Nonlinear Analysis 72 (2010) 2580–2591

Contents lists available at ScienceDirect

Nonlinear Analysis

journal homepage: www.elsevier.com/locate/na

Embeddability of mean-type mappings in a continuous iteration semigroup

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ARTICLE INFO

Article history:
Received 6 July 2009
Accepted 3 November 2009

MSC:
Primary 26A18
26E60
39B12

Keywords:
Mean
Invariant mean
Mean-type mapping
Continuous iteration semigroup
Iterability
Translation equation
Cocycle equation
Homogeneous linear functional equation

ABSTRACT

We characterize iterability in the class of homogeneous symmetric strict mean-type mappings and determine all continuous iteration semigroups of such functions in which the given mean-type mapping can be embedded.

1. Introduction

Given a topological space X we say that $F\colon (0,\infty)\times X\to X$ is a *continuous iteration semigroup* if it is continuous in the first variable and satisfies the translation equation

$$F(t, F(s,x)) = F(s+t, x). \tag{1.1}$$

A function $f\colon X\to X$ is called *embeddable* in the continuous iteration semigroup $F\colon (0,\infty)\times X\to X$ if $f = F(1,\cdot)$. The following problem is fundamental in iteration theory. Given a class $\mathcal{F}$ of self-mappings of X and a function $f\in\mathcal{F}$, embed f in a continuous iteration semigroup F such that $F(t,\cdot)\in\mathcal{F}$ for every $t\in(0,\infty)$; if such a semigroup exists, the function f is said to be *iterable in* $\mathcal{F}$.

A characterization of iterability in the class of continuous self-mappings of a closed interval was given by M.C. Zdun in [1, Theorems 1.1 and 9.1] (see also [2]). There he also described the form of all continuous semigroups of continuous mappings in which the considered function can be embedded (see [1, Theorems 5.1–8.1], also [3, Theorem 1 and the comment just before it] for a more unified formulation, and [2]). Iterability in the class of homeomorphisms of the circle was studied by Zdun in [4], whereas embeddability in the so-called disjoint groups of homeomorphisms of the circle was investigated in [5] by Ciepliński.

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0362-546X/$ – see front matter
doi:10.1016/j.na.2009.11.006

The present paper deals with iterability of mean-type mappings, which yield another important class of two dimensional transformations.

Given an interval I, a continuous function $\mu: I^2 \to I$ is called a *mean in I* if

$$\min\{x, y\} \le \mu(x, y) \le \max\{x, y\}, \quad x, y \in I.$$

It is *strict* if the above inequalities are sharp whenever $x \neq y$ and *symmetric* if

$$\mu(x, y) = \mu(y, x), \quad x, y \in I.$$

In the case where I is one of the intervals $(0, \infty)$, $(-\infty, 0)$, $\mathbb{R}$ and the function μ is positively homogeneous, that is

$$\mu(tx, ty) = t\mu(x, y), \quad x, y \in I, t \in (0, \infty),$$

the mean μ is called *homogeneous*. Clearly the classical arithmetic, geometric, and harmonic means, given by

$$A(x, y) = \frac{x+y}{2}, \qquad G(x, y) = \sqrt{xy}, \quad \text{and} \quad H(x, y) = \frac{2xy}{x+y},$$

respectively, serve as the simplest examples of strict symmetric and homogeneous means. Of course G may be considered here only in $(0, \infty)$, whereas H on $(0, \infty)$ or $(-\infty, 0)$. Whenever μ, ν are means in I, the pair (μ, ν) is said to be a *mean-type mapping in* I^2; clearly it maps I^2 into itself. We call it *strict* (resp. *symmetric, homogeneous*) if both μ, ν are strict (resp. symmetric, homogeneous).

Below we study iterability in the class of homogeneous symmetric strict mean-type mappings. In Section 2 we determine all continuous iteration semigroups of such mappings in which the given mean-type mapping can be embedded (see Theorem 2.1). Section 3 provides a characterization of iterability in that class (see Theorem 3.1). The long proof of the crucial Proposition 2.7 has been postponed to Section 4.

We begin with the following result which is fundamental for our further considerations. It is an immediate consequence of [6, Theorem 1] and [7, Theorem 2].

Lemma 1.1. *Let μ and ν be strict means in an interval I. Then there is a unique mean κ in I such that*

$$\kappa(\mu(x, y), \nu(x, y)) = \kappa(x, y), \quad x, y \in I. \tag{1.2}$$

The mean κ is strict. If μ and ν are symmetric, then so is κ. If I is one of the intervals $(0, \infty)$, $(-\infty, 0)$, $\mathbb{R}$ and the means μ and ν are homogeneous, then so is κ. Moreover, the sequence $((\mu, \nu)^n)_{n\in\mathbb{N}}$ of iterates converges to (κ, κ). If, in addition, (μ, ν) is embeddable in a continuous iteration semigroup (M, N) of mean-type self-mappings of I, then

$$\kappa(M(t, x, y), N(t, x, y)) = \kappa(x, y)$$

for every $t \in (0, \infty)$ and all $x, y \in I$.

The unique mean $\kappa: I^2 \to I$ satisfying (1.2) is called *invariant with respect to the mean-type mapping* (μ, ν).

In the rest of the paper we consider means in $(0, \infty)$. The argument in the symmetric case $I = (-\infty, 0)$ is analogous. Also the case $I = \mathbb{R}$ can be reduced to the both previous, as $\mu(J \times J) \subset J$ for any mean μ and every interval J.

2. Form of the semigroups in which the given mean-type mapping can be embedded

The first of two main results of the paper reads as follows.

