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ORIGINAL PAPER

On the non-reducibility of non-stationary correlation functions to stationary ones under a class of mean-operator transformations

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Abstract Some special functional equations involving means and related to a problem of reducibility of some classes of correlation functions are considered. We show some characterizations of the reducibility problem under several choices of the mean operators and different weak regularity assumptions imposed on the involving functions. We find that mean-generated correlation functions are completely irreducible, in the sense that, for this broad class of correlation functions, there does not exist a nontrivial solution associated to the Perrin–Senoussi problem.

Keywords Bijective deformation · Correlation functions · Functional equations · Non-stationarity · Random fields · Reducibility

1 Introduction and preliminaries

Spatial statistics is one of the major methodologies of image analysis, field trials, remote sensing, and environmental statistics. The standard methodology in spatial statistics is essentially based on the assumption of stationary and isotropic random fields. Such assumptions might not hold in large heterogeneous fields, and thus non-stationarity is one of the most challenging problems for those fields dealing with the analysis of spatial and spatio-temporal phenomena.

One important scientific field where theory and practice of random fields are combined in a unique way is the continuum mechanics discipline (Ostoja-Starzewski 1998, 2007). Continuum mechanics hinges on the concept of a Representative Volume Element playing the role of a mathematical point of a continuum field approximating the true material microstructure. Indeed, continuum mechanics is naturally suited to deal primarily with media exhibiting spatially homogeneous properties. As theoretical models it is first considered strict-sense and wide-sense stationary random fields. Many models of microstructural randomness—e.g., Boolean models and tessellations—possess such homogeneity characteristics, and they are highly desirable in stochastic homogenization. Real materials, however, often lack these nice behaviors. In this sense, we can also find a large variety of literature regarding the use and interest in using non-stationary random fields when modeling real materials (see, for example, Morikawa and Kameda 2001; Sakamoto and Ghanem 2002; Grigoriu 2003).

A classical problem in environmental transport is determining the space–time concentration resulting from the introduction of solute in a spatially variable heterogeneous flow field. If the latter has a complex spatial distribution, as is the case in every natural flow, it can be expected that the solute concentration also develops a complex spatial distribution, especially if transport is advection dominated. The applications of quantitative tools in contaminant hydrogeology involve formalized risk assessment of one sort or another, and one important difficulty that the hydrogeologists note in the application to basic flow and transport is dealing with non-stationary fields and dealing with large scale heterogeneity. This clearly limits the applicability of stochastic methods in subsurface hydrology (Ginn 2004;

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Sirin 2006). Thus, the analysis of the associated non-stationary random field through the transformation into stationarity is a key point in this context (Kapoor and Kitanidis 1997; Kabala 1997).

Environmental pollution is another large field where non-stationary random fields are considered for interpolation methods (see, as an example, Rojas-Avellaneda and Silván-Cárdenas 2006).

A dominant part of the recent statistical literature (Christakos and Hristopulos 1998; Christakos 2000) reckons that stationarity can be an unrealistic assumption with respect to the great majority of geostatistical applications. Thus, it would be desirable to have covariance models that do not depend exclusively on the separation vector between two points of the spatio-temporal domain. Unfortunately, only few models for non-stationary spatial data have been proposed.

We thus pose the natural question: given a non-stationary random field, is there any appealing statistical philosophy that allows to treat it through stationary techniques? This problem is well-known by the geostatistical community, and a fertile literature can be found under the nomenclature of Generalized Random Functions (Rozanov 1989, 1998; Pugachev and Sinitsyn 2002; Ruiz-Medina et al. 2003). These approaches are based on the fact that a non-stationary random field can be reduced to a stationary one through differentiation of some order k.

Alternatively we can talk about *direct reducibility*, based on the reduction of a non-stationary covariance function to a stationary one. For a given two-place positive definite function (i.e. a function of two arguments that is positive definite on the product space where the arguments are defined) $r : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$, find a characterization for the existence of a one-place positive definite function $R : \mathbb{R} \rightarrow \mathbb{R}$ such that $R(0) = 1$, and a bijective deformation $\Phi : \mathbb{R} \rightarrow \mathbb{R}$ such that

$$r(x, y) = R(\Phi(y) - \Phi(x)), \quad x, y \in \mathbb{R}. \tag{1}$$

Correlation functions satisfying Eq. 1 are called *stationary reducible*. The problem of reducibility of non-stationary covariance functions has been persistently emphasized by early literature, as it allows for analyzing a non-stationary phenomenon through standard stationary techniques that are much more accessible from both analytical and computational points of view.

There are two main motivations for our procedure based on proposing an alternative to non-stationary modeling through reduction to stationary situations. The first motivation comes from the fact that many models of microstructural randomness, concentration and random flows, environmental transport, local solute dispersion, water solute transport and real-world applications of stochastic hydrogeology are based on homogeneity characteristics, and they are highly desirable in stochastic homogenization, but at the same time any of these scientific disciplines need non-stationary models. The second motivation is a practical one. The statistical analysis of most of the random fields arising in the natural/environmental problems just mentioned, need smooth realizations and for which the assumption of weak stationarity is highly desirable. Furthermore, weak stationarity allows for identifying other characteristics of the phenomena under study, like regularity (in terms of mean square continuity and differentiability) and fractality (associated Haussdorff dimension). Finally, simulation of such materials works quite better in the weakly stationary case, as the turning band method shows.

The work of Sampson and Guttorp (1992) is particularly worth being mentioned as it represents the first approach dedicated to this kind of problems. The authors introduce a non-parametric approach to global estimation of the spatial covariance structure of a random function. In particular, they use the spatial dispersion as a natural metric for the spatial covariance structure and model it as a general smooth function of the geographic coordinates of station pairs. Then, Multidimensional Scaling (MDS) is used to transform the problem into one for which the covariance structure, expressed in terms of spatial dispersions, is stationary and isotropic. The Sampson and Guttorp (1992) approach follows the following intuitive approach. Suppose the covariance function associated to some spatial random field is not stationary, i.e. it does not depend on the spatial lag vector. The problem can be simplified by looking at some deformation of the geographic space into a new space that allows for the covariance to be stationary. To do this, they look for a bijection that allows for modifying the dispersion function into a stationary variogram.

Formally, Sampson and Guttorp's approach relates to the more general following problem (which will be called hereafter as the *Perrin–Senoussi problem*): let $r(\cdot,\cdot) : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ a positive definite function. Find a pair (Φ, R), for $R : \mathbb{R} \rightarrow \mathbb{R}$ and a bijection Φ such that

$$r(x, y) = R(|\Phi(y) - \Phi(x)|)$$

for all $(x, y) \in \mathbb{R} \times \mathbb{R}$.

