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UNIFORMLY CONTINUOUS COMPOSITION OPERATOR IN THE SPACE OF φ–VARIATION FUNCTIONS IN THE SENSE OF RIESZ

Abstract. Assuming that a Nemytskii operator maps a subset of the space of bounded variation functions in the sense of Riesz into another space of the same type, and is uniformly continuous, we prove that the generator of the operator is an affine function.

Key words: φ–variation in the sense of Riesz, uniformly continuous composition operator, Jensen equation.

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1. Introduction

Let I be an interval of $\mathbb{R}$, $(X, |\cdot|)$ a real normed space, C a closed subset of X, $(Y, |\cdot|)$ a real Banach space and $h : I \times C \to Y$ a given function. Denote by X^I the set of all functions $f : I \to X$ and by $H : X^I \to Y^I$ the (Nemytskii or superposition) composition operator generated by the function h defined by

$$H(f)(\cdot) = h(\cdot, f(\cdot)).$$

In this paper, we prove that if H maps some subsets of the space $RV_\varphi(I, C)$ of functions of bounded φ–variation in the sense of Riesz into space $RV_\psi(I, Y)$, and is uniformly continuous, then h, the generator function of the operator H, is an affine function of the second variable.

This generalizes the results of Chistyakov [3], and Merentes [12], where it is assumed that H is Lipschizian. The uniformly continuous composition operators were first considered in [7] for the space of differentiable functions and absolutely continuous functions, later in [8] for the space of Hölder function, and in [9] for the space of bounded variation functions.

2. Auxiliary results

By $\mathcal{F}$ we denote the family of all continuous convex functions $\varphi: [0,+\infty) \to [0,+\infty)$ such that

$$\varphi(0)=0, \quad \varphi(t)>0 \ \text{ for } \ t>0; \quad \lim_{t\to\infty}\frac{\varphi(t)}{t}=+\infty.$$

Obviously, every $\varphi\in\mathcal{F}$ is strictly increasing.

In the sequel $\varphi\in\mathcal{F}$ is fixed.

Let $I=[a,b]\subset\mathbb{R}$ be an interval. By $\mathcal{P}(I)$ we denote the family of all partitions τ of the interval I; i.e. $\tau\in\mathcal{P}(I)$ if and only if, $\tau=(\tau_i)_{i=1}^m$ for some $m\in\mathbb{N}$, and

$$a=\tau_0<\tau_1<\cdots<\tau_m=b.$$

For $f\in X^I$ and $\tau\in\mathcal{P}(I)$ we put

$$RV_\varphi(f,\tau):=\sum_{i=1}^m\varphi\left(\frac{|f(\tau_i)-f(\tau_{i-1})|}{\tau_i-\tau_{i-1}}\right)(\tau_i-\tau_{i-1}),$$

and we define

$$RV_\varphi(f):=\sup\left\{RV_\varphi(f,\tau):\ \tau\in\mathcal{P}(I)\right\}$$

that is called the Riesz φ–variation of f in I.

If $RV_\varphi(f)<\infty$, we say that f has a bounded Riesz φ–variation on I.

The set

$$RV_\varphi^*(f):=\left\{f\in X^I:\ RV_\varphi(f)<+\infty\right\}$$

is convex (cf. [2]) if φ is convex, but, in general, it is not a linear space. It is known (cf. [3]) that

$$RV_\varphi(I,X):=\left\{f\in X^I:\ \exists\lambda>0\ RV_\varphi(\lambda f)<+\infty\right\}$$

is a linear space if and only if φ satisfies the Δ_2 condition, i.e. there exists a constant $\rho_0\geq 0$ and $C>0$ such that $\varphi(2\rho)\leq C\varphi(\rho)$ for all $\rho\geq\rho_0$ and also it is a normed space with the norm

$$\|f\|_\varphi:=|f(a)|+\mathrm{p}_\varphi(f), \qquad f\in RV_\varphi(I,X),$$

where

$$\mathrm{p}_\varphi(f):=\inf\left\{\epsilon>0:\ RV_\varphi\Big(\frac{f}{\epsilon}\Big)\leq 1\right\}.$$

Moreover, if $\Big(X,|\cdot|\Big)$ is a Banach space, then so is $\Big(RV_\varphi(I,X),\ \|\cdot\|_\varphi\Big)$.

Remark 1. If $f \in RV_\varphi(I, X)$, then f is continuous in I. It is a consequence of Lemma 2.1 (d) in Chistyakov [3].

We will need the following property of $\mathbf{p}_\varphi$.

