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Iterations of the mean-type mappings

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Abstract

Let an interval $I \subset \mathbb{R}$ and $p \in \mathbb{N}$, $p \geq 2$, be fixed. Assuming that the continuous means $M_i : I^p \to I$, $i = 1, ..., p$, are such that

$$\min(x_1, ...x_p) + \max(M_1(x_1, ...x_p), ..., M_p(x_1, ...x_p))$$
$$< \min(M_1(x_1, ...x_p), ..., M_p(x_1, ...x_p)) + \max(x_1, ...x_p)$$

if not all of $x_1, ...x_p \in I$ are equal, we prove that the sequence of iterates of the mean-type mapping $(M_1, ..., M_p) : I^p \to I^p$ converges to a mean-type mapping $(K, ..., K)$, where $K : I^p \to I$ is a continuous mean. Moreover K is uniquely determined by the condition of $(M_1, ..., M_p)$-invariance. This improves an earlier result of [6] where it is assumed that at most one of the means $M_1, ..., M_p$ is not strict. As an application, for some families of mean-type mappings $(M_1, ..., M_p)$, the effective form of real continuous solutions F of the functional equation $F \circ (M_1, ..., M_p) = F$ is given. An application to the theory of iterative functional equation of the form $\varphi(t) = g(t)\varphi\left(\frac{f(t)}{g(t)}\right)$ is presented.

1 Introduction

Let an interval $I \subset \mathbb{R}$ and $p \in \mathbb{N}$, $p \geq 2$, be fixed. Assume that $M_i : I^p \to I$, $i = 1, ..., p$, are continuous means. In [8] (cf. also [10]) it was proved that if at most one of these means is not strict, then the sequence of iterates of the mean-type mapping $(M_1, ..., M_p) : I^p \to I^p$ converges to a mean-type mapping $(K, ..., K)$, where $K : I^p \to I$ is a continuous and $(M_1, ..., M_p)$-invariant mean, i.e.

$$K \circ (M_1, ..., M_p) = K.$$

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The continuity and the $(M_1, ..., M_p)$-invariance of K imply its uniqueness. Moreover, if $M_1, ..., M_p$ are strict, then so is K.

In the present paper we generalize this result. In Section 3 we show (Theorem 1) that the conclusions of the above result remain true on replacing the assumption that at most one of these means $M_1, ..., M_p$ is not strict with the following weaker and more symmetric condition: if not all $x_1, ..., x_p \in I$ are equal, then

$$\min(x_1, ..., x_p) + \max(M_1(x_1, ..., x_p), ..., M_p(x_1, ..., x_p))$$
$$< \min(M_1(x_1, ..., x_p), ..., M_p(x_1, ..., x_p)) + \max(x_1, ..., x_p).$$

Taking $I = (0, \infty)$, $p = 2$ and $M_1 = A_{[2]}$, $M_2 = G_{[2]}$, where $A_{[2]}(x, y) = \frac{x+y}{2}$, $G_{[2]}(x, y) = \sqrt{xy}$, we obtain the classical result of Gauss [5] who considered the arithmetic-geometric mean iteration in connection with elliptic integrals. In this case the $(A_{[2]}, G_{[2]})$-invariant mean K has the form

$$K(x, y) = \left(\frac{2}{\pi}\int_0^{\pi/2} \frac{dt}{\sqrt{x^2(\cos t)^2 + y^2(\sin t)^2}}\right)^{-1}, \qquad x, y > 0,$$

is denoted by $A_{[2]} \otimes G_{[2]}$. For some other applications see [1], [2], Chapter VI; [3], Chapter 4, [4], where iterations of two-dimensional continuous and strict mean-type mappings are considered. The invariant mean is also called the Gauss composition of means, the Gaussian product of means or the compound mean.

Let us note that the proportion $x : \frac{x+y}{2} = \frac{2xy}{x+y} : y$, the base of the theory of harmony made by Pythagorean school, can be written in the form

$$\sqrt{\frac{x+y}{2} \cdot \frac{2xy}{x+y}} = \sqrt{xy}.$$

Setting $H_{[2]}(x, y) = \frac{2xy}{x+y}$ the harmonic mean, we hence get $G_{[2]} \circ (A_{[2]}, H_{[2]}) = G_{[2]}$ which says that the geometric mean $G_{[2]}$ is $(A_{[2]}, H_{[2]})$-invariant. Thus the notion of invariance of a mean-type mapping has its roots in ancient times.

In Section 4 we present some general classes of continuous mean-type mappings for which the invariant mean can be easily established. Then Theorem 1 allows to determine effectively the limits of respective sequences of their iterates. We apply this fact to obtain all functions $F : I^p \to \mathbb{R}$, which are continuous on the diagonal

$$\Delta(I^p) := \{(x_1, ..., x_p) \in I^p : x_1 = ... = x_p\}$$

of the cube I^p, and satisfy the functional equation

$$F \circ (M_1, ..., M_p) = F.$$

In Section 5 we observe close relation of means and functional equations of iterative type. An application to the theory of the functional equation

$$\varphi(t) = g(t)\varphi\left(\frac{f(t)}{g(t)}\right), \qquad t > 0,$$

where the given functions f, g and the unknown function φ belong to a family of functions the class $S_1 = S_1^0(I)$ (cf. [6], p. 20) or containing this class, is presented.