Perrin and Meiring (1999) study the uniqueness of (Φ, R) under different conditions. With this method, many sets of environmental data have been analyzed: solar radiation (Sampson and Guttorp 1992), acid precipitation (Guttorp et al. 1992; Guttorp and Sampson 1994; Mardia and Goodall 1993), air pollution (Brown et al. 1994) and tropospheric ozone (Guttorp et al. 1994; Meiring 1995). In the one-dimensional case, Perrin and Senoussi (1999) give a characterization of the non-stationary correlation functions that can be reduced to stationarity via a differentiable deformation.

Sampson and Guttorp (1992) refer only to stationarity and isotropic reducibility. Perrin and Senoussi (2000) study stationarity reducibility as well without restricting to isotropic conditions, thus analyzing (1). This is the approach we are taking here.

Finally, following the work of Sampson and Guttorp on the use of MDS as a methodological approach in the analysis of non-stationary spatial covariance structures, Vera et al. (2008, 2009) propose a modification consisting of including geographical spatial constraints as they note that approximating dispersion by a non-metric MDS procedure offers, in general, low precision.

As a unified approach that takes into account all previously mentioned non-stationary problems and applications, in this paper we face a problem of reducibility that can be sketched as follows. We consider continuous non-stationary stochastic processes whose associated correlation function is generated by a two-variable mean operator M, and two continuous and non-vanishing correlation functions C_1, C_2 defined on the real line, i.e.

$$r(x,y) := M(C_1(x), C_2(y)), \quad (x,y) \in \mathbb{R} \times \mathbb{R}. \tag{2}$$

Recall that a two-variable mean operator M is a mapping $M : (0, 1] \times (0, 1] \to (0, 1]$ that satisfies the properties of commutativity, idempotency, monotonicity, and self identity (Yager 1996). It is important to note that this procedure is easily generalized to the case of r defined on $\mathbb{R}^n \times \mathbb{R}^n$, as the techniques used in this paper can be readily extended to this case. Permissibility criteria, for some classes of means M, are given in a recent paper by Porcu et al. (2009). Thus, the natural problem arises by considering the eventual stationary reducibility of correlation structures of the type $r := M$.

In particular, we pose the following problem. Take a mean $M : (0, 1] \times (0, 1] \to (0, 1]$ and $C_i : \mathbb{R} \to (0, 1]$, $i = 1,2$, continuous and non-vanishing stationary correlation functions. Does then exist a positive definite function $R : \mathbb{R} \to \mathbb{R}$ and a bijective deformation $\Phi : \mathbb{R} \to \mathbb{R}$ such that

$$M(C_1(x), C_2(y)) = R(\Phi(y) - \Phi(x)), \quad x, y \in \mathbb{R}? \tag{3}$$

Several important ingredients play a role in expression (3), and thus natural questions arise: (a) which particular types of mean operators M can be used, (b) which regularity conditions should be imposed over the involving functions R and Φ. We study particular classes of functional equations that allow to find some solutions without any smoothness assumptions on the correlation function r, which represents a novelty with respect to the rest of the literature associated to this problem. And here is one of the key points in our procedure. Previous approaches (see among others Perrin and Senoussi 1999, 2000; Perrin and Meiring 1999, 2003) have assumed smoothness on the associated random field through the correlation function. This is not physically realistic in many of the natural environmental problems we could face. And this prompted us to using mean operators so as not to rely on the smoothness assumption. In addition, the class of correlation functions generated by quasi-arithmetic operators is of particular interest for statistical modeling, as they were shown (Porcu et al. 2009) to possess some desirable mathematical features in terms of smoothness, away from the origin, of the associated Gaussian random field.

Functional equations have been widely used to give answers to important problems related to scientific disciplines as diverse as mean values theory (Matkowski 1999), group theory, ideal gas theory, conditional Cauchy equations, economy and probability distributions. We refer to Aczél (1966) and references therein for an extensive review of functional equations applied to these disciplines.

The solutions we find for correlation functions of the type (2) are actually corollaries of results that will be presented in a very general setting. In particular, we shall make reference to functional equations of the type

$$M(a(x), b(y)) = R(\Phi(y) - \Phi(x)), \quad x, y \in \mathbb{R}, \tag{4}$$

for a, b, Φ, R functions on which some weak regularity conditions will be assumed. Then, these general results will be applied to the Perrin–Senoussi problem (Perrin and Senoussi 1999, 2000; Perrin and Meiring 1999, 2003; Genton and Perrin 2004).

As for the mean M generating (2), we shall show several results of (4) under the cases of: (a) *Increasing means*; (b) *Non-increasing means* and (c) For any choice of the mean.

The remainder of the paper is organized as follows. In Sect. 2 we present some basic facts about the Perrin–Senoussi problem and the type of means used in this paper. In Sects. 3, 4 we show the general results for the solution of the functional equation (4), respectively, for the cases of increasing and non-increasing means. In Sect. 5, we give an answer for the Perrin–Senoussi problem proposed in this paper. Section 6 is dedicated to conclusions and discussion. The proofs of the theoretical results in Sects. 3, 4 can be found in the Appendix.

2 Setup

This section is largely expository and reports the basic definitions and notations about the type of means used in this paper.

As far as means operators are concerned, let $I \subset \mathbb{R}$ be an interval. A function $M : I^2 \to \mathbb{R}$ is called a *mean in an interval I* if

$$\min(u, v) \le M(u, v) \le \max(u, v), \quad u, v \in I.$$

If these inequalities are strict for all $u, v \in I$, the mean M is called *strict*. If M is a mean, then, of course, M is *reflexive* , i.e. $M(x, x) = x$ for all $x \in I$, and $M(J \times J) = J$ for every subinterval $J \subset I$; in particular, $M(I \times I) = I$. A mean M in I is *symmetric* if $M(x, y) = M(y,x)$ for all $x, y \in I$.

In this paper we shall give some theoretical results involving classes of increasing means as well as non-increasing means.

(a) Increasing means

Remark 1 Every function $M : I^2 \to \mathbb{R}$ which is reflexive (i.e. such that $M(x, x) = x$ for all $x \in I$) and (strictly) increasing with respect to each variable, is a (strict) mean in I.

In the sequel such a mean is called an increasing mean.