Theorem 2.1. *Let μ, ν be homogeneous symmetric strict means in $(0, \infty)$ and let κ be the mean invariant with respect to (μ, ν).*

Assume that (μ, ν) is embeddable in a continuous iteration semigroup (M, N) of homogeneous symmetric strict mean-type self-mappings of $(0, \infty)^2$. Then there exist numbers $a, b \in [0, \infty]$ such that

$$a \le b, \quad 1 \in \{a, b\}, \quad \text{and} \quad \frac{\mu(x, 1)}{\nu(x, 1)} \in [a, b], \quad x \in (0, \infty), \tag{2.1}$$

a continuous function $e: (0, \infty) \to (0, \infty)$ satisfying the conditions

$$e((0, \infty)) = [a, b] \cap (0, \infty), \tag{2.2}$$

$$e(x) = x, \quad x \in [a, b] \cap (0, \infty), \tag{2.3}$$

$$e(1/x) = e(x), \quad x \in (0, \infty), \tag{2.4}$$

and such that

$$\kappa(x, 1) < \kappa(y, 1) < \kappa(1/x, 1), \quad x \in (0, 1), \; y \in (\min\{e(x), 1/e(x)\}, \max\{e(x), 1/e(x)\}), \tag{2.5}$$

and a continuous strictly monotonic function $\alpha: [a, b] \to [-\infty, \infty]$ *satisfying the condition*

$$\alpha\left(\frac{\mu(x,1)}{\nu(x,1)}\right) = \min\{\alpha(e(x)) + 1, \alpha(1)\}, \quad x \in (0, \infty), \tag{2.6}$$

and such that

$$\alpha \text{ takes the greatest value at 1} \tag{2.7}$$

and

$$M(t,x,y) = \frac{\kappa(x,y)}{\kappa\left(1/F\left(t, \frac{x}{y}\right), 1\right)}, \qquad N(t,x,y) = \frac{\kappa(x,y)}{\kappa\left(F\left(t, \frac{x}{y}\right), 1\right)} \tag{2.8}$$

for all $t, x, y \in (0, \infty)$*, where*

$$F(t,x) = \alpha^{-1}\left(\min\{\alpha(e(x)) + t, \alpha(1)\}\right) \tag{2.9}$$

for all $t, x \in (0, \infty)$*.*

Conversely: if $a, b \in [0, \infty]$ *satisfy condition* (2.1)*,* $e: (0, \infty) \to (0, \infty)$ *is a continuous function and* $\alpha: [a, b] \to [-\infty, \infty]$ *is a continuous strictly monotonic function satisfying condition* (2.6) *and such that conditions* (2.2)–(2.5) *and* (2.7) *hold, then formulas* (2.8) *and* (2.9) *define a continuous iteration semigroup* (M, N) *of homogeneous symmetric strict mean-type self-mappings of* $(0, \infty)^2$ *with* $M(1, \cdot) = \mu$ *and* $N(1, \cdot) = \nu$*.*

Remark 2.2. Conditions (2.2) and (2.3) mean that e is a retraction of $(0, \infty)$ onto $[a, b] \cap (0, \infty)$.

Remark 2.3. Take any $a, b \in [0, \infty]$ such that (2.1) holds and a continuous function $e: (0, \infty) \to (0, \infty)$ satisfying conditions (2.2)–(2.4). If $a = 1$ then, by (2.2) and (2.3),

$$1 \le e(x) \le x, \quad x \in [1, \infty),$$

whence, according to (2.4),

$$1 \le e(x) \le \frac{1}{x}, \quad x \in (0, 1].$$

Similarly, if $b = 1$ then

$$x \le e(x) \le 1, \quad x \in (0, 1],$$

and

$$\frac{1}{x} \le e(x) \le 1, \quad x \in [1, \infty).$$

Therefore condition (2.5) holds when, for instance, the mean κ is strictly increasing in first (or, equivalently, in second) variable.

Example 2.4. Let μ and ν be the arithmetic mean A and the harmonic mean H, respectively. Then the geometric mean G is the unique mean κ invariant with respect to (μ, ν). Take $a = 1$, $b = \infty$, and define $e: (0, \infty) \to (0, \infty)$ by

$$e(x) = \begin{cases} 1/x, & \text{if } x \in (0, 1), \\ x, & \text{if } x \in [1, \infty). \end{cases}$$

Putting $f = \mu(\cdot, 1)/\nu(\cdot, 1)$ we have

$$f(x) = \frac{(x+1)^2}{4x}, \quad x \in (0, \infty).$$

Let α be any continuous strictly decreasing function mapping $(1, \infty)$ onto $\mathbb{R}$ and satisfying the Abel equation

$$\alpha(f(x)) = \alpha(x) + 1$$

(see [8, Theorem 2.1 and Lemma 5.1]); extend α to $[1, \infty]$ by putting $\alpha(1) = \infty$ and $\alpha(\infty) = -\infty$. Then it is clear that (2.6) holds for all $x \in [1, \infty)$. Since $f(x) = f(1/x)$ for every $x \in (0, \infty)$, we easily infer that (2.6) is satisfied also for all $x \in (0, 1)$. Making use of Theorem 2.1 one can easily check that the mapping (μ, ν) is embeddable in a continuous iteration semigroup (M, N) given by

$$M(t,x,y) = \left(xy\alpha^{-1}\left(\alpha\left(e\left(x/y\right)\right)\right) + t\right)^{1/2}$$

and

$$N(t,x,y)=\left(\frac{xy}{\alpha^{-1}\left(\alpha\left(e\left(x/y\right)\right)\right)+t}\right)^{1/2}.$$

This provides a positive solution to the problem posed by the second author in [7].

Example 2.5. A similar procedure (but with a more complicated calculation) can be repeated when μ and ν are the arithmetic mean A and the geometric mean G, respectively. It was Gauss [9] who proved still in 1799 that the formula

$$A\otimes G(x,y)=\left(\frac{2}{\pi}\int_0^{\frac{\pi}{2}}\frac{1}{x^2\cos^2 u+y^2\sin^2 u}\mathrm{d}u\right)^{-1}$$

defines the mean, named arithmetic-geometric, which is invariant with respect to (μ,ν). To find the form of the semigroups (M,N) we apply Theorem 2.1 with a, b, and e as in Example 2.4 and with $f\colon(0,\infty)\to(0,\infty)$ defined by

$$f(x)=\frac{x+1}{2\sqrt{x}}.$$

The next result provides a class of continuous iteration semigroups.

Proposition 2.6. *Let $a,b\in[0,\infty]$ satisfy the condition $a\le b$, let $e\colon(0,\infty)\to[a,b]\cap(0,\infty)$ be a continuous function such that condition (2.3) holds and α be a continuous strictly monotonic function mapping $[a,b]$ onto an interval $[p,q]\subset[-\infty,\infty]$. Then $F\colon(0,\infty)\times(0,\infty)\to(0,\infty)$, defined by the formula*

$$F(t,x)=\alpha^{-1}\left(\min\{\alpha(e(x))+t,q\}\right),$$

is a continuous iteration semigroup.