2 Preliminaries

Let $I \subset \mathbb{R}$ be an interval and $p \in \mathbb{N}$, $p \geq 2$, fixed. A function $M : I^p \to \mathbb{R}$ is said to be a *mean* on I if, for all $x_1, ..., x_p \in I$,

$$\min(x_1, ..., x_p) \leq M(x_1, ..., x_p) \leq \max(x_1, ..., x_p).$$

A mean M in I is called *strict* if these inequalities are strict whenever

$$\min(x_1, ..., x_p) < \max(x_1, ..., x_p).$$

If $I = (0, \infty)$ we say that a mean M in I is *positively homogeneous* if

$$M(tx_1, ..., tx_p) = tM(x_1, ..., x_p), \qquad t, x_1, ..., x_p > 0.$$

Note the following easy to verify

Remark 1 *Let $M : I^p \to \mathbb{R}$ be an arbitrary function. Then the following conditions are equivalent*

1. *M is a mean;*
2. *$M(J^p) \subset J$ for every subinterval $J \subset I$,*
3. *$M(J^p) = J$ for every subinterval $J \subset I$.*

Hence we have

Remark 2 *If $M : I^p \to \mathbb{R}$ is a mean then M maps I^p onto I and, moreover, M is reflexive, that is, for all $x \in I$,*

$$M(x, ..., x) = x.$$

Let us also note the following

Remark 3 *If a function $M : I^p \to \mathbb{R}$ is reflexive and (strictly) increasing with respect to each variable, then M is a (strict) mean I.*

3 Iterations of mean-type mappings and invariant means

A mapping $\mathbf{M} : I^p \to I^p$ is referred to as *mean-type* if there are some means $M_i : I^p \to I$, $i = 1, ..., p$, such that $\mathbf{M} = (M_1, ..., M_p)$. We say that the mean-type mapping $\mathbf{M}$ is *strict (positively homogeneous)* if each of its coordinate means $M_1, ..., M_p$ is strict (positively homogeneous).

Put $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$.

If $\mathbf{M} : I^p \to I^p$ is a mean-type mapping then, clearly, the sequence $(\mathbf{M}^n)_{n=0}^{\infty}$ of the iterates of $\mathbf{M}$,

$$\mathbf{M}^0 := \mathrm{Id}|_{I^p}; \qquad \mathbf{M}^{n+1} := \mathbf{M} \circ \mathbf{M}^n \quad \text{for } n \in \mathbb{N}_0,$$

is well defined.

We have the following obvious

Remark 4 *Suppose that $\mathbf{M} : I^p \to I^p$, $\mathbf{M} = (M_1, ..., M_p)$, is a mean-type mapping of I^p. Then, for each $n \in \mathbb{N}_0$,*

$$\mathbf{M}^n = (M_{n,1}, ..., M_{n,p})$$

where, for all $i = 1, ..., p$, $(x_1, ..., x_p) \in I^p$,

$$M_{i,0}(x_1, ..., x_p) = x_i,$$

and, for all $n \in \mathbb{N}_0$, $i = 1, ..., p$, $(x_1, ..., x_p) \in I^p$,

$$M_{i,n+1}(x_1, ..., x_p) = M_i(M_{1,n}(x_1, ..., x_p), ..., M_{p,n}(x_1, ..., x_p)).$$

Given a mean-type mapping $\mathbf{M} : I^p \to I^p$ and a mean $K : I^p \to I$ we say that K is *invariant with respect to the mean-type mapping* $\mathbf{M}$, briefly, $\mathbf{M}$-*invariant*, if

$$K \circ \mathbf{M} = K.$$

Remark 5 *Note that a mean $K : I^p \to I$ is $\mathbf{M}$-invariant iff the mean-type mapping $\mathbf{K} : I^p \to I^p$ defined by $\mathbf{K} = (K, ..., K)$ is $\mathbf{M}$-invariant, that is, iff $\mathbf{K} = \mathbf{K} \circ \mathbf{M}$.*

The main result of this section reads as follows:

Theorem 1 *Let $I \subset \mathbb{R}$ be an interval and $p \in \mathbb{N}$, $p \geq 2$, be fixed. Suppose that $\mathbf{M} : I^p \to I^p$, $\mathbf{M} = (M_1, ..., M_p)$, is a continuous mean-type mapping of I^p such that, for all $(x_1, ..., x_p) \in I^p \backslash \Delta(I^p)$,*

$$\max(M_1(x_1, ..., x_p), ..., M_p(x_1, ..., x_p)) - \min(M_1(x_1, ..., x_p), ..., M_p(x_1, ..., x_p))$$
$$< \max(x_1, ..., x_p) - \min(x_1, ..., x_p).$$

Then

1. *for every $n \in \mathbb{N}$, the n-th iterate $\mathbf{M}^n = (M_{n,1}, ..., M_{n,p})$, is a mean-type mapping of I^p;*

2. *there is a continuous mean $K : I^p \to I$ such that the sequence of iterates $(\mathbf{M}^n)_{n=0}^{\infty}$ converges, uniformly on compact subsets of I^p, to the mean-type mapping $\mathbf{K} : I^p \to I^p$, $\mathbf{K} = (K_1, ..., K_p)$ such that*
$$K_1 = ... = K_p = K;$$

3. *$\mathbf{K} : I^p \to I^p$ is $\mathbf{M}$-invariant, that is,*
$$\mathbf{K} = \mathbf{K} \circ \mathbf{M}$$
or, equivalently, the mean K is $\mathbf{M}$-invariant;

4. *a continuous $\mathbf{M}$-invariant mean (mean-type mapping) is unique;*

5. *if $\mathbf{M}$ is strict then so is K (and $\mathbf{K}$);*

6. *if $M_1, ..., M_p$ are (strictly) increasing with respect to each variable then so is K;*

7. *if $I = (0, \infty)$ and $\mathbf{M}$ is positively homogeneous, then every iterate of $\mathbf{M}$ and K are positively homogeneous.*

Proof. From the definition of the mean we infer that the basic assumption of our result can be formulated as follows: for all $x_1, ..., x_p \in I$, the inequality

$$\min(x_1, ..., x_p) < \max(x_1, ..., x_p)$$

implies that

$$\min(x_1, ..., x_p) \leq \min(M_1(x_1, ..., x_p), ..., M_p(x_1, ..., x_p))$$

or

$$\max(M_1(x_1,...,x_p),...,M_p(x_1,...,x_p)) < \max(x_1,...,x_p).$$

To avoid writing too long expression we assume that $p = 2$. It is easy to see that the same idea works in general case.