Let $I \subset \mathbb{R}$ be an interval, $\varphi : I \to \mathbb{R}$ be a continuous and strictly monotonic function and $p \in (0, 1)$. Then the function $M_p^{[\varphi]} : I^2 \to \mathbb{R}$ defined by

$$M_p^{[\varphi]}(x,y) := \varphi^{-1}(p\varphi(x) + (1-p)\varphi(y)), \quad x, y \in I, \tag{5}$$

is a strict mean on I.

The mean $M_p^{[\varphi]}$ is called *weighted quasi-arithmetic*, the function φ its generator, and the numbers p and $1 - p$ its weights. Note that $M_p^{[\varphi]}$ is symmetric if and only if $p = \frac{1}{2}$. In this case we write $M^{[\varphi]}$ instead of $M_p^{[\varphi]}$, and call the mean *quasi-arithmetic*, in which case the inequalities

$$\min(x,y) \le M^{[\varphi]}(x,y) \le \max(x,y), \quad x, y \in I$$

are strict for all $x, y \in I$, $x \neq y$. The function φ is called a *generator* of the quasi-arithmetic mean.

Remark 2 $M^{[\varphi]}$ is symmetric, continuous and $M^{[\varphi]}(J) = J$ for every interval $J \subset I$.

Note that every weighted quasi-arithmetic mean is an increasing mean.

Remark 3 Let $I \subset \mathbb{R}$ be an interval and let $\varphi, \psi : I \to \mathbb{R}$ be continuous and strictly monotonic functions. Then $M_p^{[\psi]} = M_p^{[\varphi]}$ if and only if

$$\psi(x) = a_1\varphi(x) + a_2, \quad x \in I,$$

for some $a_1, a_2 \in \mathbb{R}$, $a_1 \neq 0$.

An important example of increasing mean, that is not a quasi-arithmetic, is the *logarithmic mean* $L : (0, \infty)^2 \to (0, \infty)$, defined by the formula

$$L(x,y) := \begin{cases} \frac{x-y}{\log x - \log y}, & x \neq y \\ x, & x = y \end{cases}. \tag{6}$$

This mean is a member of a broader class of the Lagrangean means which are increasing.

(b) Means which are not increasing

Let us now introduce the Beckenbach–Gini class of means.

Remark 4 Let $I \subset \mathbb{R}$ be an interval and let $f, g : I \to (0, \infty)$ be continuous functions such that $\frac{f}{g}$ is strictly monotonic. Then the function $M_{f,g} : I^2 \to \mathbb{R}$ defined by

$$M_{f,g}(x,y) := \left(\frac{f}{g}\right)^{-1}\left(\frac{f(x) + f(y)}{g(x) + g(y)}\right), \quad x, y \in I, \tag{7}$$

is a strict symmetric mean in I.

The function $M_{f,g}$ is called a Beckenbach–Gini mean of the generators f and g.

For $g(x) = 1$, $(x \in I)$, the $M_{f,g}$ becomes the quasi-arithmetic mean M_f. Taking $g(x) = x$, $f(x) = x^2$ for $x \in I$, we obtain the *contra-harmonic mean* $M : (0, \infty)^2 \to (0, \infty)$,

$$M(x,y) := \frac{x^2 + y^2}{x + y}, \quad x, y > 0. \tag{8}$$

It can be characterised as a unique mean which, together with the harmonic mean $H(x,y) = \frac{2xy}{x+y}$, forms a mean-type mapping (H, M) for which the arithmetic A is *invariant*, briefly, A is *(H, M)-invariant* (therefore sometimes M is called a A-complementary mean for H). Note that the contra-harmonic mean is not increasing.

We end this section with the following remark concerning invariant means as explained below.

Remark 5 Consider $K, L : I^2 \to I$ continuous strictly increasing means in I. There exists a unique continuous (K, L)-invariant mean $M : I^2 \to I$, i.e. $M \circ (K, L) = M$, and the sequence $(K, L)^n$, $n \in N$, of iterates of the mean-type mapping (K, L) is pointwise convergent in I^2 to (M, M), that is

$$\lim_{n\to\infty} (K, L)^n = (M, M) \text{ in } I^2.$$

In addition, M is strict if both K and L are strict (cf. Matkowski 1999, 2006).

Since the composition of increasing functions is an increasing function, the mean M is also increasing. Now, the functional equation

$$M(K(a(x), b(y)), L(a(x), b(y))) = R(\Phi(y) - \Phi(x)),$$
$$x, y \in [0, \infty),$$

reduces to the functional equation considered in the subsequent Corollary 1, due to the (K, L)-invariance of the mean M.

3 Theoretical results related to increasing mean-type functional equations

In this section we shall deal with functional equations of the type (4) assuming that M is an increasing mean. Theorems 1 and 2 find an answer for the general case under different regularity assumptions on the involving functions. Corollary 1 and Theorem 3 treat analogously two very important special cases giving rise to important conclusions. In particular, the case of quasi-arithmetic means will be emphasised.

Recall that all the proofs of the theoretical results following in the subsequent sections of this paper can be found in the Appendix.

Remark 6 Assume that some functions $a, b : \mathbb{R} \to I$, $\Phi : \mathbb{R} \to \mathbb{R}$, R and a mean M satisfy Eq. 4. If $\Phi(\mathbb{R})$, the range of Φ, contains a non-empty open interval, then the set

$$J_\Phi := \Phi(\mathbb{R}) - \Phi(\mathbb{R}) = \{u - v : u, v \in \Phi(\mathbb{R})\}$$

is a neighbourhood of 0 and is contained in the domain of the function R.

Theorem 1 *Let $I \subseteq \mathbb{R}$ be an interval, and $M : I^2 \to I$ be a strictly increasing mean. Let $\Phi : [0, \infty) \to \mathbb{R}$ be arbitrary and $R : \mathbb{R} \to \mathbb{R}$ an even function. Assume that one of the functions $a, b : [0, \infty) \to I$ is monotonic. If (4) holds, then a,b are constant on $[0, \infty)$ and so is R on the set $\Phi(\mathbb{R}) - \Phi(\mathbb{R})$.*

Theorem 2 *Let $I \subseteq \mathbb{R}$ be an interval, and $M : I^2 \to I$ be a strictly increasing mean. Let $\Phi : [0, \infty) \to \mathbb{R}$ and $R : \mathbb{R} \to \mathbb{R}$ be arbitrary functions. Assume that the functions $a, b : [0, \infty) \to I$ are monotonic in the same sense. If (4) holds, then a, b are constant on $[0, \infty)$ and so is R on the set $\{\Phi(y) - \Phi(x) : x, y \geq 0\}$.*

3.1 M is weighted quasi-arithmetic

Let us now consider Eq. 4, assuming $M := M_p^{[\varphi]}$, the weighted quasi-arithmetic mean as defined in Eq. 5.