Proof. It is sufficient to check that F is a solution of the translation equation (1.1). At first observe that all the values of F are in $[a,b]$, and thus, by (2.3), we have

$$e\left(F(s,x)\right)=F(s,x),\quad s,x\in(0,\infty). \tag{2.10}$$

Take any $s,t,x\in(0,\infty)$. If $\alpha(e(x))+s\le q$ then $F(s,x)=\alpha^{-1}\left(\alpha(e(x))+s\right)$, whence, by (2.10),

$$\begin{aligned}F(t,F(s,x))&=\alpha^{-1}\left(\min\{\alpha(F(s,x))+t,q\}\right)\\&=\alpha^{-1}\left(\min\{\alpha(e(x))+s+t,q\}\right)=F(s+t,x).\end{aligned}$$

Otherwise $\alpha(e(x))+s>q$ and $\alpha(e(x))+s+t>q$, so

$$F(t,F(s,x))=F(t,\alpha^{-1}(q))=\alpha^{-1}(q)=F(s+t,x),$$

which completes the proof. □

The proposition proved below provides a great part of the assertion of Theorem 2.1.

Proposition 2.7. *Let μ, ν be homogeneous symmetric strict means in $(0,\infty)$. Assume that (μ,ν) is embeddable in a continuous iteration semigroup (M,N) of homogeneous symmetric strict mean-type self-mappings of $(0,\infty)^2$. Then*

(i) *there exist numbers $a,b\in[0,\infty]$ such that (2.1) holds and a continuous function $e\colon(0,\infty)\to(0,\infty)$ satisfying conditions (2.2)–(2.4);*

(ii) *the function $F=M(\cdot,\cdot,1)/N(\cdot,\cdot,1)$ is a continuous iteration semigroup,*

$$F(1,x)=\frac{\mu(x,1)}{\nu(x,1)},\quad x\in(0,\infty),$$

and

$$\lim_{t\to0}F(t,x)=e(x)\quad\textit{and}\quad\lim_{t\to\infty}F(t,x)=1,\quad x\in(0,\infty); \tag{2.11}$$

(iii) *there exists a continuous strictly monotonic function α mapping $[a,b]$ onto an interval $[p,q]\subset[-\infty,\infty]$ such that $\alpha(1)=q$ and*

$$F(t,x)=\alpha^{-1}\left(\min\{\alpha(e(x))+t,q\}\right),\quad t,x\in(0,\infty); \tag{2.12}$$

(iv)

$$N(t,x,1)=\frac{\kappa(x,1)}{\kappa(F(t,x),1)},\quad t,x\in(0,\infty), \tag{2.13}$$

where κ is the unique mean which is invariant with respect to (μ,ν).

The proof of Proposition 2.7 is postponed to the last section.

Remark 2.8. If μ, ν, M, N, e, and F are as in Theorem 2.1, then, by Proposition 2.7,

$$\lim_{t\to 0} F(t,x) = e(x) \quad \text{and} \quad \lim_{t\to\infty} F(t,x) = 1, \quad x \in (0,\infty).$$

Assume, for instance, that $a = 1$. Take any $x, y \in (0,\infty)$, $x \geq y$. Then, by (2.8), we have

$$\lim_{t\to\infty} M(t,x,y) = \lim_{t\to\infty} \frac{\kappa(x,y)}{\kappa\left(1/F\left(t,\frac{x}{y}\right),1\right)} = \frac{\kappa(x,y)}{\kappa(1,1)} = \kappa(x,y)$$

and

$$\lim_{t\to\infty} N(t,x,y) = \lim_{t\to\infty} \frac{\kappa(x,y)}{\kappa\left(F\left(t,\frac{x}{y}\right),1\right)} = \frac{\kappa(x,y)}{\kappa(1,1)} = \kappa(x,y).$$

If, in addition $x \leq by$, then

$$\begin{aligned}\lim_{t\to 0} M(t,x,y) &= \lim_{t\to 0} \frac{\kappa(x,y)}{\kappa\left(1/F\left(t,\frac{x}{y}\right),1\right)} = \frac{\kappa(x,y)}{\kappa\left(1/e\left(\frac{x}{y}\right),1\right)} \\ &= \frac{\kappa(x,y)}{\kappa\left(\frac{y}{x},1\right)} = \frac{x\kappa(x,y)}{\kappa(y,x)} = x = \max\{x,y\}\end{aligned}$$

and

$$\begin{aligned}\lim_{t\to 0} N(t,x,y) &= \lim_{t\to 0} \frac{\kappa(x,y)}{\kappa\left(F\left(t,\frac{x}{y}\right),1\right)} = \frac{\kappa(x,y)}{\kappa\left(e\left(\frac{x}{y}\right),1\right)} \\ &= \frac{\kappa(x,y)}{\kappa\left(\frac{x}{y},1\right)} = \frac{y\kappa(x,y)}{\kappa(x,y)} = y = \min\{x,y\}.\end{aligned}$$

We conclude this section with proving Theorem 2.1.

Proof of Theorem 2.1. Assume that (μ, ν) is embeddable in a continuous iteration semigroup (M, N) of homogeneous symmetric strict mean-type self-mappings of $(0,\infty)^2$. Define a function $F: (0,\infty)\times(0,\infty) \to (0,\infty)$ by the formula

$$F(t,x) = \frac{M(t,x,1)}{N(t,x,1)}. \tag{2.14}$$

By Proposition 2.7 there exist numbers $a, b \in [0,\infty]$ satisfying condition (2.1), a continuous function $e: (0,\infty) \to (0,\infty)$ with properties (2.2)–(2.4), and a continuous strictly monotonic function α mapping $[a,b]$ onto an interval $[p,q] \subset [-\infty,\infty]$, satisfying condition (2.12), that is also Eq. (2.6), and such that $\alpha(1) = q$ and F can be written in form (2.9). Clearly α satisfies (2.7); moreover (2.13) holds.

