Let $I, J \subset \mathbb{R}$ be intervals. By J^I denote the set of all functions $\varphi : I \to J$. For a given function $h : I \times J \to \mathbb{R}$, the mapping $H : J^I \to \mathbb{R}^I$ defined by

$$H(\varphi)(x) := h(x, \varphi(x)), \quad \varphi \in J^I,$$

is called a superposition (or Nemytskij) operator of a generator h.

The superposition operators play important role in the theory of differential equations, integral equations and functional equations. In [7] (1982) it has been proved that if a superposition operator maps the Banach space $BV(I, \mathbb{R})$ of the bounded variation functions into itself and is globally Lipschitzian with respect to BV-norm, i.e., for an $L \geq 0$,

$$\|H(\varphi) - H(\psi)\|_{BV} \leq L \|\varphi - \psi\|_{BV}, \quad \varphi, \psi \in BV(I, \mathbb{R}),$$

then for each $x \in I$, $x > \inf I$, and for every $y \in \mathbb{R}$ there exists the limit

$$h_-(x, y) := \lim_{u \to x-} h(u, y)$$

and, for some left-hand side continuous functions $a, b \in BV(I, \mathbb{R})$,

$$h_-(x, y) = a(x)y + b(x), \quad x \in I, \quad y \in \mathbb{R},$$

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(cf. also V. V. Chistyakov [3] for a generalization). Analogous results hold true for some other function Banach spaces ([1], [4], [6], cf. also J. Appell and P. P. Zabrejko [2]).

In the present paper, applying the theory of Jensen functional equation, we prove that if the operator H mapping the set $BV(I,J)$ of all bounded variation functions $\varphi : I \to J$ into the Banach space $BV(I,\mathbb{R})$ satisfies the inequality

$$\|H(\varphi) - H(\psi)\|_{BV} \le \gamma\left(\|\varphi - \psi\|_{BV}\right), \quad \varphi, \psi \in BV(I,J),$$

where the function $\gamma : [0,\infty) \to [0,\infty)$ is continuous at 0 and $\gamma(0) = 0$, then

$$h_-(x,y) = a(x)y + b(x), \quad x \in I, \quad y \in J,$$

for some $a, b \in BV(I,\mathbb{R})$.

Applying this we prove the main result saying that the generator h of any superposition operator mapping $BV(I,J)$ into $BV(I,\mathbb{R})$ and uniformly continuous with respect to the norm $\|\cdot\|_{BV}$ must be also of this form.

2 Results

Let $I, J \subset \mathbb{R}$ be intervals and let $x_0 \in I$. By $BV(I,\mathbb{R})$ we denote the Banach space of all functions of bounded variation $\varphi : I \to \mathbb{R}$ with the norm

$$\|\varphi\|_{BV} := |\varphi(x_0)| + \mathrm{Var}(\varphi),$$

where

$$\mathrm{Var}(\varphi) := \sup \sum_{i=1}^{n} |\varphi(x_i) - \varphi(x_{i-1})| < \infty$$

and the supremum is taken over all $n \in \mathbb{N}$ and all strictly increasing sequences $x_i \in I$, $i = 0, 1, \dots, n$. For an interval $J \subset \mathbb{R}$ we put

$$BV(I,J) := \{\varphi \in BV(I,\mathbb{R}) : \varphi(I) \subset J\}.$$

Theorem 2.1 *Let $I, J \subset \mathbb{R}$ be intervals, and $h : I \times J \to \mathbb{R}$. Suppose that $\gamma : [0,\infty) \to [0,\infty)$ is continuous at 0 and $\gamma(0) = 0$. If the superposition operator H of the generator h maps the set $BV(I,J)$ into the Banach space $BV(I,\mathbb{R})$ and satisfies the inequality*

$$\|H(\varphi) - H(\psi)\|_{BV} \le \gamma\left(\|\varphi - \psi\|_{BV}\right), \quad \varphi, \psi \in BV(I,J), \tag{2.1}$$

then

1. *for each $x \in I$, $x > \inf I$, and for every $y \in J$ there exists the limit*

$$h_-(x,y) := \lim_{u \to x-} h(u,y)$$

and, for some left-hand side continuous functions $a, b \in BV(I,\mathbb{R})$,

$$h_-(x,y) = a(x)y + b(x), \quad x \in I, \quad y \in J;$$

2. *for each $x \in I$, $x < \sup I$, and for every $y \in J$ there exists the limit*

$$h_+(x,y) := \lim_{u \to x+} h(u,y)$$

and, for some right-hand side continuous functions $a, b \in BV(I,\mathbb{R})$,

$$h_+(x,y) = a(x)y + b(x), \quad x \in I, \quad y \in J.$$

P r o o f. Without any loss of generality we can assume that $I = [0,1]$ and that $x_0 = 0$. Thus