As a consequence of Theorem 1, we get the following result.

Corollary 1 *Let $I \subset \mathbb{R}$ and let $\varphi : I \to \mathbb{R}$ be continuous and strictly monotonic. Assume that $\Phi : \mathbb{R} \to \mathbb{R}$ is an arbitrary bijection and $R : \mathbb{R} \to \mathbb{R}$ an even function. If the functions $a, b : [0, \infty) \to I$ satisfy the functional equation*

$$M_p^{[\varphi]}(a(x), b(y)) = R(\Phi(y) - \Phi(x)), \quad x, y \in [0, \infty), \tag{9}$$

then a, b and R are constant functions.

It should be stressed that in Theorem 1 and Corollary 1 we assume that the function R is even. Under this assumption the functions a and b have to be constant. Omitting this assumption, under some other conditions on the involving functions, we obtain non-trivial solutions as shown in subsequent Theorem 3. In the case when M is weighted quasi-arithmetic, and under some conditions, one can describe effective formulas for all functions satisfying Eq. 4.

Theorem 3 *Let $I \subset \mathbb{R}$ be an interval, let $\varphi : I \to \mathbb{R}$ be continuous and strictly monotonic and $p \in (0, 1)$. Assume that $\Phi : \mathbb{R} \to \mathbb{R}$ is continuous and non-constant. Then a continuous at least at one point function $R : (\Phi(\mathbb{R}) - \Phi(\mathbb{R})) \to \mathbb{R}$ and the functions $a, b : \mathbb{R} \to I$ satisfy the functional equation (9) if and only if there exist some $k, l \in \mathbb{R}$ such that*

$$a(x) = \varphi^{-1}(-l\Phi(x) + k) \quad b(x) = \varphi^{-1}(l\Phi(x) - k), \quad x \in \mathbb{R}, \tag{10}$$

and

$$R(u) = \varphi^{-1}\left(\frac{l}{2}u\right), \quad u \in \Phi(\mathbb{R}) - \Phi(\mathbb{R}).$$

Remark 7 As a consequence of the above result we get the invertibility of the function R. This is proved in the Appendix.

Remark 8 Let $I \subset \mathbb{R}$ be an interval such that $(0, 1] \subset I$, let $\varphi : I \to \mathbb{R}$ be a continuous and strictly monotonic function and $C_1 : \mathbb{R} \to (0, 1]$ be such that $C_1(0) = 1$. Applying Theorem 3 with $l = 1$, $k = 0$ and setting $a := C_1$ we get $C_1 = \varphi^{-1} \circ (-\Phi)$. It follows that $\Phi = -\varphi \circ C_1$ and, consequently, $b = \varphi^{-1} \circ \Phi = \varphi^{-1} \circ (-\varphi \circ C_1)$. Write $C_2 := \varphi^{-1} \circ (-\varphi \circ C_1)$. Thus $\Phi = \varphi \circ C_2$. Since

$$R(u) = \varphi^{-1}\left(\frac{l}{2}u\right), \quad u \in \Phi(\mathbb{R}) - \Phi(\mathbb{R}),$$

we hence get

$$\begin{aligned} R(\Phi(y) - \Phi(x)) &= \varphi^{-1}\left(\frac{l}{2}[\varphi(C_2(y)) + \varphi(C_1(x))]\right) \\ &= M_p^{[\varphi]}(C_1(x), C_2(y)) \end{aligned}$$

for all $x, y \in \mathbb{R}$.

3.2 M is the logarithmic mean

Let us consider the special case $M := L$ where L is the logarithmic mean as defined in Eq. 6.

Theorem 4 *Let L be the logarithmic mean as defined in Eq. 6. Suppose that $\Phi : \mathbb{R} \to \mathbb{R}$ is non-constant and continuous. Then a continuous function $R : (\Phi(\mathbb{R}) - \Phi(\mathbb{R})) \to (0, \infty)$ and the functions $a, b : \mathbb{R} \to (0, 1]$ such that*

$$a(0) = 1 = b(0)$$

satisfy the functional equation

$$L(a(x), b(y)) = R(\Phi(y) - \Phi(x)), \quad x, y \in \mathbb{R}, \tag{11}$$

if and only if

$$R(u) = 1, \quad u \in \Phi(\mathbb{R}) - \Phi(\mathbb{R}),$$

and

$$a(x) = b(x) = 1, \quad x \in \mathbb{R}.$$

4 Theoretical results related to mean-type functional equations for means which are not increasing

We have assumed in Sect. 3 that the basic mean M is increasing. In the case when M does not show this property, the functional equation (4) is more difficult to consider. It is known that some Gini means are not increasing; we begin this section with a special case of the Beckenbach–Gini mean.

4.1 M is the Beckenbach–Gini mean

Let $M_{f,g}$ be defined as in Eq. 7. Consider the functional equation

$$M_{f,g}(a(x),b(y)) = R(\Phi(y) - \Phi(x)), \quad x,y \in \mathbb{R},$$

where $a,b:\mathbb{R} \to I, \Phi:\mathbb{R} \to \mathbb{R}$ and $R:(\Phi(\mathbb{R}) - \Phi(\mathbb{R})) \to \mathbb{R}$ are unknown functions.

The means $M_{f,g}$, in general, are not increasing, and the above functional equation is more difficult to examine. To show this, in the next section we consider this equation in a special case when $f(x) = x^2$ and $g(x) = x$ for $x \in I$, i.e. when $M_{f,g}$ is the contra-harmonic mean. Note also that, by definition of $M_{f,g}$, and setting $\psi := \frac{f}{g} \circ R$, and $y = x$, we get

$$\frac{f(a(x)) + f(b(x))}{g(a(x)) + g(b(x))} = c, \quad x \in \mathbb{R},$$

for some $c \in \mathbb{R}$.

This problem is much more difficult to treat, even if we assume additionally that $f(x) = xg(x)$ and $g: I \to (0,\infty)$ is continuous. Define $M_g: I^2 \to \mathbb{R}$ by

$$M_g(x,y) := \frac{xg(x) + yg(y)}{g(x) + g(y)}, \quad x,y \in I. \tag{12}$$

Clearly, M_g is a special Beckenbach–Gini mean and, in general, it is not increasing.