To obtain representation (2.8) of M and N at first observe that, by (2.13) and (2.14) and the homogeneity and symmetry of κ, we have

$$M(t,x,1) = \frac{\kappa(x,1)}{\kappa(1/F(t,x),1)}, \quad t,x \in (0,\infty). \tag{2.15}$$

Then the homogeneity of M gives

$$M(t,x,y) = yM\left(t,\frac{x}{y},1\right) = y\frac{\kappa\left(\frac{x}{y},1\right)}{\kappa\left(1/F\left(t,\frac{x}{y}\right),1\right)} = \frac{\kappa(x,y)}{\kappa\left(1/F\left(t,\frac{x}{y}\right),1\right)}$$

for all $t, x, y \in (0,\infty)$. Similarly, since N is homogeneous, (2.13) implies

$$N(t,x,y) = yN\left(t,\frac{x}{y},1\right) = y\frac{\kappa\left(\frac{x}{y},1\right)}{\kappa\left(F\left(t,\frac{x}{y}\right),1\right)} = \frac{\kappa(x,y)}{\kappa\left(F\left(t,\frac{x}{y}\right),1\right)}$$

whenever $t, x, y \in (0,\infty)$.

To prove property (2.5) consider, for instance, the case $a = 1$. Then, by (2.2), we have $e(x) \geq 1$, $x \in (0, \infty)$. Take any $u \in (0, 1)$ and $v \in (1/e(u), e(u))$. First assume that $v < 1$. Then $1/v \in (1, e(u))$. By Proposition 2.7(ii)

$$\text{for every } x \in (0,\infty) \text{ the function } F(\cdot, x) \text{is continuous,} \quad \lim_{t\to 0} F(t,x) = e(x) \quad \text{and} \quad \lim_{t\to\infty} F(t,x) = 1. \tag{2.16}$$

Using that property we find a $t \in (0, \infty)$ such that $F(t, u) = 1/v$. Hence and from (2.15), as $M(t, \cdot)$ is a strict mean, we get

$$u < \frac{\kappa(u, 1)}{\kappa(v, 1)} < 1.$$

Therefore $\kappa(u, 1) < \kappa(v, 1)$ and

$$\kappa(v, 1) < \frac{\kappa(u, 1)}{u} = \kappa\left(1, \frac{1}{u}\right) = \kappa\left(\frac{1}{u}, 1\right).$$

Now assume that $v > 1$. Then, by (2.16), we have $v = F(t, 1/u)$ for some $t \in (0, \infty)$. Since $N(t, \cdot)$ is a strict mean, condition (2.13) gives

$$1 < \frac{\kappa(1/u, 1)}{\kappa(v, 1)} < \frac{1}{u},$$

and thus $\kappa(v, 1) < \kappa(1/u, 1)$ as well as

$$\kappa(u, 1) = \kappa(1, u) = u\kappa\left(\frac{1}{u}, 1\right) < \kappa(v, 1).$$

Observe also that, since κ is a strict mean, we have

$$\kappa(u, 1) < 1 = \kappa(1, 1) < \kappa\left(\frac{1}{u}, 1\right).$$

Consequently, we come to (2.5). In the case $b = 1$ the argument is analogous.

To prove the converse take any numbers $a, b \in [0, \infty]$ satisfying condition (2.1), a continuous function $e: (0, \infty) \to (0, \infty)$, and a continuous strictly monotonic solution $\alpha: [a, b] \to [-\infty, \infty]$ of Eq. (2.6) such that conditions (2.2)–(2.5) and (2.7) are fulfilled. By Proposition 2.6 the function F, defined by formula (2.9), is a continuous iteration semigroup. Moreover, it follows from equalities (2.9) and (2.6)that

$$F(1, x) = \frac{\mu(x, 1)}{\nu(x, 1)}, \quad x \in (0, \infty). \tag{2.17}$$

We check that for every $t \in (0, \infty)$ equalities (2.8) define strict means $M(t, \cdot)$ and $N(t, \cdot)$ in $(0, \infty)^2$. Take any $t, x \in (0, \infty)$. First assume that $x \in (0, 1)$. It follows from (2.9) that either $F(t, x) = e(x) = 1$, or $F(t, x)$, $1/F(t, x) \in (\min\{e(x), 1/e(x)\}, \max\{e(x), 1/e(x)\})$. Therefore, by (2.5), we have

$$\kappa(x, 1) < \kappa(1/F(t, x), 1) < \kappa\left(\frac{1}{x}, 1\right)$$

and

$$\kappa(x, 1) < \kappa(F(t, x), 1) < \kappa\left(\frac{1}{x}, 1\right),$$

whence

$$x < x\frac{\kappa(1/x, 1)}{\kappa(1/F(t, x), 1)} = \frac{\kappa(x, 1)}{\kappa(1/F(t, x), 1)} < 1$$

and

$$x < x\frac{\kappa(1/x, 1)}{\kappa(F(t, x), 1)} = \frac{\kappa(x, 1)}{\kappa(F(t, x), 1)} < 1,$$

i.e., according to (2.13) and (2.15),

$$x < M(t, x, 1) < 1 \quad \text{and} \quad x < N(t, x, 1) < 1.$$

Similarly, one can show that if $x \in (1, \infty)$, then

$$1 < M(t, x, 1) < x \quad \text{and} \quad 1 < N(t, x, 1) < x.$$

Consequently, as κ is homogeneous, it follows from formulas (2.8) that $M(t,\cdot)$ and $N(t,\cdot)$ are strict means. Since κ is symmetric and homogeneous, clearly so are the means $M(t,\cdot)$ and $N(t,\cdot)$.