Part 1 is an immediate consequence of the definition of a mean.

Assume that $M, N : I^2 \to I$ are continuous means satisfying the above condition. Thus, for all $x, y \in I$, if

$$\min(x, y) < \max(x, y)$$

then

$$\min(x, y) < \min(M(x, y), N(x, y)). \tag{1}$$

or

$$\max(M(x, y), N(x, y)) < \max(x, y). \tag{2}$$

Consider the sequence $(M, N)^n$, $n \in \mathbb{N}$, of the iterates of the mean-type mapping $(M, N) : I^2 \to I^2$.

Putting (cf. Remark 4)

$$M_0(x, y) := x, \quad N_0(x, y) := y, \quad x, y \in I,$$

and

$$(M_n, N_n) := (M, N)^n, \qquad n \in \mathbb{N}_0,$$

we have

$$M_{n+1} = M \circ (M_n, N_n), \qquad N_{n+1} = N \circ (M_n, N_n), \qquad n \in \mathbb{N}_0. \tag{3}$$

Define

$$\alpha_n := \min(M_n, N_n), \qquad \beta_n := \max(M_n, N_n), \qquad n \in \mathbb{N}_0.$$

Taking into account the definition of the mean, we have

$$\alpha_n \le \alpha_{n+1} \le \beta_{n+1} \le \beta_n, \qquad n \in \mathbb{N}_0,$$

whence

$$\alpha := \sup\{\alpha_n : n \in \mathbb{N}_0\} = \lim_{n\to\infty} \alpha_n, \qquad \beta := \inf\{\beta_n : n \in \mathbb{N}_0\} = \lim_{n\to\infty} \beta_n$$

and

$$\alpha \le \beta.$$

We shall show that $\alpha = \beta$. Assume, for an indirect argument, that

$$\alpha(x_0, y_0) < \beta(x_0, y_0),$$

for some $x_0, y_0 \in I$. Then, by (1) and (2), we would have

$$\alpha(x_0, y_0) < \min[M(\alpha(x_0, y_0), \beta(x_0, y_0)), N(\alpha(x_0, y_0), \beta(x_0, y_0))] \leq \beta(x_0, y_0)$$

or

$$\alpha(x_0, y_0) \leq \max[M(\alpha(x_0, y_0), \beta(x_0, y_0)), N(\alpha(x_0, y_0), \beta(x_0, y_0))] < \beta(x_0, y_0).$$

Since $\min(x, y) = \min(y, x)$ and $\max(x, y) = \max(y, x)$, we also have

$$\alpha(x_0, y_0) < \min[M(\beta(x_0, y_0), \alpha(x_0, y_0)), N(\beta(x_0, y_0), \alpha(x_0, y_0))] \leq \beta(x_0, y_0)$$

or

$$\alpha(x_0, y_0) \leq \max[M(\beta(x_0, y_0), \alpha(x_0, y_0)), N(\beta(x_0, y_0), \alpha(x_0, y_0)) < \beta(x_0, y_0).$$

By the continuity of M and N there is $\delta > 0$ such that, for all $(u, v) \in I^2$, the following implication holds true

$$\begin{gathered}\alpha(x_0, y_0) - \delta < u < \beta(x_0, y_0) \quad \text{and} \quad \alpha(x_0, y_0) < v < \beta(x_0, y_0) + \delta \\ \Downarrow \\ \{\alpha(x_0, y_0) < \min(M(u, v), N(u, v)) < \beta(x_0, y_0) + \delta \\ \text{or} \\ \alpha(x_0, y_0) - \delta < \max(M(u, v), N(u, v)) < \beta(x_0, y_0)\}.\end{gathered} \tag{4}$$

By the definitions of α and β there is $n_0 \in N$ such that, for all $n \in \mathbb{N}$, $n \geq n_0$,

$$\alpha(x_0, y_0) - \delta < \alpha_n(x_0, y_0) < \alpha(x_0, y_0)$$

and

$$\beta(x_0, y_0) < \beta_n(x_0, y_0) < \beta(x_0, y_0) + \delta.$$

For all $n \in \mathbb{N}_0$,

$$\alpha_n(x_0, y_0) = \min(M(\alpha_n(x_0, y_0), N_n(x_0, y_0))$$

and

$$\beta_n(x_0, y_0) = \max(M_n(x_0, y_0), N_n(x_0, y_0)).$$

By implication (4), we would have

$$\alpha(x_0, y_0) < M(M_n(x_0, y_0), N_n(x_0, y_0)) < \beta(x_0, y_0), \qquad n \geq n_0.$$

or

$$\alpha(x_0, y_0) < N(M_n(x_0, y_0), N_n(x_0, y_0)) < \beta(x_0, y_0), \qquad n \geq n_0,$$

that is, for all $n \geq n_0$,

$$\alpha(x_0, y_0) < M_{n+1}(x_0, y_0) < \beta(x_0, y_0),$$

or

$$\alpha(x_0, y_0) < N_{n+1}(x_0, y_0) < \beta(x_0, y_0)$$

whence, by the definition of the sequences α_n and β_n, for all $n \geq n_0$,

$$\alpha(x_0, y_0) < \alpha_{n+1}(x_0, y_0) < \beta(x_0, y_0),$$

or

$$\alpha(x_0, y_0) < \beta_{n+1}(x_0, y_0) < \beta(x_0, y_0).$$

This contradicts to the definition of $\alpha(x_0, y_0)$ and $\beta(x_0, y_0)$ and proves that $\alpha = \beta$. Since, for every $n \in \mathbb{N}$, the functions α_n and β_n are continuous, the sequence (α_n) is increasing and the sequence (β_n) is decreasing, the function α is lower semicontinuous and β is upper semicontinuous. It follows that the function

$$K(x, y) := \alpha(x, y), \qquad x, y \in I,$$

is continuous. By Dini's Theorem, the convergence of the monotonic sequences of continuous functions to a continuous function is uniform on compact sets. It is obvious that K is a mean in I. From the definition of the sequences (α_n) and (β_n) we have

$$\alpha_n \leq M_n \leq \beta_n, \qquad \alpha_n \leq N_n \leq \beta_n, \qquad n \in \mathbb{N}_0.$$

It follows that

$$\lim_{n\to\infty} M_n = K = \lim_{n\to\infty} N_n,$$

and, obviously, this convergence is uniform on compact sets. Consequently,

$$\lim_{n\to\infty} (M, N)^n = \lim_{n\to\infty} (M_n, N_n) = (K, K), \tag{5}$$

uniformly on compact subsets of I^2. This completes the proof of part 2.

