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Uniformly continuous composition operators in the space of bounded φ-variation functions[☆]

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ABSTRACT

We prove that, under some general assumptions, a generator of any uniformly continuous Nemytskii operator, mapping a subset of space of bounded variation functions in the sense of Wiener into another space of this type, must be an affine function. As a special case, we obtain an earlier result from Matkowski (in press) [4].

1. Introduction

Let X, Y be real normed spaces and C be a closed convex subset of X. For a fixed real interval I denote by X^I (or Y^I) the set of all functions $f : I \longrightarrow X$ (or $f : I \longrightarrow Y$). If $h : I \times C \longrightarrow Y$ is a given function, then the operator $H : X^I \longrightarrow Y^I$ defined by the formula

$$(Hf)(t) = h(t, f(t)), \quad t \in I \tag{1}$$

is called the Nemytskii composition operator generated by the function h.

Let $(BV_\varphi(I, X), \|\cdot\|_\varphi)$ be the Banach space of functions $f : I \to X$ which are of bounded φ-variation in the sense of Wiener, where the norm $\|\cdot\|_\varphi$ is defined with the aid of Luxemburg–Nakano–Orlicz seminorm [1–3].

Assume that H maps the set of functions $f \in BV_\varphi(I, X)$ such that $f(I) \subset C$ into $BV_\varphi(I, Y)$. In the present paper, we prove that, if H is uniformly continuous, then the left and right regularization of its generator h with respect for the first variable are affine functions in the second variable. This extends the main result of paper [4].

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2. Preliminaries

In this section we present some definitions and preliminary results related with bounded φ-variation functions in the sense of Wiener.

Let $\mathcal{F}$ be the set of all convex functions $\varphi : [0, \infty) \longrightarrow [0, \infty)$ such that: $\varphi(0^+) = 0$ and $\lim_{t\to\infty} \varphi(t) = \infty$. Then we have that

Remark 2.1. If $\varphi \in \mathcal{F}$, then φ is continuous and strictly increasing. Indeed, the continuity of φ at each point $t > 0$ follows from its convexity and continuity at 0 from the assumption $\varphi(0) = \varphi(0^+) = 0$. Suppose that $\varphi(t_1) \geq \varphi(t_2)$ for some $0 < t_1 < t_2$. Then

$$\frac{\varphi(t_1) - \varphi(0)}{t_1 - 0} = \frac{\varphi(t_1)}{t_1} > \frac{\varphi(t_2)}{t_2} = \frac{\varphi(t_2) - \varphi(0)}{t_2 - 0},$$

contradicting the convexity of φ.

Definition 2.2. Let $\varphi \in \mathcal{F}$ and $(X, |\cdot|)$ be a real normed space. A function $f \in X^I$ is of bounded φ-variation in the sense of Wiener in I, if

$$v_\varphi(f) = v_\varphi(f, I) := \sup_\xi \sum_{i=1}^{m} \varphi\big(|f(t_i) - f(t_{i-1})|\big) < \infty, \tag{2}$$

where the supremum is taken over all increasing finite sequences $\xi = (t_i)_{i=0}^m$, $t_i \in I$, $m \in \mathbb{N}$.

For $\varphi(t) = t^p$ ($t \geq 0$, $p \geq 1$), condition (2) coincides with the classical concept of variation in the sense of Jordan [5, Chapter 8] whenever $p = 1$, and in the sense of Wiener [6] if $p > 1$. The general Definition 2.2 was introduced by Young [7].

It is known that for all $a, b, c \in I$, $a \leq c \leq b$ we have $v_\varphi(f, [a, c]) \leq v_\varphi(f, [a, b])$ (that is, v_φ is increasing with respect to the interval) and $v_\varphi(f, [a, c]) + v_\varphi(f, [c, b]) \leq v_\varphi(f, [a, b])$.

We will denote by $V_\varphi(I, X)$ the set of all functions $f \in X^I$ with bounded φ-variation in Wiener sense. This is a symmetric and convex set; but it is not necessarily a linear space. In fact, Musielak and Orlicz proved the following statement: this class of functions is a vector space if, and only if, φ satisfies the δ_2 condition [8]. We denote by $BV_\varphi(I, X)$ the linear space of all functions $f \in X^I$ such that $v_\varphi(\lambda f) < \infty$ for some constant $\lambda > 0$.