$$\|\varphi\|_{BV} = |\varphi(0)| + \mathrm{Var}(\varphi).$$

For arbitrary $y \in J$ the function $\varphi : [0,1] \to J$ defined by $\varphi(t) := y$ for $t \in [0,1]$ is of bounded variation. It follows that $H(\varphi) = h(\cdot, y) \in BV([0,1], \mathbb{R})$. By Jordan's decomposition theorem, the function $h(\cdot, y)$ is a difference of two increasing functions. Thus the function $h(\cdot, y)$ has the left-sided limit $h_-(x,y) := \lim_{u\to x-} h(u,y)$ at every point $x \in (0,1]$, and the right-sided limit $h_+(x,y) := \lim_{u\to x+} h(u,y)$ at every point $x \in [0,1)$. Note that for each $y \in J$ the function $h_-(\cdot, y)$ is left-continuous and $h_-(\cdot, y) \in BV([0,1], \mathbb{R})$. Similarly, for each $y \in J$ the function $h_+(\cdot, y)$ is right-continuous and $h_+(\cdot, y) \in BV([0,1], \mathbb{R})$.

Take $x \in (0,1]$, $n \in \mathbb{N}$, $y_1, y_2, \bar{y}_1, \bar{y}_2 \in J$, $x_1, x_2, \ldots, x_{2n}$ such that

$$a < x_1 < x_2 < \cdots < x_{2n} < x,$$

and define the functions $\varphi, \psi : [0,1] \to \mathbb{R}$ by

$$\varphi(t) := \begin{cases} y_1 & \text{for} \quad t \notin \{x_2, x_4, \ldots, x_{2n}\}, \\ y_2 & \text{for} \quad t \in \{x_2, x_4, \ldots, x_{2n}\}, \end{cases} \qquad \psi(t) := \begin{cases} \bar{y}_1 & \text{for} \quad t \notin \{x_2, x_4, \ldots, x_{2n}\}, \\ \bar{y}_2 & \text{for} \quad t \in \{x_2, x_4, \ldots, x_{2n}\}. \end{cases}$$

Note that $\varphi, \psi \in BV([0,1], \mathbb{R})$ and

$$\|\varphi - \psi\|_{BV} = |y_1 - \bar{y}_1| + 2n\,|y_2 - \bar{y}_2 - y_1 + \bar{y}_1|\,.$$

From the definition of the norm $\|\cdot\|_{BV}$ we have

$$\sum_{i=1}^{n} |h(x_{2i}, y_2) - h(x_{2i}, \bar{y}_2) - h(x_{2i-1}, y_1) + h(x_{2i-1}, \bar{y}_1)| \le \mathrm{Var}\,(H(\varphi) - H(\psi))\,.$$

Hence, making use of (2.1) and the definition of the norm $\|\cdot\|_{BV}$, we obtain the inequality

$$\begin{aligned} &\sum_{i=1}^{n} |h(x_{2i}, y_2) - h(x_{2i}, \bar{y}_2) - h(x_{2i-1}, y_1) + h(x_{2i-1}, \bar{y}_1)| \\ &\quad \le \gamma(|y_1 - \bar{y}_1| + 2n\,|y_2 - \bar{y}_2 - y_1 + \bar{y}_1|) \end{aligned}$$

for all $x \in (0,1]$, $n \in \mathbb{N}$, $y_1, y_2, \bar{y}_1, \bar{y}_2 \in J$, and $x_1, x_2, \ldots, x_{2n}$ such that $a < x_1 < x_2 < \cdots < x_{2n} < x$. Since for each $y \in J$ the function $h_-(\cdot, y)$ is left-continuous, letting $x_1 \to x$, we hence get

$$n\,|h_-(x, y_2) - h_-(x, \bar{y}_2) - h_-(x, y_1) + h_-(x, \bar{y}_1)| \le \gamma(|y_1 - \bar{y}_1| + 2n\,|y_2 - \bar{y}_2 - y_1 + \bar{y}_1|)$$

for all $x \in (0,1]$, $n \in \mathbb{N}$, and $y_1, y_2, \bar{y}_1, \bar{y}_2 \in J$.

Taking $u, v \in J$ and substituting here

$$y_1 := \frac{u+v}{2}, \quad y_2 := v, \quad \bar{y}_1 := u, \quad \bar{y}_2 := \frac{u+v}{2},$$

we get the inequality

$$n\left|2h_-\left(x, \frac{u+v}{2}\right) - h_-\,(x,u) - h_-\,(x,v)\right| \le \gamma\left(\left|\frac{u-v}{2}\right|\right).$$

Since $n \in \mathbb{N}$ is arbitrary, it follows that

$$2h_-\left(x, \frac{u+v}{2}\right) - h_-\,(x,u) - h_=\,(x,v) = 0, \quad x \in (0,1], \quad u, v \in J,$$

that is, for every $x \in (0,1]$, the function $h_-(x, \cdot)$ satisfies the Jensen functional equation.