Remark 9 Note that the arithmetic mean $\mathcal{A}(x,y) := \frac{x+y}{2}$, $x, y \in I$, is invariant with respect to the mean-type mapping $(M_g, M_{1/g})$, briefly $\mathcal{A}$ is $(M_g, M_{1/g})$-invariant, which means that

$$\mathcal{A} \circ (M_g, M_{1/g}) = \mathcal{A}.$$

In the sequel we assume that $(0,1] \subseteq I$.

Theorem 5 *Let $I \subset \mathbb{R}$ be an interval such that $(0,1] \subseteq I$ and let $g: I \to (0,\infty)$. Assume that $\Phi: \mathbb{R} \to \mathbb{R}$ is non-constant and continuous. Then a continuous at least at one point (or measurable) function $R:(\Phi(\mathbb{R}) - \Phi(\mathbb{R})) \to I$ and the functions $a,b:\mathbb{R} \to \mathbb{I}$ such that*

$$a(0) = 1 = b(0)$$

satisfy the functional equation

$$M_g(a(x),b(y)) + M_{1/g}(a(x),b(y)) = R(\Phi(y) - \Phi(x)), \quad x,y \in \mathbb{R}, \tag{13}$$

for M_g, $M_{1/g}$ as defined in Eq. 12, *if and only if there exists an $l \in \mathbb{R}$ such that*

$$R(x) = lx + 2, \quad x \in \Phi(\mathbb{R}) - \Phi(\mathbb{R}),$$

$$a(x) = -l\Phi(x) + 1, \quad b(x) = l\Phi(x) + 1, \quad x \in \mathbb{R}.$$

In the same way we can prove the following general result.

Theorem 6 *Let $I \subset \mathbb{R}$ be an interval such that $(0,1] \subseteq I$ and let $M, N: I^2 \to I$ be two means in I such that the arithmetic mean $\mathcal{A}$ is (M,N)-invariant. Assume that $\Phi: \mathbb{R} \to \mathbb{R}$ is non-constant and continuous. Then a continuous at least at one point (or measurable) function $R:(\Phi(\mathbb{R}) - \Phi(\mathbb{R})) \to I$ and the continuous functions $a,b:\mathbb{R} \to I$ such that*

$$a(0) = 1 = b(0)$$

satisfy the functional equation

$$M(a(x)),b(y)) + N(a(x),b(y)) = R(\Phi(y) - \Phi(x)), \quad x,y \in \mathbb{R}.$$

if and only if there is an $l \in \mathbb{R}$ such that

$$R(x) = lx + 2, \quad x \in \Phi(\mathbb{R}) - \Phi(\mathbb{R}),$$

$$a(x) = -l\Phi(x) + 1, \quad b(x) = l\Phi(x) + 1, \quad x \in \mathbb{R}.$$

4.2 A special case: M is the contra-harmonic mean

Theorem 7 *Let $I \subset \mathbb{R}$ be an interval such that $(0,1] \subseteq I$. Assume that $\Phi: \mathbb{R} \to \mathbb{R}$ is non-constant and continuous. Then a continuous function $R:(\Phi(\mathbb{R}) - \Phi(\mathbb{R})) \to I$ and the functions $a,b:\mathbb{R} \to I$ such that*

$$a(0) = 1 = b(0)$$

satisfy the functional equation (4), *for M a contra-harmonic mean if and only if,*

$$R(u) = 1, \quad u \in \Phi(\mathbb{R}) - \Phi(\mathbb{R}),$$

and

$$a(x) = b(x) = 1, \quad x \in \mathbb{R}.$$

5 Consequences for irreducible correlation functions

Let $a := C_1$ and $b := C_2$, for $C_1, C_2: \mathbb{R} \to (0,1]$ continuous and non-vanishing correlation functions. The theoretical results shown in the previous sections give an answer to the problem in Eq. 3, when $r := M \circ (C_1, C_2)$, for $r: \mathbb{R}^2 \to (0,1]$ a non-stationary correlation function. The solution can be postulated as follows:

Whenever $M \circ (C_1, C_2) : \mathbb{R} \times \mathbb{R} \to (0, 1]$ *is positive definite on* $\mathbb{R} \times \mathbb{R}$*, then it is irreducible, in the sense that there does not exist a non-trivial solution for the problem in* Eq. 3, *under the settings imposed in Theorems 1–4 and 5–7 and Corollary 1.*

Let us analyse the statistical consequences of the previous conclusion by making reference to the general results. In Theorems 1, 2, and Corollary 1, we found that the set of solutions associated to the problem (3) is empty, in the sense that if C_1, C_2 and R are constant functions, then they are not positive definite. Theorem 3 gives a non-trivial solution, but unfortunately one can easily see that, by Eq. 10, C_1, C_2 are mutually exclusive, in the sense that, if one of them is positive definite, then the other is not. Similar remarks apply to Theorems 4–7.

Some comments are in order. It should be stressed that permissibility criteria for a quasi-arithmetic correlation function have been found by Porcu et al. (2009). We conclude that this broad class is irreducible. Moreover, observe that this class of correlation functions includes as special cases two celebrated constructions, that are the linear combination and the tensorial product of correlation functions, respectively, $M_p^{[A]} = pC_1 + (1-p)C_2$ and $M_p^{[G]} := C_1^p C_2^{1-p}$, $p \in (0, 1)$. This construction, especially known in the geostatistical community, turns out to be irreducible.

Finally, observe that one can easily show that the extension to higher dimensional spaces works mutatis mutandis. It is sufficient to assume that $C_i : \mathbb{R}^d \to (0, 1]$, $i = 1, 2$ in Eq. 3 are motion and rotation invariant, that is isotropic, in the sense that they depend on their vector argument through its Euclidean distance.

6 Conclusions

Understanding when stationarity and isotropy are reasonable assumptions against non-stationarity in the context of spatial or space–time statistics is a key question in practical analysis. It is quite evident that these assumptions might not hold in large heterogeneous fields, which is usually the case. In this paper we have motivated the case when having a non-stationary random field we can transform or reduce it into a stationary entity. In particular we make use of mean operators.

We have found that mean-generated correlation functions are completely irreducible, in the sense that, for this broad class of correlation functions, there does not exist a non-trivial solution associated to the Perrin–Senoussi problem. Thus, random fields generated by mean-type correlation functions cannot be statistically treated with standard reduction techniques.

In this paper we have used functional equations theory to give a solution for a problem of reducibility, under the choice that the correlation function is generated by some mean operator.