In the linear space $BV_\varphi(I, X)$, the function $\|\cdot\|_\varphi$ defined by

$$\|f\|_\varphi := |f(a)| + p_\varphi(f), \quad f \in BV_\varphi(I, X),$$

where

$$p_\varphi(f) := p_\varphi(f, I) = \inf\left\{\epsilon > 0 : v_\varphi(f/\epsilon) \leq 1\right\}, \quad f \in BV_\varphi(I, X), \tag{3}$$

is a norm (see for instance [8]).

For $X = \mathbb{R}$, the linear normed space $(BV_\varphi(I, \mathbb{R}), \|\cdot\|_\varphi)$ was studied by Musielak and Orlicz [8], Ciemnoczołowski and Orlicz [9], and Maligranda and Orlicz [10]. In particular, it is shown in [10] that the space $(BV_\varphi(I, \mathbb{R}), \|\cdot\|_\varphi)$ is a Banach algebra. The functional $p_\varphi(\cdot)$ defined by (3) is called *the Luxemburg–Nakano–Orlicz seminorm* [1–3].

In what follows, the symbol $BV_\varphi(I, C)$ stands for the set of all functions $f \in BV_\varphi(I, X)$ such that $f : I \longrightarrow C$ and C is a subset of X.

Lemma 2.3 (*Chistyakov [11, Lemma 1]*). *For $f \in BV_\varphi(I, X)$, we have:*

(a) *if $t, t' \in I$, then $\|f(t) - f(t')\| \leq \varphi^{-1}(1)p_\varphi(f)$;*
(b) *if $p_\varphi(f) > 0$ then $v_\varphi(f/p_\varphi(f)) \leq 1$;*
(c) *for $\lambda > 0$,*
 (c1) *$p_\varphi(f) \leq \lambda$ if and only if $v_\varphi(f/\lambda) \leq 1$;*
 (c2) *if $v_\varphi(f/\lambda) = 1$ then $p_\varphi(f) = \lambda$.* ∎

Property (a) in Lemma 2.3 implies that any function $f \in BV_\varphi(I, X)$ is bounded. Indeed, we have $\|f\| \leq \|f(a)\| + \|f(t) - f(a)\|$, whence

$$\|f\|_\infty \leq \|f(a)\| + \varphi^{-1}(1)p_\varphi(f) < \infty.$$

If $(X, |\cdot|)$ is a Banach space and $f \in BV_\varphi(I, X)$, then

$$f^-(t) := \lim_{s\uparrow t} f(s), \quad t \in I^-,$$

exists and is called the *left regularization* of f [12].

Let $BV_\varphi^-(I, X)$ denote the subset in $BV_\varphi(I, X)$ that consists of those functions that are left continuous on $I^- := I \setminus \{\inf I\}$.

Lemma 2.4 (*Chistyakov [11, Lemma 6]*). *If X is a Banach space and $f \in BV_\varphi(I, X)$, then $f^- \in BV_\varphi^-(I, X)$.* ∎

Thus, if a function has a bounded φ-variation, then its left regularization is a left continuous function.

3. The composition operator

Our main result reads as follows:

Theorem 3.1. *Let $(X, |\cdot|_X)$, $(Y, |\cdot|_Y)$ be real normed spaces and let C be a closed convex subset of X. Suppose that $\varphi \in \mathcal{F}$ and $h : I \times C \longrightarrow Y$. If a composition operator $H : C^I \longrightarrow Y^I$ generated by h, maps $BV_\varphi(I, C)$ into $BV_\varphi(I, Y)$ and is uniformly continuous, then the left regularization of h, i.e. the function $h^- : I^- \times X \longrightarrow Y$, defined by*

$$h^-(t, y) := \lim_{s \uparrow t} h(s, y), \quad t \in I^-;\ y \in C,$$

exists and

$$h^-(t, y) = A(t)y + B(t), \quad t \in I^-,\ y \in C,$$

for some $A : I^- \longrightarrow \mathcal{L}(X, Y)$[1] *and $B \in BV_\varphi(I^-, Y)$. Moreover the functions A and B are left continuous in I^-.*

Proof. For every $y \in C$, the constant function $f(t) = y$ $(t \in I)$ belongs to $BV_\varphi(I, C)$. Since H maps $BV_\varphi(I, C)$ into $BV_\varphi(I, Y)$, it follows that the function $t \mapsto h(t, y)$ $(t \in I)$ belongs to $BV_\varphi(I, Y)$. Now, by Lemma 2.4, the completeness of $(Y, |\cdot|_Y)$ implies the existence of the left regularization h^- of h.