We show that (M,N) is a continuous iteration semigroup. It is sufficient to verify that it satisfies the translation equation (1.1). At first observe that for all $t,x\in(0,\infty)$, by equalities (2.8) and the homogeneity of κ, we have

$$\frac{M(t,x,1)}{N(t,x,1)}=\frac{\kappa(F(t,x),1)}{\kappa(1/F(t,x),1)}=F(t,x),$$

i.e. relation (2.14). Now, for all $s,t,x,y\in(0,\infty)$, as κ is invariant with respect to (M,N), the means $M(s,\cdot)$ and $N(s,\cdot)$ are homogeneous and F satisfies the translation equation, we have

$$\begin{aligned}M(t,M(s,x,y),N(s,x,y)) &= \frac{\kappa(M(s,x,y),N(s,x,y))}{\kappa\left(1/F\left(t,\frac{M(s,x,y)}{N(s,x,y)}\right),1\right)}\\ &= \frac{\kappa(x,y)}{\kappa\left(1/F\left(t,F\left(s,\frac{x}{y}\right)\right),1\right)}\\ &= \frac{\kappa(x,y)}{\kappa\left(1/F\left(s+t,\frac{x}{y}\right),1\right)}=M(s+t,x,y)\end{aligned}$$

and

$$\begin{aligned}N(t,M(s,x,y),N(s,x,y)) &= \frac{\kappa(M(s,x,y),N(s,x,y))}{\kappa\left(F\left(t,\frac{M(s,x,y)}{N(s,x,y)}\right),1\right)}\\ &= \frac{\kappa(x,y)}{\kappa\left(F\left(t,F\left(s,\frac{x}{y}\right)\right),1\right)}\\ &= \frac{\kappa(x,y)}{\kappa\left(F\left(s+t,\frac{x}{y}\right),1\right)}=N(s+t,x,y),\end{aligned}$$

which was to be proved.

Finally, by (2.8) and (2.17), the homogeneity of μ,ν,κ, and the invariance of κ, we obtain

$$\begin{aligned}M(1,x,y) &= \frac{\kappa(x,y)}{\kappa\left(1/F\left(1,\frac{x}{y}\right),1\right)}=\frac{\kappa(x,y)}{\kappa\left(\frac{\nu(x/y,1)}{\mu(x/y,1)},1\right)}=\frac{\kappa(x,y)}{\kappa\left(\frac{\nu(x,y)}{\mu(x,y)},1\right)}\\ &= \frac{\mu(x,y)\kappa(x,y)}{\kappa\left(\nu(x,y),\mu(x,y)\right)}=\mu(x,y)\end{aligned}$$

and

$$\begin{aligned}N(1,x,y) &= \frac{\kappa(x,y)}{\kappa\left(F\left(1,\frac{x}{y}\right),1\right)}=\frac{\kappa(x,y)}{\kappa\left(\frac{\mu(x/y,1)}{\nu(x/y,1)},1\right)}=\frac{\kappa(x,y)}{\kappa\left(\frac{\mu(x,y)}{\nu(x,y)},1\right)}\\ &= \frac{\nu(x,y)\kappa(x,y)}{\kappa\left(\mu(x,y),\nu(x,y)\right)}=\nu(x,y)\end{aligned}$$

for all $x,y\in(0,\infty)$. This completes the proof. □

3. Characterization of iterability of mean-type mappings

The result of this section reads as follow.

Theorem 3.1. *Let μ,ν be homogeneous symmetric strict means in $(0,\infty)$ and let κ be the mean invariant with respect to (μ,ν). Put $f=\mu(\cdot,1)/\nu(\cdot,1)$. The mapping (μ,ν) is iterable in the class of homogeneous symmetric strict mean-type self-mappings of $(0,\infty)^2$ if, and only if,*

(i) *there exist numbers $a,b\in[0,\infty]$ such that (2.1) holds and a continuous function $e\colon(0,\infty)\to(0,\infty)$ satisfying conditions (2.2)–(2.4);*

(ii)

$$a\le f(a+)\le f(x)\le f(b-)\le b,\quad x\in(0,\infty),\tag{3.1}$$

the function $f|_{[a,b]\cap(0,\infty)}$ is increasing, and there is at most one interval of constancy of f, which, in addition, contains 1; moreover,

$$f(e(x)) = f(x), \quad x \in (0,\infty); \tag{3.2}$$

(iii) *condition (2.5) holds.*

We start with a simple lemma.

Lemma 3.2. *Let $(M,N)\colon (0,\infty)\times(0,\infty)^2 \to (0,\infty)^2$ satisfy the translation equation and assume that for every $t\in(0,\infty)$ the functions $M(t,\cdot)$ and $N(t,\cdot)$ are positively homogeneous. Then the function $F\colon (0,\infty)\times(0,\infty)\to(0,\infty)$, defined by formula (2.14), satisfies the translation equation and the function $N(\cdot,\cdot,1)$ is a solution of the cocycle equation*

$$G(s,x)G(t,F(s,x)) = G(s+t,x). \tag{3.3}$$

Proof. Define $A\colon (0,\infty)\times(0,\infty)\to(0,\infty)$ and $B\colon (0,\infty)\times(0,\infty)\to(0,\infty)$ by

$$A(t,x) = M(t,x,1) \quad \text{and} \quad B(t,x) = N(t,x,1), \tag{3.4}$$

respectively. For all $s,t,x,y\in(0,\infty)$ we have

$$(M,N)(t,(M,N)(s,x,y)) = (M,N)(s+t,x,y),$$

that is

$$M(t,M(s,x,y),N(s,x,y)) = M(s+t,x,y)$$

and

$$N(t,M(s,x,y),N(s,x,y)) = N(s+t,x,y).$$

Putting here $y=1$ we obtain

$$M(t,A(s,x),B(s,x)) = A(s+t,x)$$

and

$$N(t,A(s,x),B(s,x)) = B(s+t,x)$$

for all $t,x,y\in(0,\infty)$, which, by (2.14) and the positive homogeneity of $M(t,\cdot)$ and $N(t,\cdot)$, means that (A,B) satisfies the equation

$$B(s,x)A(t,F(s,x)) = A(s+t,x)$$

and B is a solution of the cocycle equation

$$B(s,x)B(t,F(s,x)) = B(s+t,x).$$

Hence we infer also that F satisfies the translation equation. □

Proposition 3.3. *Let μ,ν be homogeneous symmetric strict means in $(0,\infty)$ and let κ be the mean invariant with respect to (μ,ν). Put $f=\mu(\cdot,1)/\nu(\cdot,1)$. If the mapping (μ,ν) is iterable in the class of homogeneous symmetric strict mean-type self-mappings of $(0,\infty)^2$, then*

(i) *there exist numbers $a,b\in[0,\infty]$ such that (2.1) holds and a continuous function $e\colon(0,\infty)\to(0,\infty)$ satisfying conditions (2.2)–(2.4);*

(ii) *condition (3.1) holds, 1 is a unique fixed point of f,*

$$f(x) > x, \quad x\in(0,1), \quad \text{and} \quad f(x) < x, \quad x\in(1,\infty), \tag{3.5}$$

the function $f|_{[a,b]\cap(0,\infty)}$ is increasing, and there is at most one interval of constancy of f, which in addition, contains 1; moreover, condition (3.2) is satisfied.