Put $\mathbf{K} := (K, K)$. From (5) and (3) we have

$$\begin{aligned}\mathbf{K} &= (K, K) = \lim_{n\to\infty} (M_{n+1}, N_{n+1}) = \lim_{n\to\infty} (M_n \circ (M, N), N_n \circ (M, N)) \\ &= \left(\lim_{n\to\infty} M_n \circ (M, N), \lim_{n\to\infty} N_n \circ (M, N)\right) = (K \circ (M, N), K \circ (M, N)) \\ &= \mathbf{K} \circ (M, N),\end{aligned}$$

which proves conclusion 3.

Assume that $\mathbf{L} = (L, L) : I^2 \to I^2$ is a continuous and (M, N)-invariant mean-type mapping, that is $\mathbf{L} = \mathbf{L} \circ (M, N)$. Hence, by induction,

$$\mathbf{L} = \mathbf{L} \circ (M, N)^n, \quad n \in \mathbb{N}.$$

Letting $n \to \infty$, and applying the reflexivity of the means, we obtain

$$\begin{aligned}\mathbf{L} &= \lim_{n\to\infty} \mathbf{L} \circ (M, N)^n = \mathbf{L} \circ (M_\lambda, M_{1-\lambda}) = \mathbf{L} \circ \mathbf{K} = (L \circ (K, K), L \circ (K, K)) \\ &= (K, K) = \mathbf{K},\end{aligned}$$

which completes the proof of conclusion 4.

Since the remaining pars are obvious, the proof is completed. □

Theorem 1 improves the main result of [8], Theorem 1 (cf. also [10]) where a much stronger condition, that at most one of its coordinate means $M_1, ..., M_p$ is not strict, is assumed. In the case when $p = 2$ and the coordinate means are strict the suitable result is well known and frequently applied (cf. [1], [2], Chapter VI, [3], Chapter 4, [4]; and [9]).

Remark 6 *The condition assumed in Theorem 1 is indispensable.*

To show it consider three examples.

Example 1 *Let* $\mathbf{M} : I^p \to I^p$, $\mathbf{M} = (M_1, ..., M_p)$, *be a mean-type mapping such that*

$$M_1(x_1, ..., x_p) = \min(x_1, ..., x_p), \qquad M_p(x_1, ..., x_p) = \max(x_1, ..., x_p).$$

Then, for all $x_1, ..., x_p \in I$,

$$\begin{aligned}&\min(x_1, ..., x_p) + \max(M_1(x_1, ..., x_p), ..., M_p(x_1, ..., x_p)) \\ &= M_1(x_1, ..., x_p) + M_p(x_1, ..., x_p) \\ &= \min(M_1(x_1, ..., x_p), ..., M_p(x_1, ..., x_p)) + \max(x_1, ..., x_p),\end{aligned}$$

so the assumption of Theorem 1 is not fulfilled. Since the means $K = M_1$ *and* $K = M_p$ *are* $\mathbf{M}$*-invariant, the uniqueness of the continuous* $\mathbf{M}$*-invariant mean is failed.*

Example 2 *Take* $p = 3$, $I = \mathbb{R}$, *and* $\mathbf{M} := (M_1, M_2, M_3)$ *where* $M_1(x_1, x_2, x_3) := \min(x_1, x_2, x_3)$, $M_2(x_1, x_2, x_3) := x_1 + x_2 + x_3 - \min(x_1, x_2, x_3) - \max(x_1, x_2, x_3)$, $M_3(x_1, x_2, x_3) := \max(x_1, x_2, x_3)$. *Then each of the means* M_1, M_3 *and* $A_{[3]}(x_1, x_2, x_3) = \frac{x_1+x_2+x_3}{3}$ *is* $\mathbf{M}$*-invariant.*

Example 3 *Take* $p = 2$, $I = \mathbb{R}$, *and* $\mathbf{M} := (M_1, M_2)$, *where* $M_1(x_1, x_2) = \min(x_1, x_2)$, *and* $M_2(x_1, x_2) = \max(x_1, x_2)$. *Then each of the means* M_1, M_2 *and* $A_2(x_1, x_2) := \frac{x_1+x_2}{2}$ *is* $\mathbf{M}$*-invariant.*

In each of these examples the mean-type mapping $\mathbf{M}$ does not fulfill the basic assumption of Theorem 1 and the $\mathbf{M}$-invariant mean is not unique.

4 Some special mean-type mappings and applications to functional equations of several variables

In this section we consider some general classes of mean-type mappings for which the invariant means can be effectively determined and we present some applications of Theorem 1 in the theory of functional equations.

We begin with the case when the arithmetic mean is invariant.

Remark 7 *Let* $M : I^2 \to I$ *be an arbitrary mean in an interval* I. *Then*

$$M(x, y) = \lambda(x, y)x + (1 - \lambda(x, y))y, \qquad x, y \in I,\ x \neq y,$$

where

$$\lambda(x, y) := \frac{M(x, y) - y}{x - y} \in [0, 1], \qquad x, y \in I,\ x \neq y.$$

It follows that every mean can be written in the form

$$M(x, y) = \lambda(x, y)x + (1 - \lambda(x, y))y, \qquad x, y \in I,$$

where $\lambda : I^2 \to [0, 1]$.