By assumption, H is uniformly continuous on $BV_\varphi(I, C)$. Let $\omega : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ be the *modulus continuity* of H that is

$$\omega(\rho) := \sup\left\{\|H(f_1) - H(f_2)\|_\varphi : \|f_1 - f_2\|_\varphi \le \rho; f_1, f_2 \in BV_\varphi(I, C)\right\}, \quad \text{for } \rho > 0.$$

Hence we get

$$\|H(f_1) - H(f_2)\|_\varphi \le \omega(\|f_1 - f_2\|_\varphi), \quad \text{for } f_1, f_2 \in BV_\varphi(I, C). \tag{4}$$

From the definition of the norm $\|\cdot\|_\varphi$, we obtain

$$p_\varphi(H(f_1) - H(f_2)) \le \|H(f_1) - H(f_2)\|_\varphi, \quad \text{for } f_1, f_2 \in BV_\varphi(I, C). \tag{5}$$

From (4), (5) and Lemma 2.3 (c1), if $\omega(\|f_1 - f_2\|_\varphi) > 0$, then

$$v_\varphi\left(\frac{H(f_1) - H(f_2)}{\omega(\|f_1 - f_2\|_\varphi)}\right) \le 1. \tag{6}$$

Therefore, for any $\alpha_1 < \beta_1 < \alpha_2 < \beta_2 < \cdots < \alpha_m < \beta_m$, $\alpha_i, \beta_i \in I$, $i \in \{1, 2, \ldots, m\}$, $m \in \mathbb{N}$, the definitions of the operator H and the functional $v_\varphi(\cdot)$ imply

$$\sum_{i=1}^{m} \varphi\left(\frac{|h(\beta_i, f_1(\beta_i)) - h(\beta_i, f_2(\beta_i)) - h(\alpha_i, f_1(\alpha_i)) + h(\alpha_i, f_2(\alpha_i))|}{\omega(\|f_1 - f_2\|_\varphi)}\right) \le 1. \tag{7}$$

For $\alpha, \beta \in \mathbb{R}$, $\alpha < \beta$, we define functions $\eta_{\alpha,\beta} : \mathbb{R} \longrightarrow [0, 1]$ by putting

$$\eta_{\alpha,\beta}(t) := \begin{cases} 0 & \text{if } t \le \alpha \\ \dfrac{t - \alpha}{\beta - \alpha} & \text{if } \alpha \le t \le \beta \\ 1 & \text{if } \beta \le t. \end{cases} \tag{8}$$

Let us fix $t \in I^-$. For arbitrary finite sequence $\inf I < \alpha_1 < \beta_1 < \alpha_2 < \beta_2 < \cdots < \alpha_m < \beta_m < t$ and $y_1, y_2 \in C$, $y_1 \neq y_2$, the functions $f_1, f_2 : I \longrightarrow X$ defined by

$$f_\ell(\tau) := \frac{1}{2}\left(\eta_{\alpha_i,\beta_i}(\tau)(y_1 - y_2) + y_\ell + y_2\right), \quad \tau \in I,\ \ell = 1, 2, \tag{9}$$

belong to the space $BV_\varphi(I, C)$. From (9), we have

$$f_1(\cdot) - f_2(\cdot) = \frac{y_1 - y_2}{2},$$

therefore

$$\|f_1 - f_2\|_\varphi = \left|\frac{y_1 - y_2}{2}\right|;$$

[1] $\mathcal{L}(X, Y)$ denote the space of all linear mappings $A : X \longrightarrow Y$.

moreover

$$f_1(\beta_i) = y_1; \qquad f_2(\beta_i) = \frac{y_1+y_2}{2}; \qquad f_1(\alpha_i) = \frac{y_1+y_2}{2}; \qquad f_2(\alpha_i) = y_2.$$

Using (7), we hence get

$$\sum_{i=1}^{m} \varphi\left(\frac{\left|h(\beta_i, y_1) - h\left(\beta_i, \frac{y_1+y_2}{2}\right) - h\left(\alpha_i, \frac{y_1+y_2}{2}\right) + h(\alpha_i, y_2)\right|}{\omega\left(\left|\frac{y_1-y_2}{2}\right|\right)}\right) \leq 1. \tag{10}$$