A point of interest for future researches could be to inspect whether a function of the form of a quasi-arithmetic weighted mean composed with the correlation functions is permissible for a set of (possibly negative) weights. In this case a non-trivial solution to the problem of reducibility could be easily found. To this aspect we shall dedicate future researches.

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Appendix: Proofs

Proof of Theorem 1

Proof Setting $y = x$ into Eq. 4 we get

$$M(a(x), b(x)) = R(0), \quad x \geq 0. \tag{14}$$

Interchanging the roles of x and y in (4) and by the assumption that R is even we have

$$M(a(x), b(y)) = M(a(y), b(x)), \quad x, y \geq 0. \tag{15}$$

Assume that a is increasing on $[0, \infty)$. We shall show that b must be decreasing on the same interval. For an indirect proof, assume that it is not the case. So there would exist $x_1, x_2 \geq 0$ s.t. $x_1 < x_2$ and $b(x_1) < b(x_2)$. Now, applying in turn (14), the strict monotonicity of M w.r.t. the second variable, formula (15), and again the strict monotonicity of M w.r.t. the second variable, we obtain

$$\begin{aligned} R(0) &= M(a(x_1), b(x_1)) < M(a(x_1), b(x_2)) \\ &= M(a(x_2), b(x_1)) < M(a(x_2), b(x_2)) = R(0), \end{aligned}$$

which is a contradiction.

Now, to show that a is constant, assume that there exist some $0 \leq x_1 < x_2$ such that $a(x_1) < a(x_2)$. Now, applying in turn (14), the strict monotonicity of M w.r.t. the first variable, formula (15), and again the strict monotonicity of M w.r.t. the first variable, we obtain

$$\begin{aligned} R(0) &= M(a(x_1), b(x_1)) < M(a(x_2), b(x_1)) \\ &= M(a(x_1), b(x_2)) < M(a(x_2), b(x_2)) = R(0). \end{aligned}$$

This contradiction shows that a is constant in $[0, \infty)$. From Eq. 14, it follows that R is constant.

The case of a decreasing is omitted as it can be shown by the same arguments. Thus, the proof is completed. □

Proof of Theorem 2

Proof Setting $y = x$ we obtain Eq. 14. Assume that a, b are not-decreasing on $[0, \infty)$. Thus,

$$a(x_1) \le a(x_2) \quad \text{and} \quad b(x_1) \le b(x_2), \quad 0 \le x_1 \le x_2.$$

For an indirect proof, assume that there exist $x_1, x_2 \in [0, \infty)$, $x_1 \le x_2$, such that either $a(x_1) < a(x_2)$ or $b(x_1) < b(x_2)$. Now, applying in turn (14), the strict monotonicity of M w.r.t. both variables we get

$$R(0) = M(a(x_1), b(x_1)) < M(a(x_2), b(x_2)) = R(0),$$

which is a contradiction. Since the remaining statement is obvious, the proof is completed. □

Proof of Corollary 1

Proof Setting $y = x$ in Eq. 9 and by the definition of $M_p^{[\varphi]}$ we get

$$p\varphi(a(x)) + ((1-p)\varphi(b(x))) = R(0), \quad x \ge 0. \tag{16}$$

Interchanging the role of the variables x and y, and taking into account that R is even, we get

$$M_p^{[\varphi]}(a(x), b(y)) = M_p^{[\varphi]}(a(y), b(x)), \quad x, y \ge 0.$$

From the definition of $M_p^{[\varphi]}$, we can write this equality in the form

$$p\varphi(a(x)) + (1-p)\varphi(b(y)) = p\varphi(a(y)) + (1-p)\varphi(b(x)), \quad x, y \ge 0,$$

which implies that

$$p\varphi(a(x)) - (1-p)\varphi(b(x)) = p\varphi(a(y)) - (1-p)\varphi(b(y)), \quad x, y \ge 0,$$

which means that there exists a constant c such that

$$p\varphi(a(x)) - (1-p)\varphi(b(x)) = c, \quad x, y \ge 0. \tag{17}$$

Equations 16, 17 imply that $\varphi \circ a$ and $\varphi \circ b$ are constant in $[0, \infty)$. Since φ is one-to-one, it follows that a,b are constant in $[0, \infty)$. Thus, by Eq. 9 the function R is constant too, which completes the proof. □

Proof of Theorem 3

Proof By the definition of the quasi-arithmetic mean $M_p^{[\varphi]}$ we can write Eq. 9 in the form

$$p\varphi(a(x)) + (1-p)\varphi(b(y)) = \varphi(R(\Phi(y) - \Phi(x))), \quad x, y \in \mathbb{R}. \tag{18}$$

Setting here $y = x$ we get

$$p\varphi(a(x)) + (1-p)\varphi(b(x)) = \varphi((R(0)), \quad x \in \mathbb{R}.$$

Since the functions φ and $a_1\varphi + a_2$, $a_1 \ne 0$, generate the same quasi-arithmetic mean (cf. Remark 3), we can assume, without any loss of generality, that

$$\varphi(R(0)) = 0.$$

From this equation we get

$$(1-p)\varphi(b(x)) = -p\varphi(a(x)), \quad x \in \mathbb{R},$$

and setting

$$\psi := \varphi \circ R \tag{19}$$

we can write (18) in the form

$$p(\varphi(a(x)) - \varphi(a(y))) = \psi(\Phi(y) - \Phi(x)), \quad x, y \in \mathbb{R}. \tag{20}$$

We can also assume, without any loss of generality, that $\Phi(0) = 0$,

and setting $y = 0$ in (20) we get

$$p(\varphi(a(x)) - \varphi(a(0))) = \psi(-\Phi(x)), \quad x \in \mathbb{R},$$

whence

$$\varphi(a(x)) = \frac{\psi(-\Phi(x)) + k}{p}, \quad x \in \mathbb{R}, \tag{21}$$

where

$$k = p\varphi(a(0)).$$

Now, from (20) and (21), we get

$$\psi(\Phi(y) - \Phi(x)) = \psi(-\Phi(x)) - \psi(-\Phi(y)), \quad x, y \in \mathbb{R}.$$