We shall show that for each $x \in (0,1]$ the function $h_-(x,\cdot)$ is continuous in the interval J. To this end take an $x \in [0,1]$, $y, \bar{y} \in J$ and define $\varphi, \psi : [0,1] \to J$ by

$$\varphi(t) := y, \quad \psi(t) := \begin{cases} y & \text{for} \quad t \neq x, \\ \bar{y} & \text{for} \quad t = x, \end{cases} \qquad t \in [0,1].$$

Then, of course, $\varphi, \psi \in BV(I,J)$ and in the case when $x \in (0,1)$ we have

$$\|\varphi - \psi\|_{BV} = 2\,|y - \bar{y}|\,, \quad \|H(\varphi) - H(\psi)\|_{BV} = 2\,|h(x,y) - h(x,\bar{y})|\,,$$

and, by (2.1),

$$2\,|h(x,y) - h(x,\bar{y})| \leq \gamma\,(2\,|y-\bar{y}|)\,, \quad x \in (0,1), \quad y,\bar{y} \in J,$$

whence

$$2\,|h_-(x,y) - h_-(x,\bar{y})| \leq \gamma\,(2\,|y-\bar{y}|)\,, \quad x \in (0,1], \quad y,\bar{y} \in J.$$

Since, by assumption, the function γ is continuous at 0 and $\gamma(0) = 0$, we hence get the continuity of the function $h_-(x,\cdot)$ in J. Now, applying the classical result on Jensen functional equation (cf. M. Kuczma [4], p. 315, Theorem 1) we infer that for every $x \in (0,1]$ there are $a(x), b(x) \in \mathbb{R}$ such that

$$h_-(x,y) = a(x)y + b(x), \quad y \in J.$$

Since $h_-(\cdot,c) \in BV(I,\mathbb{R})$ for every $c \in J$, the functions a and b are left continuous in $(0,1]$ and of bounded variation. This completes the proof of part 1.

As the proof of part 2 is analogous, we omit it. □

Taking $J = \mathbb{R}$ and $\gamma(t) = Lt \ \ (t \geq 0)$ for some $L \geq 0$ one gets a result presented in [7] (cf. also [2], p. 175).

Applying Theorem 2.1 we obtain the main result:

Theorem 2.2 *Let $I, J \subset \mathbb{R}$ be intervals, I compact, and $h : I \times J \to \mathbb{R}$. If the superposition operator H of the generator h mapping the set $BV(I,J)$ into the Banach space $BV(I,\mathbb{R})$ is uniformly continuous with respect to the norm of the space $BV(I,\mathbb{R})$ then*

1. *for each $x \in I$, $x > \inf I$, and for every $y \in J$ there exists the limit*

$$h_-(x,y) := \lim_{u \to x-} h(u,y)$$

and, for some left-hand side continuous functions $a, b \in BV(I,\mathbb{R})$,

$$h_-(x,y) = a(x)y + b(x), \quad x \in I, \quad y \in J;$$

2. *for each $x \in I$, $x < \sup I$, and for every $y \in J$ there exists the limit*

$$h_+(x,y) := \lim_{u \to x+} h(u,y)$$

and, for some right-hand side continuous functions $a, b \in BV(I,\mathbb{R})$,

$$h_+(x,y) = a(x)y + b(x), \quad x \in I, \quad y \in J.$$

Proof. Assume that H is uniformly continuous. Then for every $\varepsilon > 0$ there is a $\delta > 0$ such that for all $\varphi, \psi \in BV(I,J)$,

$$\|\varphi - \psi\|_{BV} \leq \delta \Longrightarrow \|H(\varphi) - H(\psi)\|_{BV} \leq \varepsilon$$

and, consequently, the function $\gamma : [0,\infty) \to [0,\infty)$, defined by the formula

$$\gamma(t) := \sup\left\{\|H(\varphi) - H(\psi)\|_{BV} : \|\varphi - \psi\|_{BV} = t\right\}, \quad t \geq 0,$$

is continuous at 0 and $\gamma(0) = 0$. Since

$$\|H(\varphi) - H(\psi)\|_{BV} \leq \gamma\left(\|\varphi - \psi\|_{BV}\right), \quad \varphi, \psi \in BV(I,J),$$

the result follows from Theorem 2.1. □

Applying the same idea, one can prove the counterparts of Theorems 2.1 and 2.2 for some other function spaces with norms essentially stronger then the supremum norm.

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