Lemma 1 (cf. Chistyakov, [4], Lemma 3.4). *Let $\varphi \in \mathcal{F}$ and $f \in RV_\varphi^*(I, X)$.*
If $r > 0$, then $V_\varphi(f/r) \le 1$ if and only if $\mathbf{p}_\varphi(f) \le r$.

3. Main result

For a set $C \subset X$ we put

$$RV_\varphi(I, C) := \left\{ f \in RV_\varphi(I, X) : \ f(I) \subset C \right\}.$$

By $\mathcal{A}(X, Y)$ denote the space of all additive mappings $A : X \to Y$, and by $\mathcal{L}(X, Y)$ denote the space of all linear mappings from X into Y.

The main result reads as follows.

Theorem 1. *Let $I = [a, b]$ and $\varphi, \psi \in \mathcal{F}$. Suppose that $(X, |\cdot|)$ is a linear real normed space, $(Y, |\cdot|)$ is a real Banach space, $C \subset X$ is a closed and convex set, and $h : I \times C \to Y$ is a function. If the composition operator H given by*

$$H(f)(t) := h(t, f(t)), \quad t \in I, \ f \in X^I,$$

maps the set $RV_\varphi(I, C)$ into $RV_\psi(I, Y)$ and H is uniformly continuous, then there are the functions $A : I \to \mathcal{A}(X, Y)$ and $B \in RV_\psi(I, Y)$ such that

$$h(t, x) = A(t)x + B(t), \quad t \in I, \ x \in C.$$

Moreover, if $0 \in C$ and $Int(C) \neq \emptyset$, then $A : I \to \mathcal{L}(X, Y)$ and $B \in RV_\psi(I, Y)$.

Proof. For every $x \in C$, the constant function $I \ni t \mapsto x$ belongs to $RV_\varphi(I, C)$. Since the Nemytskii operator H maps the space $RV_\varphi(I, C)$ into $RV_\psi(I, Y)$, it follows that for every $x \in C$ the function $h(\cdot, x)$ belongs to $RV_\psi(I, Y)$.

The uniform continuity of H on $RV_\varphi(I, C)$ implies that

$$\|H(f_1) - H(f_2)\|_\psi \le \omega(\|f_1 - f_2\|_\varphi) \quad \text{for} \quad f_1, f_2 \in RV_\varphi(I, C),$$

where $\omega : \mathbb{R}_+ \to \mathbb{R}_+$ is the modulus continuity of H, i.e.

$$\omega(\rho) := \sup\left\{ \|H(f_1) - H(f_2)\|_\psi : \|f_1 - f_2\|_\varphi \le \rho; \ f_1, \ f_2 \in RV_\varphi(I, C) \right\} \quad \text{for } \rho > 0.$$

By the definition of the norm $\|\cdot\|_\psi$, we obtain

$$\mathbf{p}_\psi(H(f_1)-H(f_2)) \le \|H(f_1)-H(f_2)\|_\psi \quad \text{for} \quad f_1, f_2 \in RV_\varphi(I,C).$$

Hence, in view of Lemma 1,

$$RV_\psi\left(\frac{H(f_1)-H(f_2)}{\omega(\|f_1-f_2\|_\varphi)}\right) \le 1 \quad \text{if} \quad \omega(\|f_1-f_2\|_\varphi) > 0.$$

Therefore, by the definitions of $RV_\psi(\cdot)$ and H, for any $f_1, f_2 \in RV_\varphi(I,C)$ and $\alpha, \beta \in I$, $\alpha < \beta$, we get

$$\psi\left(\frac{|h(\beta, f_1(\beta)) - h(\beta, f_2(\beta)) - h(\alpha, f_1(\alpha)) + h(\alpha, f_2(\alpha))|}{\omega(|f_1-f_2|)(\beta-\alpha)}\right) \le \frac{1}{\beta-\alpha},$$

whence, by taking the inverse function ψ^{-1} from both sides, we obtain

$$\begin{aligned}(1) \qquad &|h(\beta, f_1(\beta)) - h(\beta, f_2(\beta)) - h(\alpha, f_1(\alpha)) + h(\alpha, f_2(\alpha))| \\ &\qquad \le \omega(|f_1-f_2|)(\beta-\alpha)\psi^{-1}(1/(\beta-\alpha)).\end{aligned}$$

For $\alpha, \beta \in [a,b]$, $\alpha < \beta$, define the function $\eta_{\alpha,\beta} : \mathbb{R} \to [0,1]$ by

$$\eta_{\alpha,\beta}(t) := \begin{cases} 0 & \text{if } \ t \le \alpha \\ \dfrac{t-\alpha}{\beta-\alpha} & \text{if } \ \alpha \le t \le \beta \\ 1 & \text{if } \ \beta \le t. \end{cases}$$