Since, by assumption, the function Φ is continuous and non-constant, its range $\Phi(\mathbb{R})$ is a non-trivial interval containing 0. Thus

$$\psi(v-u) = \psi(-u) - \psi(-v), \quad u, v \in \Phi(\mathbb{R}). \tag{22}$$

Setting here $u = v = 0$, we obtain $\psi(0) = 0$. Hence, setting $u = 0$ in (22), we obtain

$$\psi(v) = -\psi(-v), \quad v \in \Phi(\mathbb{R}).$$

It follows that

$$\psi(v-u) = \psi(v) - \psi(u), \quad u, v \in \Phi(\mathbb{R}),$$

that is ψ is additive. Since R is continuous at least at one-point, we infer that so is ψ and, consequently, ψ is a linear function (cf. Aczél 1966; Kuczma 1985), that is

$$\psi(u) = lu, \quad u \in \Phi(\mathbb{R}) - \Phi(\mathbb{R}),$$

for some $l \in \mathbb{R}$, $l \neq 0$. From (19) we have

$$R(u) = \varphi^{-1}(lu), \quad u \in \Phi(\mathbb{R}) - \Phi(\mathbb{R}). \tag{23}$$

From (21) we get

$$a(x) = \varphi^{-1}(-l\Phi(x) + k), \quad x \in \mathbb{R}, \tag{24}$$

and from (18)

$$b(x) = \varphi^{-1}(l\Phi(x) - k), \quad x \in \mathbb{R}. \tag{25}$$

As it can be easily checked that the functions a, b, R given by the formulas (23), (24) and (25) satisfy Eq. 9, and thus the proof is complete. □

Proof of Remark 7

Proof Set, without loss of generality, $p = 1/2$ in Eq. 5. Note that the invertibility of Φ would be too restrictive. Indeed, assuming the invertibility of Φ, Eq. 9 could be written in the form

$$M_p^{[\varphi]}(a(\Phi^{-1}(u)), b(\Phi^{-1}(v))) = R(v-u), \quad u, v \in \Phi(\mathbb{R}) \tag{26}$$

whence

$$\varphi(a(\Phi^{-1}(u))) + \varphi(b(\Phi^{-1}(v))) = 2\varphi(R(v-u)), \quad u, v \in \Phi(\mathbb{R}). \tag{27}$$

Setting

$$\gamma := 2\varphi \circ R \tag{28}$$

we can write this equation in the form

$$\varphi(a(\Phi^{-1}(u))) + \varphi(b(\Phi^{-1}(v))) = \gamma(v-u), \quad u, v \in \Phi(\mathbb{R}).$$

Setting $v := u$ we get

$$\varphi(b(\Phi^{-1}(u))) = c - \varphi(a(\Phi^{-1}(u))), \quad u \in \Phi(\mathbb{R}), \tag{29}$$

where

$$c := \gamma(0).$$

Hence, by (29), we get

$$\varphi(a(\Phi^{-1}(u))) + c - \varphi(a(\Phi^{-1}(v))) = \gamma(v-u), \quad u, v \in \Phi(\mathbb{R}),$$

or, equivalently,

$$[\varphi(a(\Phi^{-1}(u))) - c] - [\varphi(a(\Phi^{-1}(v))) - c] = \gamma(v-u) - c, \quad u, v \in \Phi(\mathbb{R}).$$

Setting

$$\alpha(u) := \varphi(a(\Phi^{-1}(u))) - c, \quad u \in \Phi(\mathbb{R}); \qquad \beta(u) := \gamma(-u) - c, \quad u \in \Phi(\mathbb{R}) - \Phi(\mathbb{R}), \tag{30}$$

we can write this equation in the form

$$\alpha(u) - \alpha(v) = \beta(u-v), \quad u, v \in \Phi(\mathbb{R}).$$

It follows that

$$\alpha(u+v) = \alpha(u) + \beta(v), \quad u \in \Phi(\mathbb{R}), \quad v \in \Phi(\mathbb{R}) - \Phi(\mathbb{R}).$$

Assume that Φ is continuous. Then $\Phi(\mathbb{R})$ is an interval and $\Phi(\mathbb{R}) - \Phi(\mathbb{R})$ a symmetric interval with respect to 0.

Assume that the function a is continuous. Now one can prove that

$$\alpha(u) = a_1 u + a_2 \quad \text{for } u \in \Phi(\mathbb{R}), \quad \text{and} \quad \beta(u) = a_1 u \quad \text{for } u \in \Phi(\mathbb{R}) - \Phi(\mathbb{R}),$$

for some $a_1, a_2 \in \mathbb{R}$, $a_1 \neq 0$. From (30) and (29) we obtain

$$\varphi(a(\Phi^{-1}(u))) = a_1 u + a_2 + c \quad \text{for } u \in \Phi(\mathbb{R});$$
$$R(u) = \varphi^{-1}\left(-\frac{a_1}{2}u + c\right) \quad \text{for } u \in \Phi(\mathbb{R}) - \Phi(\mathbb{R}),$$

whence

$$a(x) = \varphi^{-1}(a_1\Phi(x) + a_2 + c) \quad \text{for } x \in \mathbb{R},$$
$$\varphi(u) = R^{-1}\left(\frac{-a_1 u + c}{2}\right) \quad \text{for } u \in \Phi(\mathbb{R}) - \Phi(\mathbb{R}).$$

which shows that a is invertible. Here a stands for C_1 which, by assumption, is not a one-to-one function. □

Proof of Theorem 4

Proof Assume that some functions R, Φ, a and b satisfy Eq. 11. For $y = x$ we have

$$L(a(x), b(x)) = R(0), \quad x \in (0, \infty).$$

Since $a(0) = 1 = b(0)$, we hence get $R(0) = 1$, whence

$$L(a(x), b(x)) = 1, \quad x \in (0, \infty).$$

Assume that there is an $x \in \mathbb{R}$ such that $a(x) \neq b(x)$. Then

$$\frac{a(x) - b(x)}{\log a(x) - \log b(x)} = 1,$$

that is

$$\log a(x) - a(x) = \log b(x) - b(x).$$

Since the function $u \to \log u - u$ is strictly increasing in (0, 1], we would get $a(x) = b(x)$, which is a contradiction. □

Proof of Theorem 5

Proof By the definition of contra-harmonic mean M in Eq. 8, and setting $y = x$ in (9), we get

$$\frac{a(x)^2+b(x)^2}{a(x)+b(x)}=R(0),\quad x\in\mathbb{R}.$$

Since, by assumption, $a(0)=1=b(0)$, we hence get $R(0)=1$.

Consequently,

$$\frac{a(x)^2+b(x)^2}{a(x)+b(x)}=1,\quad x\in\mathbb{R}. \tag{31}$$

Without any loss of generality we can assume that $\Phi(0)=0$. Setting $y=0$ in (8) gives

$$\frac{a(x)^2+1}{a(x)+1}=R(-\Phi(x)),\quad x\in\mathbb{R},$$

whence

$$a(x)=\frac{1}{2}(R(-\Phi(x))+\kappa_a K(-\Phi(x))), \tag{32}$$

where

$$K(u):=\sqrt{[R(u)]^2+4R(u)-4}$$

and $\kappa_a=1$ or $\kappa_a=-1$.