If $\lambda : I^2 \to [0, 1]$ *is a continuous function then the function* $M_\lambda : I^2 \to I$ *defined by*

$$M_\lambda(x, y) := \lambda(x, y)x + (1 - \lambda(x, y))y, \qquad x, y \in I,$$

is a continuous mean in I. *If, moreover ,* $\lambda : I^2 \to (0, 1)$ *then* M_λ *and* $M_{1-\lambda}$ *are strict. In particular, the mean-type mapping* $(M_\lambda, M_{1-\lambda}) : I^2 \to I^2$ *satisfies the assumptions of Theorem 1. Since the arithmetic mean* $A_{[2]}$ *is* $(M_\lambda, M_{1-\lambda})$*-invariant, in view of Theorem 1, the sequence of iterates* $((M_\lambda, M_{1-\lambda})^n)$ *converges (uniformly on compact subsets of* I*) and*

$$\lim_{n\to\infty} (M_\lambda, M_{1-\lambda})^n = (A_{[2]}, A_{[2]}).$$

Applying Theorem 1 we prove the following

Proposition 1 *Let $I \subset \mathbb{R}$ be an interval and let $\Delta(I^2) := \{(x,x) : x \in I\}$ be the diagonal of I^2. Suppose that a function $F : I^2 \to \mathbb{R}$ is continuous on $\Delta(I^2)$. Then F satisfies the functional equation*

$$F(M_\lambda(x,y), M_{1-\lambda}(x,y)) = F(x,y), \qquad x, y \in I, \tag{5}$$

if, and only if, there is a single variable continuous function $f : I \to \mathbb{R}$ such that

$$F(x,y) = f\left(\frac{x+y}{2}\right), \qquad x, y \in I. \tag{6}$$

Proof. Assume that F satisfies equation (5). Hence, by induction,

$$F(x,y) = F \circ [(M_\lambda, M_{1-\lambda})^n](x,y), \qquad n \in \mathbb{N},\ x, y \in I.$$

By Remark 7 and the continuity of F on $\Delta(I^2)$, letting $n \to \infty$, we get

$$F(x,y) = F\left(\frac{x+y}{2}, \frac{x+y}{2}\right), \qquad x, y \in I.$$

Setting $f(x) := F(x,x)$ for $x \in I$ we get (6). The converse implication is easy to verify. □

The above remark and proposition can be easily generalized.

Remark 8 *Let $I \subset \mathbb{R}$ be an interval and $p \in \mathbb{N}$, $p \geq 2$. Assume that $\lambda_i : I^p \to (0,1)$, $i = 1, ..., p$, are continuous and such that*

$$\sum_{i=1}^{p} \lambda_i(x_1, ..., x_p) = 1, \qquad x_1, ..., x_p \in I.$$

Then the function $M_{\lambda_1,...,\lambda_p} : I^p \to I$ defined by

$$M_{\lambda_1,...,\lambda_p} := \sum_{i=1}^{p} x_i \lambda_i(x_1, ..., x_p), \qquad x_1, ..., x_p \in I.$$

is a continuous and strict mean in I. It is easy to see that the mean-type mapping

$$\mathbf{M} := (M_{\lambda_1,...,\lambda_p}, M_{\lambda_2,...,\lambda_p,\lambda_1}, M_{\lambda_3,...,\lambda_p,\lambda_1,\lambda_2}, ..., M_{\lambda_p,\lambda_1,\lambda_2,...,\lambda_{p-1}}) : I^p \to I^p$$

satisfies the assumptions of Theorem 1. Since the arithmetic mean

$$A_{[p]}(x_1, ..., x_p) = \frac{x_1, ..., x_p}{p}$$

is $\mathbf{M}$*-invariant, in view of Theorem 1, the sequence of iterates* $(\mathbf{M}^n)_{n=0}^{\infty}$ *converges and*

$$\lim_{n\to\infty} \mathbf{M}^n = (A_{[p]}, ..., A_{[p]}).$$

Similarly as Proposition 1, we can prove

Proposition 2 *Let* $I \subset \mathbb{R}$ *be an interval and let* $\Delta(I^p) := \{(x_1, ..., x_p) \in I^p : x_1 = .. = x_p\}$*. Suppose that a function* $F : I^p \to \mathbb{R}$ *is continuous on* $\Delta(I^p)$*. Then* F *satisfies the functional equation*

$$F \circ (M_{\lambda_1,...,\lambda_p}, M_{\lambda_2,...,\lambda_p,\lambda_1}, ..., M_{\lambda_p,\lambda_1,\lambda_2,...,\lambda_{p-1}}) = F$$

if, and only if, there is a continuous single variable function $f : I \to \mathbb{R}$ *such that*

$$F(x_1, ..., x_p) = f\left(A_{[p]}(x_1, ..., x_p)\right), \qquad x_1, ..., x_p \in I.$$

Now we consider some mean-type mappings for which the geometric mean is invariant.

Remark 9 *Let* $I \subset (0, \infty)$ *be an interval and* $p \in \mathbb{N}$, $p \geq 2$ *be fixed. Assume that* $\varphi_i : I^p \to (0, \infty)$, $i = 1, ..., p$, *are continuous. Then, for each* $i = 1, ..., p$, *the function* $M_i : I^p \to (0, \infty)$ *defined by*

$$M_i(x_1, ..., x_p) := \frac{\sum_{k=1}^{p} \left(\prod_{j=0}^{i-1} x_{k+i+j-1}\right) \varphi_k(x_1, ..., x_p)}{\sum_{k=1}^{p} \left(\prod_{j=0}^{i-2} x_{k+i+j-1}\right) \varphi_k(x_1, ..., x_p)},$$

where $x_{p+j} = x_j$ *for* $j = 1, 2, ...,$ *is a continuous and strict mean. Note that the geometric mean*

$$G_{[p]}(x_1, ..., x_p) = \sqrt[p]{x_1 \cdot ... \cdot x_p}$$

is $(M_1, ..., M_p)$*-invariant. Thus, in view of Theorem 1,*

$$\lim_{n\to\infty} (M_1, ..., M_p)^n = \left(G_{[p]}, ..., G_{[p]}\right).$$

The invariance identity

$$G_{[p]} \circ (M_1, ..., M_p) = G_{[p]}$$

generalizes the harmony proportion.