Proof. Let (μ,ν) be embeddable in a continuous iteration semigroup (M,N) of homogeneous symmetric strict mean-type self-mappings of $(0,\infty)^2$. Making use of Lemma 3.2 we infer that $F\colon(0,\infty)\times(0,\infty)\to(0,\infty)$, given by (2.14), is a continuous iteration semigroup. Moreover, for every $x\in(0,\infty)$ we have

$$F(1,x) = \frac{M(1,x,1)}{N(1,x,1)} = \frac{\mu(x,1)}{\nu(x,1)},$$

that is condition (2.17) holds. Clearly $f(1)=1$. If $f(x)=x$ for some $x\in(0,\infty)\setminus\{1\}$, say $x\in(0,1)$, then

$$x < \mu(x,1) = x\nu(x,1) < x,$$

which is impossible. This proves that 1 is a unique fixed point of f.

To prove (i) define functions $A, B: (0,\infty) \times (0,\infty) \to (0,\infty)$ by formulas (3.4). Fix a positive integer n. Since $M(t,\cdot)$ and $N(t,\cdot)$ are means for every $t \in (0,\infty)$, we have

$$A((0,\infty) \times [1/n, 1]) \subset [1/n, 1], \qquad A((0,\infty) \times [1, n]) \subset [1, n]$$

and

$$B((0,\infty) \times [1/n, 1]) \subset [1/n, 1], \qquad B((0,\infty) \times [1, n]) \subset [1, n],$$

whence

$$F((0,\infty) \times [1/n, n]) \subset [1/n, n].$$

Therefore $F_n := F|_{(0,\infty)\times[1/n,n]}$ is a continuous iteration semigroup. By [1, Corollary 1.1] for every $x \in [1/n, n]$ there exists the limit

$$e_n(x) := \lim_{t\to 0} F_n(t,x) = \lim_{t\to 0} F(t,x). \tag{3.6}$$

Putting

$$a_n = \inf e_n([1/n, n]) \quad \text{and} \quad b_n = \sup e_n([1/n, n]) \tag{3.7}$$

we have $1/n \le a_n \le b_n \le n$. According to [1, Lemma 7.1] the function $e_n: [1/n, n] \to [a_n, b_n]$ is continuous and

$$e_n(x) = x, \quad x \in [a_n, b_n], \tag{3.8}$$

and, by [1, formulas (8.1), (10.1) and Theorem 4.1], we have

$$F(t, e_n(x)) = F(t,x), \quad t \in (0,\infty),\ x \in [1/n, n]. \tag{3.9}$$

Observe that $F(t,1) = 1$ for every $t \in (0,\infty)$, so $e_n(1) = 1$ and, consequently,

$$a_n \le 1 \le b_n. \tag{3.10}$$

Moreover, due to [1, formulas (8.1), (10.1) and Theorem 3.1],

$$a_n \le f(a_n) \le f(x) \le f(b_n) \le b_n, \qquad x \in [1/n, n], \tag{3.11}$$

the function $f|_{[a_n,b_n]}$ is increasing, and there is at most one interval of constancy of $f|_{[1/n,n]}$, which, in addition, contains 1. As 1 is the unique fixed point of f it follows from (3.11) that

$$f(x) > x, \quad x \in [1/n, 1), \quad \text{and} \quad f(x) < x, \quad x \in (1, n].$$

Since n is here an arbitrary positive integer, this means that condition (3.5) holds.

Define a function $e: (0,\infty) \to (0,\infty)$ by

$$e(x) = \lim_{t\to 0} F(t,x).$$

For all $t, x \in (0,\infty)$, by the homogeneity and symmetry of the means $M(t,\cdot)$ and $N(t,\cdot)$, we have

$$M(t,x,1) = xM(t,1,1/x) = xM(t,1/x,1)$$

and

$$N(t,x,1) = xN(t,1,1/x) = xN(t,1/x,1),$$

and thus $F(t,x) = F(t,1/x)$. Hence we obtain condition (2.4). Moreover, e is continuous. By virtue of (3.6) and (3.7) the sequence $(a_n)_{n\in\mathbb{N}}$ is decreasing, whereas $(b_n)_{n\in\mathbb{N}}$ is increasing. Put

$$a = \lim_{n\to\infty} a_n \quad \text{and} \quad b = \lim_{n\to\infty} b_n.$$

Then, by (3.10), we have

$$0 \le a \le 1 \le b \le \infty$$

and, in view of (3.8), we get property (2.3). Now, (3.7) and (2.3) yield equality (2.2). Moreover, due to (3.9) we see that

$$F(t, e(x)) = F(t,x), \quad t, x \in (0,\infty). \tag{3.12}$$

As $1 \in [a,b]$ it follows from conditions (2.3) and (2.4) and the monotonicity of $(a_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$ that $a_n = 1$ for every $n \in \mathbb{N}$ or $b_n = 1$ for every $n \in \mathbb{N}$. Therefore $a = 1$ or $b = 1$; in particular, (i) follows.