Let $x_1, x_2 \in C$, $x_1 \ne x_2$. Note that the functions $f_1, f_2 : I \to X$ given by

$$(2) \qquad f_j(t) := \frac{1}{2}\left[\eta_{\alpha,\beta}(t)(x_1-x_2) + x_j + x_2\right], \quad t \in I, \ \ j = 1, 2,$$

belong to $RV_\varphi(I,C)$,

$$f_1(\beta) = x_1, \quad f_2(\beta) = \frac{x_1+x_2}{2}, \quad f_1(\alpha) = \frac{x_1+x_2}{2}, \quad f_2(\alpha) = x_2,$$

and, since

$$f_1(t) - f_2(t) = \frac{x_1-x_2}{2}, \quad t \in I,$$

we have

$$\|f_1-f_2\|_\varphi = \frac{|x_1-x_2|}{2}.$$

Substituting the functions f_1, f_2 in (1), we obtain

$$\begin{aligned}(3) \qquad &\left|h(\beta, x_1) - h\left(\beta, \frac{x_1+x_2}{2}\right) - h\left(\alpha, \frac{x_1+x_2}{2}\right) + h(\alpha, x_2)\right| \\ &\qquad \le \omega\left(\frac{|x_1-x_2|}{2}\right)(\beta-\alpha)\psi^{-1}(1/(\beta-\alpha)).\end{aligned}$$

Since for any $x \in C$ the constant function $t \mapsto x \quad (t \in I)$ belongs to $RV_\varphi(I,C)$ and H maps $RV_\varphi(I,C)$ into $RV_\psi(I,Y)$, the function $h(\cdot,x)$, belongs to $RV_\psi(I,Y)$ for any $x \in C$. Since $\psi \in \mathcal{F}$, we have

$$\lim_{\rho\to\infty} \frac{\rho}{\psi(\rho)} = \lim_{r\to 0} r\psi^{-1}\Big(\frac{1}{r}\Big) = 0.$$

Hence

$$\lim_{\beta-\alpha\to 0} (\beta-\alpha)\psi^{-1}(1/(\beta-\alpha)) = 0.$$

Now take $t \in I$ and $\alpha \le t \le \beta$, $\alpha < \beta$, $\alpha, \beta \in I$. Letting $\beta - \alpha$ tend to zero in (3), and making use of the continuity of the function $h(\cdot,x)$ for any $x \in C$ (cf. *Remark 1*), we get

$$h\Big(t, \frac{x_1+x_2}{2}\Big) = \frac{h(t,x_1)+h(t,x_2)}{2},$$

for all $t \in I$ and $x_1, x_2 \in C$.

Thus, for each $t \in I$ the function $h(t,\cdot)$ satisfies the *Jensen functional* equation in C. Hence, by the standard argument (cf. Kuczma [5]), we conclude that there exist an additive function $A(t) : X \longrightarrow Y$ and $B(t) \in Y$ such that

$$h(t,x) = A(t)x + B(t), \quad t \in I, \quad x \in C$$

which finishes the proof of the first part of our result.

Since $0 \in C$, the constant zero function belongs to $RV_\varphi(I,C)$. Setting this function in the just proved formula and taking into account that H maps $RV_\varphi(I,C)$ into $RV_\psi(I,Y)$, we infer that $H(0) = h(\cdot,0) = B$ belongs to $RV_\psi(I,Y)$. The uniform continuity of operator $H : RV_\varphi(I,C) \longrightarrow RV_\psi(I,C)$ implies the continuity of the additive function $A(t)$ for $t \in I$. Consequently, $A(t) \in \mathcal{L}(X,Y)$ for each $t \in I$. This completes of proof. ∎

Remark 2. In the proof of the theorem we apply the uniform continuity of the operator H only on the set $Z \subset RV_\varphi(I,C)$ such that $f \in Z$ if there are $\alpha, \beta \in I$, $\alpha < \beta$ such that

$$f(t) = \frac{1}{2}\big[\eta_{\alpha,\beta}(t)(x_1 - x_2) + x + x_2\big], \quad t \in I,$$

where $\eta_{\alpha,\beta}$ is defined by (2), $x_1, x_2 \in C$ and $x = x_1$ or $x = x_2$.

Thus the assumption of the uniform continuity of H on $RV_\varphi(I,C)$ in the theorem can be replaced by a weaker condition of the uniform continuity of H on Z.

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