In the case $p = 2$, setting $M := M_1$, $N := M_2$, $\varphi := \varphi_1$, $\psi := \varphi_2$, $x := x_1$, $y := x_2$, we have

$$M(x,y) = \frac{x\varphi(x,y) + y\psi(x,y)}{\varphi(x,y) + \psi(x,y)}, \qquad N(x,y) = \frac{yx\varphi(x,y) + xy\psi(x,y)}{x\varphi(x,y) + y\psi(x,y)}.$$

Taking here $I = (0, \infty)$, $\varphi(x,y) = \psi(x,y) = 1$ for $x, y \in (0, \infty)$, we have $M = A_{[2]}$, $N = H_{[2]}$. Hence we get

$$G_{[2]} \circ (A_{[2]}, H_{[2]}) = G_{[2]}, \tag{7}$$

the invariance mentioned in the Introduction, equivalent to the Pythagorean harmony proportion, and, moreover

$$\lim_{n \to \infty} (A_{[2]}, H_{[2]})^n = G_{[2]}.$$

In the case $p = 3$, setting $M := M_1$, $N := M_2$, $P := M_3$, $\varphi := \varphi_1$, $\psi := \varphi_2$, $\gamma := \varphi_3$, $x := x_1$, $y := x_2$, $z := x_3$, we have

$$\begin{aligned} M(x,y,z) &= \frac{x\varphi(x,y,z) + y\psi(x,y,z) + z\gamma(x,y,z)}{\varphi(x,y,z) + \psi(y,y,z) + \gamma(x,y,z)}, \\ N(x,y,z) &= \frac{yx\varphi(x,y,z) + zy\psi(x,y,z) + xz\gamma(x,y,z)}{x\varphi(x,y,z) + y\psi(x,y,z) + z\gamma(x,y,z)}, \\ P(x,y,z) &= \frac{zyx\varphi(x,y,z) + xzy\psi(x,y,z) + yxz\gamma(x,y,z)}{yx\varphi(x,y,z) + zy\psi(x,y,z) + xz\gamma(x,y,z)}. \end{aligned}$$

Taking $I = (0, \infty)$, $\varphi(x,y,z) = \psi(x,y,z) = \gamma(x,y,z) = 1$ for $x, y, z \in (0, \infty)$, we have

$$\begin{aligned} M(x,y,z) &= A_{[3]}(x,y,z) := \frac{x+y+z}{3}, \\ N(x,y,z) &= \frac{yx + zy + xz}{x+y+z} \\ P(x,y,z) &= H_{[3]}(x,y,z) := \frac{zyx + xzy + yxz}{yx + zy + xz} = \frac{3}{\frac{1}{x} + \frac{1}{y} + \frac{1}{z}}, \end{aligned}$$

where $A_{[3]}$ is the arithmetic mean and $H_{[3]}$ the harmonic mean. Applying the last remark we obtain

$$G_{[3]} \circ \left(A_{[3]}, N, H_{[3]}\right) = G_{[3]} \tag{8}$$

and

$$\lim_{n\to\infty}\left(A_{[3]}, N, H_{[3]}\right)^n = G_{[3]}.$$

The identity (8) can be seen as a three-dimensional counterpart of two-dimensional harmony relation (7).

Taking $I=(0,\infty)$, $\varphi_j = 1$ for $j = 1, ..., p$, in Remark 9, we obtain a p-dimensional counterpart of harmony relation (7) (see also [10]).

Applying Theorem 1 and Remark 9 we obtain the following

Proposition 3 *A function $F : I^p \to \mathbb{R}$, continuous on $\Delta(I^p)$, satisfies the functional equation*

$$F\circ(M_1, ..., M_p) = F$$

if, and only if, there is a continuous function $f : I \to \mathbb{R}$ such that

$$F(x_1, ..., x_p) = f\left(G_{[p]}(x_1, ..., x_p)\right), \qquad x_1, ..., x_p \in I.$$

In connection with invariance of the mean-type mappings defined in Remark 9, the following problem arises. What are necessary and sufficient conditions for the $(M_1, ..., M_p)$-invariance of the geometric mean $G_{[p]}$ if, for some $\varphi_{i,j} : I^p \to (0,\infty)$, $i, j \in \{1, ..., p\}$,

$$M_i(x_1, ..., x_p) = \frac{\sum_{j=1}^{p} x_j\varphi_{i,j}(x_1, ..., x_p)}{\sum_{j=1}^{p}\varphi_{i,j}(x_1, ..., x_p)}, \qquad x_1, ..., x_p \in I?$$

In the case $p = 2$ the answer gives the following

Proposition 4 *Let $M, N : I^2 \to I$ be given by*

$$M(x,y) = \frac{x\alpha(x,y) + y\beta(x,y)}{\alpha(x,y)+\beta(x,y)}, \qquad N(x,y) = \frac{x\varphi(x,y) + y\psi(x,y)}{\varphi(x,y)+\psi(x,y)},$$

for some $\alpha, \beta, \varphi, \psi : I^2 \to (0,\infty)$. Then the following conditions are equivalent:

1. *the geometric mean $G_{[2]}$ is (M,N)-invariant;*
2. *for all $x, y \in I$,*

$$\psi(x,y) = \frac{x\alpha(x,y)\varphi(x,y)}{y\beta(x,y)}, \qquad x, y \in I;$$

3. the means M and N have the forms

$$M(x,y) = \frac{x\alpha(x,y) + y\beta(x,y)}{\alpha(x,y) + \beta(x,y)}, \quad N(x,y) = \frac{yx\alpha(x,y) + xy\beta(x,y)}{x\alpha(x,y) + y\beta(x,y)}, \quad x,y \in I;$$

4. $M = M_1$ and $N = M_2$ where M_1 and M_2 are defined in Proposition 9 with $p = 2$.