Similarly, setting $x=0$ in (8) and then replacing y by x gives

$$\frac{b(x)^2+1}{b(x)+1}=R(\Phi(x)),\quad x\in\mathbb{R},$$

whence

$$b(x)=\frac{1}{2}(R(\Phi(x))+\kappa_b K(\Phi(x))), \tag{33}$$

where $\kappa_b=1$ or $\kappa_b=-1$.

Making use of relation (32), we obtain from Eq. 8

$$\frac{(R(-\Phi(x))+\kappa_a K(-\Phi(x)))^2+(R(\Phi(y))+\kappa_b K(\Phi(y)))^2}{R(-\Phi(x))+\kappa_a K(-\Phi(x))+R(\Phi(y))+\kappa_b K(\Phi(y))}=2R(\Phi(y)-\Phi(x)),\quad x,y\in\mathbb{R}. \tag{34}$$

Substituting the right-hand sides of (32) and (33) into Eq. 31 we get, after obvious simplification,

$$[R(-\Phi(x))]^2+R(-\Phi(x))+[R(\Phi(x))]^2+R(\Phi(x))=4,\quad x\in\mathbb{R}. \tag{35}$$

Assume that $\kappa_a=\kappa_b$. Interchanging x and y in (34) we see that the left-hand side does not change. It follows that the function is even on its range, whence, by (35),

$$[R(\Phi(x))]^2+R(\Phi(x))=2,\quad x\in\mathbb{R}.$$

Since $R\circ\Phi$ is continuous and $R\circ\Phi(0)=1$, it follows that R is a constant function equal to 1.

Thus, if $R\neq 1$ then $\kappa_a\kappa_b=-1$. Assuming (without any loss of generality) that $\kappa_a=1$ and $\kappa_b=-1$, we get from (34)

$$\frac{(R(-\Phi(x))+K(-\Phi(x)))^2+(R(\Phi(y))-K(\Phi(y)))^2}{R(-\Phi(x))+K(-\Phi(x))+R(\Phi(y))-K(\Phi(y))}=2R(\Phi(y)-\Phi(x)),$$
$$x,y\in\mathbb{R}.$$

Putting here $u:=\Phi(x)$ and $v:=\Phi(y)$ we hence get

$$\frac{(R(-u)+K(-u))^2+(R(v)-K(v))^2}{R(-u)+K(-u)+R(v)-K(v)}=2R(v-u),$$
$$u,v\in\Phi(\mathbb{R}).$$

Replacing $-u$ by u we see that Φ satisfies the functional equation

$$\frac{(R(u)+K(u))^2+(R(v)-K(v))^2}{R(u)+K(u)+R(v)-K(v)}=2R(u+v),$$
$$u,v\in\Phi(\mathbb{R}).$$

Since $R((u+v)+w)=R(u+(v+w))$, we hence get

$$\frac{(R(u+v)+K(u+v))^2+(R(w)-K(w))^2}{R(u+v)+K(u+v)+R(w)-K(w)}=\frac{(R(u)+K(u))^2+(R(v+w)-K(v+w))^2}{R(u)+K(u)+R(v+w)-K(v+w)}$$

for all $u,v,w,u+v,v+w\in\Phi(\mathbb{R})$. Setting here $u=w=0$ and taking into account that $R(0)=1$, we hence get

$$\frac{(R(v)+K(v))^2}{R(v)+K(v)}=\frac{2^2+(R(v)-K(v))^2}{2+R(v)-K(v)}$$

whence, for all $v\in\Phi(\mathbb{R})$,

$$(R(v)+K(v))(2+R(v+w)-K(v))=2^2+(R(v)-K(u+v))^2,$$

which reduces to the equation

$$(R(v)+1)\sqrt{[R(v)]^2+4R(v)-4}=[R(v)]^2+3R(v)-2,$$
$$v\in\Phi(\mathbb{R}).$$

Taking the power two of both sides we get

$$R(v)=1,\quad v\in\Phi(\mathbb{R}).$$

Now from (8) we get

$$\frac{a(x)^2+b(y)^2}{a(x)+b(y)}=1,\quad x,y\in\mathbb{R},$$

which implies that $a(x)=b(x)=1$ for all $x\in\mathbb{R}$. This completes the proof. □

Proof of Theorem 6

Proof By Remark 9, the arithmetic mean is $(M_g, M_{1/g})$-invariant. Therefore Eq. 13 is equivalent to the equation

$$a(x) + b(y) = R(\Phi(y) - \Phi(x)), \quad x, y \in \mathbb{R}. \tag{36}$$

Putting here $y = x$ we get

$$a(x) + b(x) = R(0), \quad x \in \mathbb{R},$$

so the left-hand side is a constant function. Since, by assumption, $a(0) + b(0) = 2$, we hence get

$$R(0) = 2$$

and, consequently,

$$b(x) = 2 - a(x), \quad x \in \mathbb{R}. \tag{37}$$

Now from (36) we get

$$a(x) + 2 - a(y) = R(\Phi(y) - \Phi(x)), \quad x, y \in \mathbb{R}.$$

Without any loss of generality we can assume that $\Phi(0) = 0$ as, in the opposite case, we could replace Φ by $\Phi - \Phi(0)$. Setting in this equation separately $y = 0$ and $x = 0$, we obtain

$$a(x) + 1 = R(-\Phi(x)), \quad x \in \mathbb{R}, \tag{38}$$

and

$$3 - a(y) = R(\Phi(y)), \quad y \in \mathbb{R},$$

whence, by (37),

$$R(-\Phi(x)) - 2 + R(\Phi(y)) = R(\Phi(y) - \Phi(x)), \quad x, y \in \mathbb{R}.$$

Thus the function $R - 2$ is additive on the set $\Phi(\mathbb{R}) - \Phi(\mathbb{R})$ which contains a neighbourhood of 0. It follows that there exists an $l \in \mathbb{R}$ such that

$$R(x) = lx + 2, \quad x \in \Phi(\mathbb{R}) - \Phi(\mathbb{R}).$$

From (38) and (37) we get

$$a(x) = -l\Phi(x) + 1, \quad b(x) = l\Phi(x) + 1, \quad x \in \mathbb{R}.$$

□

Proof of Theorem 7

Omitted.

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