What we have just proved, condition (3.11) satisfied for all $n \in \mathbb{N}$, and (3.12) imply also (ii). □

Proof of Theorem 3.1. The necessity of conditions (i)–(iii) follows immediately from Proposition 2.7 and Theorem 2.1. Conversely, assume (i)–(iii) and consider, for instance, the case $a = 1$; if $b = \infty$ put additionally $f(b) = \lim_{x\to\infty} f(x)$. By virtue of [1, Theorems 9.1 and 1.1] the function $f|_{[a,b]}$ is iterable in the class of continuous self-mappings of $[a,b]$. Making use of [3, Theorem 1] we can find a continuous strictly monotonic function α mapping $[a,b]$ onto an interval $[p,q] \subset [-\infty,\infty]$

such that

$$f(x) = \alpha^{-1}\left(\min\{\alpha(x)+1, q\}\right), \quad x \in [a, b].$$

If α were increasing, then, as $a = 1$, it would follow from (3.5) that

$$\min\{\alpha(x)+1, q\} < \alpha(x), \quad x \in (a, b],$$

which is impossible. Thus α is decreasing and $\alpha(a) = q$, that is condition (2.7) holds. This and (3.2) imply that α satisfies Eq. (2.6). Now to complete the proof it is sufficient to apply Theorem 2.1. □

4. Proof of Proposition 2.7

Assertion (i) follows from Proposition 3.3 immediately. To prove (iii) assume, for instance, that $a = 1$. Introduce a_n, b_n, and F_n as in the proof of Proposition 3.3. Put $f = F(1, \cdot)$. Making use of [3, Theorem 1], for every $n \in \mathbb{N}$ we find a continuous strictly monotonic function α_n mapping $[1, b_n]$ onto an interval $[p_n, q_n] \subset [-\infty, \infty]$ such that

$$F_n(t, x) = \alpha_n^{-1}\left(\min\{\alpha_n(x)+t, q_n\}\right), \quad t \in (0, \infty),\ x \in [1, b_n]. \tag{4.1}$$

Using induction and replacing, if necessary, α_{n+1} by α_{n+1} shifted by a constant we may additionally assume that

$$\alpha_{n+1}(b_n) = \alpha_n(b_n), \quad n \in \mathbb{N}.$$

Fix a positive integer n. Then, by (4.1), (2.17) and (3.5), we have

$$\alpha_n^{-1}\left(\min\{\alpha_n(x)+1, q_n\}\right) = F_n(1, x) = f(x) < x, \quad x \in (1, b_n].$$

If α_n were increasing the above condition would mean that

$$\min\{\alpha_n(x)+1, q_n\} < \alpha_n(x), \quad x \in (1, b_n],$$

which is impossible. Therefore α_n is decreasing.

Put

$$c_n = \inf\{x \in (1, b_n] : \alpha_{n+1}|_{[x,b_n]} = \alpha_n|_{[x,b_n]}\}$$

and suppose that $c_n > 1$. Then

$$\alpha_{n+1}(c_n) = \alpha_n(c_n) < \min\{\alpha_{n+1}(1), \alpha_n(1)\} = \min\{q_{n+1}, q_n\}.$$

Fix a positive $t_0 < \min\{q_{n+1} - \alpha_{n+1}(c_n), q_n - \alpha_n(c_n)\}$. If $t \in (0, t_0]$, then, by (4.1),

$$\begin{aligned}\alpha_{n+1}(F(t, c_n)) &= \alpha_{n+1}(F_{n+1}(t, c_n)) = \min\{\alpha_{n+1}(c_n)+t, q_{n+1}\}\\ &= \alpha_{n+1}(c_n)+t = \alpha_n(c_n)+t = \min\{\alpha_n(c_n)+t, q_n\}\\ &= \alpha_n(F_n(t, c_n)) = \alpha_n(F(t, c_n))\end{aligned}$$

and, in particular, $\alpha_n(F(t, c_n)) > \alpha_n(c_n)$, whence $F(t, c_n) < c_n$. Since, by (3.8) and (3.6),

$$c_n = e_n(c_n) = \lim_{t\to 0} F(t, c_n),$$

this means that

$$\alpha_{n+1}(x) = \alpha_n(x), \quad x \in (F(t_0, c_n), c_n),$$

contrary to the definition of c_n. Therefore $c_n = 1$. Consequently, we have

$$\alpha_{n+1}|_{[1,b_n]} = \alpha_n, \quad n \in \mathbb{N}. \tag{4.2}$$

In particular, $q_{n+1} = \alpha_{n+1}(1) = \alpha_n(1) = q_n$ for every $n \in \mathbb{N}$, so we can put $q := q_n$. Since

$$p_{n+1} = \alpha_{n+1}(b_{n+1}) \le \alpha_{n+1}(b_n) = \alpha_n(b_n) = p_n, \quad n \in \mathbb{N},$$

the sequence $(p_n)_{n\in\mathbb{N}}$ has a limit $p \in [-\infty, \infty]$. By (4.2) the formula

$$\alpha(x) = \begin{cases}\alpha_n(x), & \text{if } x \in [1, b_n],\ n \in \mathbb{N},\\ p, & \text{if } x = b,\end{cases}$$

defines a function. It is a continuous strictly decreasing function mapping $[1, b]$ onto $[p, q]$ and, by (4.1),

$$F(t, x) = \alpha^{-1}(\min\{\alpha(x)+t, q\}), \quad t \in (0, \infty), x \in [1, b].$$

Hence, taking into account (3.12) and (2.2), we get

$$F(t, x) = \alpha^{-1}(\min\{\alpha(e(x))+t, q\}), \quad t, x \in (0, \infty),$$

which ends the proof of (iii) and condition (2.11). Thus we have completed also the proof of (ii).

Finally we pass to the proof of assertion (iv). To this aim assume, for instance, that $a = 1$. At first we prove that

$$N(t, x, 1) = \frac{\kappa(x, 1)}{\kappa(F(t, x), 1)} \tag{4.3}$$

for every $t \in (0,\infty)$ and $x \in [1,b] \cap (0,\infty)$. This is clear if $b = 1$. So assume that $b > 1$ and take any $n \in \mathbb{N}$ such that $b_n > 1$. As we know α_n is decreasing. So if $t \in (0,\infty)$ and $x \in [1,b_n)$ then, by (4.1), we get $F(t,x) \in [1,x]$. Therefore $F((0,\infty)\times[1,b_n)) \subset [1,b_n)$; in particular, $f([1,b_n)) \subset [1,b_n)$, and, by Lemma 3.2, the function $N(\cdot,\cdot,1)|_{(0,\infty)\times[1,b_n)}$ satisfies the cocycle equation (3.3). Making use of [3, Corollary 3] we find a continuous function $\varphi_n\colon [1,b_n) \to (0,\infty)$ and a number $c \in (0,\infty)$ such that for all $t \in (0,\infty)$ and $x \in [1,b_n)$ we have

$$N(t,x,1) = \begin{cases} \dfrac{\varphi_n(F(t,x))}{\varphi_n(x)}, & \text{if } \alpha_n(x)+t \le q, \\ \dfrac{\varphi_n(1)}{\varphi_n(x)} c^{\alpha_n(x)+t-q}, & \text{if } \alpha_n(x)+t > q. \end{cases}$$