Since the constant functions belong to the space $BV_\varphi(I, C)$ and H maps $BV_\varphi(I, C)$ into $BV_\varphi(I, Y)$, it follows that the function $t \mapsto h(t, y)$ $(t \in I)$ belongs to $BV_\varphi(I, Y)$ for all $y \in C$. From the continuity of φ and the definition of h^-, passing to the limit in (10) when $\alpha_1 \uparrow t$, we obtain that

$$\sum_{i=1}^{m} \varphi\left(\frac{\left|h^-(t, y_1) - h^-\left(t, \frac{y_1+y_2}{2}\right) - h^-\left(t, \frac{y_1+y_2}{2}\right) + h^-(t, y_2)\right|}{\omega\left(\left|\frac{y_1-y_2}{2}\right|\right)}\right) \leq 1.$$

that is

$$\varphi\left(\frac{\left|h^-(t, y_1) - 2h^-\left(t, \frac{y_1+y_2}{2}\right) + h^-(t, y_2)\right|}{\omega\left(\left|\frac{y_1-y_2}{2}\right|\right)}\right) \leq \frac{1}{m}.$$

Hence, since $m \in \mathbb{N}$ is arbitrary,

$$\varphi\left(\frac{\left|h^-(t, y_1) - 2h^-\left(t, \frac{y_1+y_2}{2}\right) + h^-(t, y_2)\right|}{\omega\left(\left|\frac{y_1-y_2}{2}\right|\right)}\right) = 0,$$

and, as $\varphi(z) = 0$ only if $z = 0$, we obtain

$$\left|h^-(t, y_1) - 2h^-\left(t, \frac{y_1+y_2}{2}\right) + h^-(t, y_2)\right| = 0.$$

Therefore

$$h^-\left(t, \frac{y_1+y_2}{2}\right) = \frac{h^-(t, y_1) + h^-(t, y_2)}{2}$$

for all $t \in I^-$ and all $y_1, y_2 \in C$.

Thus, for each $t \in I^-$, the function $h^-(t, \cdot)$ satisfies the Jensen functional equation in C. Modifying a little the standard argument (cf. Kuczma [13]), we conclude that, for each $t \in I^-$, there exist $A(t) : C \longrightarrow \mathcal{L}(X, Y)$ and $B(t) \in Y$ such that $h^-(t, y) = A(t)y + B(t)$.

The uniform continuity of the operator $H : BV_\varphi(I, C) \longrightarrow BV_\varphi(I, Y)$ implies the continuity of the additive function $A(t)$. Consequently $A(t) \in \mathcal{L}(X, Y)$, for each $t \in I^-$. ■

Remark 3.2. Obviously, the counterpart of Theorem 3.1 for the right regularization h^+ of h defined by

$$h^+(t, y) := \lim_{s \downarrow t} h(s, y); \quad t \in I^+ := I \setminus \{\sup I\},$$

is also true.

Remark 3.3. Taking $X = Z = \mathbb{R}$, $\varphi := id|_{[0,+\infty)}$ in Theorem 3.1 and $C := J$ where $J \subset \mathbb{R}$ is an interval we obtain the main result from [4].

Remark 3.4. Theorem 3.1 extends also the result of Matkowski and Miś [12] concerning the Lipschitzian Nemytskii operator (of also Appell and Zabrejko [14], p. 175).

Remark 3.5. In the proof of Theorem 3.1 we apply the uniform continuity of the operator H only on the set of functions $U \subset BV_\varphi(I, C)$ such that $f \in U$ if, and only if, there are $\alpha, \beta \in I$, $\alpha < \beta$, such that

$$f(t) = \frac{1}{2}\left[\eta_{\alpha,\beta}(t)(y_1 - y_2) + y + y_2\right], \quad t \in I,$$

where $\eta_{\alpha,\beta}$ is defined by (8), $y_1, y_2 \in C$ and $y = y_1$ or $y = y_2$.

Thus the assumption of the uniform continuity of H on $BV_\varphi(I, C)$ in Theorem 3.1 can be replaced by a weaker condition of the uniform continuity of H on U.

Remark 3.6. Theorem 3.1 remains true on replacing the space $BV_\varphi(I, Y)$ by a space $BV_\psi(I, Y)$ with an arbitrary $\psi \in \mathcal{F}$.

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