Proof. Writing the equality $G_{[2]} \circ (M,N) = G_{[2]}$ in the explicit form, after simple calculations, we obtain

$$(y-x)\left[y\beta(x,y)\psi(x,y) - x\alpha(x,y)\varphi(x,y)\right] = 0, \qquad x,y \in I,$$

which is equivalent to the equality $\psi(x,y) = \frac{x\alpha(x,y)\varphi(x,y)}{y\beta(x,y)}$ for all $x,y \in I$. The remaining equivalences are easy to verify. □

We end up this section with the following simple

Remark 10 *Let $\varphi, \psi : I \to (0,\infty)$. A mean $M : I^2 \to I$,*

$$M(x,y) = \frac{x\varphi(x,y) + y\psi(x,y)}{\varphi(x,y) + \psi(x,y)}, \qquad x,y \in I,$$

is symmetric iff the function $\frac{\psi}{\varphi}$ is symmetric, that is iff

$$\frac{\psi(x,y)}{\varphi(x,y)} = \frac{\psi(y,x)}{\varphi(y,x)}, \qquad x,y \in I.$$

5 Application in the theory of iterative functional equations

In the theory of functional equation in a single variable:

$$F(x,\varphi(x),\varphi[f(x)]) = 0,$$

where φ is an unknown function, the class $\mathbf{S}_\xi^n(I)$ of functions, from which the given function f is taken, plays important role (cf. [6]).

Let $I \subset \mathbb{R}$ be an interval, $\xi \in \mathrm{cl}\, I$. In its original form, $\mathbf{S}_\xi^n(I)$ was defined (cf. [6], p. 20) as the class of functions $f : I \to \mathbb{R}$ which are n-times continuously differentiable in I and fulfill the conditions:

$$\begin{aligned} (f(x) - x)(\xi - x) > 0 \quad &\text{for} \quad x \in I,\ x \neq \xi, \\ (f(x) - \xi)(\xi - x) < 0 \quad &\text{for} \quad x \in I,\ x \neq \xi. \end{aligned}$$

It was observed (cf. [7]) that these two inequalities are equivalent to the simpler condition:

$$0 < \frac{f(x) - \xi}{x - \xi} < 1 \quad \text{for} \quad x \in I,\ x \neq \xi.$$

The next remarks show that there are close relations between the class $\mathbf{S}_\xi^0(I)$ and the family of means.

Remark 11 *A function $f \in \mathbf{S}_\xi^0(I)$ if, and only, if $f : I \to \mathbb{R}$ is continuous and*

$$\min(x, \xi) < f(x) < \max(x, \xi) \quad \textit{for} \quad x \in I,\ x \neq \xi.$$

Remark 12 *Let $M : I^2 \to I$ be a strict and continuous mean in an interval I. Then, for every $\xi \in \operatorname{cl} I$, the function $f(x) := M(x, \xi)$, for $x \in I$, belongs to $\mathbf{S}_\xi^0(I)$.*

Applying Theorem 1, we get

Remark 13 *(cf. [6], p. 20) If $f \in \mathbf{S}_\xi^0(I)$ then*

$$\lim_{n\to\infty} f^n(t) = \xi, \qquad t \in (0, \infty),$$

where f^n denotes the nth iterate of f.

Proof. Assume that $\xi \in \mathbb{R}$ is the left end-point of I and define

$$M(x, y) := \begin{cases} f(\xi + x - y) - \xi + y & \text{for } \xi \leq y \leq x \\ f(\xi + y - x) - \xi + x & \text{for } \xi \leq x \leq y \end{cases}.$$

It can be easily verified that M is well defined in I^2. The assumption $f \in \mathbf{S}_\xi^0(I)$ implies that M is a strict and continuous mean in I. Moreover, since $\xi \leq x$ for all $x \in I$, we hence get

$$M(x, \xi) = f(x), \qquad x \in I.$$

Put $N(x, y) := y$ for all $x, y \in I$. Clearly, N is a continuous (not strict) mean in I. Consider the mean-type mapping $(M, N) : I^2 \to I^2$. Note that

$$(M, N)^n(x, \xi) = (f^n(x), \xi), \qquad n \in \mathbb{N},\ x \in I. \tag{9}$$

Indeed, by the definitions of M, N and f, for $n = 1$ we have

$$(M, N)^1(x, \xi) = (M(x, \xi), \xi) = (f(x), \xi) = \left(f^1(x), \xi\right), \qquad x \in I.$$

so (9) holds true for $n = 1$. Assume that (9) is true for some $n \in \mathbb{N}$. Then

$$\begin{aligned}(M,N)^{n+1}(x,\xi) &= (M(M,N)^n(x,\xi)), N(M,N)^n(x,\xi))) \\ &= (M(f^n(x),\xi)), N(f^n(x),\xi)) = (f^{n+1}(x),\xi),\end{aligned}$$

and the induction proves (9). Since M is strict, the mean-type mapping (M,N) satisfies the conditions of Theorem 1 with $p = 2$, $M_1 = M$, $M_2 = N$. Thus

$$\lim_{n\to\infty}(M,N)^n = (K,K)$$

where K is a unique mean continuous mean. Consequently,

$$\lim_{n\to\infty}(f^n(x),\xi) = \lim_{n\to\infty}(M,N)^n(x,\xi) = (K((x,\xi),K(x,\xi)).$$

It follows that $K(x,\xi) = \xi$ and $\lim_{n\to\infty} f^n(x) = \xi$.