Observe that, in view of (4.1), for all $t \in (0,\infty)$ and $x \in (1,b_n)$ we get $F(t,x) = 1$ whenever $\alpha_n(x)+t > q$, and

$$\max\{0, \alpha_n(x)+t-q\} = \alpha_n(x)+t-\min\{\alpha_n(x)+t, q\} = \alpha_n(x)+t-\alpha_n(F(t,x)).$$

Therefore

$$N(t,x,1) = \frac{\varphi_n(F(t,x))}{\varphi_n(x)} c^{-[\alpha(F(t,x))-\alpha(x)-t]}, \quad t \in (0,\infty),\ x \in (1,b_n). \tag{4.4}$$

In particular,

$$\frac{\varphi_n(F(1,x))}{\varphi_n(x)} = N(1,x,1)c^{\alpha(f(x))-\alpha(x)-1} = \nu(x,1)c^{\alpha(f(x))-\alpha(x)-1}, \quad x \in (1,b_n),$$

which means that φ_n satisfies the linear functional equation

$$\varphi(f(x)) = g(x)\varphi(x)$$

with $g\colon (1,b_n) \to (0,\infty)$ given by $g(x) = \nu(x,1)c^{\alpha(f(x))-\alpha(x)-1}$. Then, by induction, for every $m \in \mathbb{N}$ we have

$$\varphi_n(f^m(x)) = \varphi_n(x)\prod_{i=0}^{m-1} g(f^i(x)), \quad x \in (1,b_n),$$

that is

$$\prod_{i=0}^{m-1} g(f^i(x)) = \frac{\varphi_n(f^m(x))}{\varphi_n(x)}, \quad x \in (1,b_n).$$

It follows from Proposition 3.3. (ii) that

$$1 \le f(x) < x, \quad x \in (1,b_n),$$

whence

$$\lim_{m\to\infty} f^m(x) = 1, \quad x \in (1,b_n).$$

Consequently, by the continuity of φ_n,

$$\lim_{m\to\infty}\prod_{i=0}^{m-1} g(f^i(x)) = \frac{\varphi_n(1)}{\varphi_n(x)}, \quad x \in (1,b_n). \tag{4.5}$$

On the other hand, using induction and the homogeneity of the means μ and ν, one can check that for every $m \in \mathbb{N}$ we have

$$f^m(x) = \frac{\mu\left((\mu,\nu)^{m-1}(x,1)\right)}{\nu\left((\mu,\nu)^{m-1}(x,1)\right)}, \quad x \in (0,\infty),$$

and then

$$\prod_{i=0}^{m-1} g(f^i(x)) = \nu\left((\mu,\nu)^{m-1}(x,1)\right) c^{\alpha(f^m(x))-\alpha(x)-m}, \quad x \in (1,b_n). \tag{4.6}$$

By the continuity of ν and due to Lemma 1.1, for every $x \in (1,b_n)$ we obtain

$$\lim_{m\to\infty} \nu\left((\mu,\nu)^{m-1}(x,1)\right) = \nu(\kappa(x,1),\kappa(x,1)) = \kappa(x,1). \tag{4.7}$$

According to (4.5) the limit

$$\lim_{m\to\infty}\prod_{i=0}^{m-1} g\left(f^i(x)\right)$$

is positive and finite for every $x \in (1, b_n)$, so it follows from (4.6) and (4.7) that

$$\lim_{m\to\infty} c^{\alpha(f^m(x))-\alpha(x)-m} \in (0, \infty), \quad x \in (1, b_n).$$

Hence, as $\lim_{m\to\infty} f^m(x) = 1$ for all $x \in (1, b_n)$ and $\alpha(1) = q$, we infer that either q is finite and $c = 1$, or $q = \infty$ and, by (2.2) and assertion (iii),

$$f^m(x) = \alpha^{-1}(\alpha(x) + m), \quad x \in (1, b_n), \; m \in \mathbb{N}.$$

In both cases

$$c^{\alpha(f^m(x))-\alpha(x)-m} = 1, \quad x \in (1, b_n), \; m \in \mathbb{N},$$

which, according to conditions (4.5)–(4.7), gives

$$\varphi_n(x) = \frac{\varphi_n(1)}{\kappa(x, 1)}, \quad x \in (1, b_n).$$

Moreover, it follows from (4.4) that (4.3) holds for every $t \in (0, \infty)$ and $x \in [1, b_n)$. As $\lim_{n\to\infty} b_n = b$, representation (4.3) holds true for all $t \in (0, \infty)$ and $x \in [1, b] \cap (0, \infty)$. Now take any $t \in (0, \infty)$ and arbitrary $x \in (0, \infty)$. Fix any $s \in (0, t)$. Then, since $F(s, x) \in [1, b]$ and the function $N(\cdot, \cdot, 1)$ satisfies the cocycle equation (3.3), what we have just proved gives

$$\begin{aligned} N(t, x, 1) &= N(s, x, 1)N(t-s, F(s, x), 1) \\ &= N(s, x, 1)\frac{\kappa(F(s, x), 1)}{\kappa(F(t-s, F(s, x)), 1)} \\ &= N(s, x, 1)\frac{\kappa(M(s, x, 1)/N(s, x, 1), 1)}{\kappa(F(t, x), 1)} \\ &= \frac{\kappa(M(s, x, 1), N(s, x, 1))}{\kappa(F(t, x), 1)} = \frac{\kappa(x, 1)}{\kappa(F(t, x), 1)}, \end{aligned}$$

that is again (4.3). This completes the proof of assertion (iv). □

Remark 4.1. One could expect that the embeddability of a mean-type mapping in a continuous iteration group should be much easier to settle. It turns out that such a problem is not well posed as, which is easy to verify, in general the inverse of a strict mean-type mapping is not a mean-type mapping.

Acknowledgement

The authors are grateful to the referee for all the comments and suggestions which improved the paper.

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