In the case when ξ is the right end-point of I we can argue similarly. □

Definition 1 *For $f : (0,\infty)^2 \to (0,\infty)$ the function $f^* : (0,\infty)^2 \to (0,\infty)$ given by*

$$f^*(t) = tf\left(\frac{1}{t}\right), \qquad t > 0,$$

is called a conjugate of f.

Remark 14 *Note that $(f^*)^* = f$ and $f \in \mathbf{S}_1^0((0,\infty))$ if and only if $f^* \in \mathbf{S}_1^0((0,\infty))$.*

In the sequel we denote $\mathbf{S}_1^0((0,\infty))$ by $\mathbf{S}_1$.

Let us note the following easy to verify.

Remark 15 *If $f \in \mathbf{S}_1$ then the function $M : (0,\infty)^2 \to (0,\infty)$ defined by*

$$M(x,y) := yf\left(\frac{x}{y}\right), \qquad x,y \in (0,\infty),$$

is a continuous, strict and homogeneous mean. Moreover

$$M(x,y) = xf^*\left(\frac{y}{x}\right), \qquad x,y \in (0,\infty).$$

Conversely, if a two-variable function $M : (0,\infty)^2 \to (0,\infty)$ is a continuous, strict and homogeneous mean, then $f(t) := M(t,1)$ for all $t > 0$ belongs to $\mathbf{S}_1$,

$$M(x,y) = yf\left(\frac{x}{y}\right), \qquad x,y \in (0,\infty);$$

and, moreover,

$$f(t) = M(t,1), \qquad f^*(t) = M(1,t), \qquad \text{for all } t > 0.$$

Now we prove

Theorem 2 *Let $f, g \in \mathbf{S}_1$. Then*

1. *the linear homogeneous functional equation*

$$\varphi(t) = g(t)\varphi\left(\frac{f(t)}{g(t)}\right), \qquad t \in (0, \infty),$$

has a unique solution $\varphi \in \mathbf{S}_1$ and

$$\varphi(t) = \lim_{n\to\infty} \varphi_n(t), \qquad t \in (0, \infty),$$

where $\varphi_1 = f$ and

$$\varphi_{n+1}(t) = g(t)\varphi_n\left(\frac{f(t)}{g(t)}\right), \qquad t \in (0, \infty),\ n \in \mathbb{N};$$

2. *the functional equation*

$$\psi(t) = f(t)\psi\left(\frac{g(t)}{f(t)}\right), \qquad t \in (0, \infty),$$

has a unique solution $\psi \in \mathbf{S}_1$ and

$$\psi(t) = \lim_{n\to\infty} \psi_n(t), \qquad t \in (0, \infty),$$

where $\psi_1 = g$ and

$$\psi_{n+1}(t) = f(t)\psi_n\left(\frac{g(t)}{f(t)}\right), \qquad t \in (0, \infty),\ n \in \mathbb{N};$$

3. *the functions φ and ψ are equal.*

Proof. Take $f, g \in \mathbf{S}_1$. In view of Remark 14, the functions $M, N : (0, \infty)^2 \to (0, \infty)$,

$$M(x, y) := yf\left(\frac{x}{y}\right), \qquad N(x, y) := yg\left(\frac{x}{y}\right), \qquad x, y \in (0, \infty),$$

are continuous, strict and homogeneous means in $(0, \infty)$.

Applying the main result of [8] (or Theorem 1) we conclude that there exists a unique continuous (M, N)-invariant mean K, i.e.,

$$K(x, y) = K\left(M(x, y), N(x, y)\right), \quad x, y > 0.$$

Moreover K is strict and homogeneous. By the homogeneity of K and the definition of M and N, this equality can be written in the form

$$yK\left(\frac{x}{y},1\right) = yg\left(\frac{x}{y}\right) K\left(\frac{yf\left(\frac{x}{y}\right)}{yg\left(\frac{x}{y}\right)},1\right), \qquad x,y \in (0,\infty),$$

which, after setting $\varphi(t) := K(t,1)$, $t > 0$, and $t := \frac{x}{y}$, reduces to the equality

$$\varphi(t) = g(t)\varphi\left(\frac{f(t)}{g(t)}\right), \qquad t \in (0,\infty).$$

By Remark 14, $\varphi \in S_1$. Moreover the uniqueness of K implies the uniqueness of φ.

By Theorem 1, for every $n \in \mathbb{N}$, we have $(M,N)^n = (M_n, N_n)$ where M_n and N_n are strict continuous and homogeneous means. Setting $\varphi_n(t) := M_n(t,1)$, $n \in \mathbb{N}$, we get

$$M_n(x,y) = y\varphi_n\left(\frac{x}{y}\right), \qquad x,y \in (0,\infty),\ n \in \mathbb{N}.$$

It follows that the equality

$$M_{n+1}(x,y) = M_n(M(x,y), N(x,y)), \qquad x,y \in (0,\infty),$$

can be written in the form

$$y\varphi_{n+1}\left(\frac{x}{y}\right) = yg\left(\frac{x}{y}\right)\varphi_n\left(\frac{yf\left(\frac{x}{y}\right)}{yg\left(\frac{x}{y}\right)}\right), \qquad x,y \in (0,\infty),\ n \in \mathbb{N},$$

which is equivalent to the equality

$$\varphi_{n+1}(t) = g(t)\varphi_n\left(\frac{f(t)}{g(t)}\right), \qquad t \in (0,\infty),\ n \in \mathbb{N}.$$

By Theorem 1, the sequence of iterates $(M,N)^n$, $n \in \mathbb{N}$, converges to the mean-type mapping (K,K). The above formulas imply that this convergence is equivalent to the convergence of the $(\varphi_n)_{n\in\mathbb{N}}$ to the function φ. This completes the proof of the first result.

We omit an analogous argument for part 2. The third part is obvious. □

Applying Theorem 1 we prove the following

Theorem 3 *Let $f, g \in \mathbf{S}_1$. Suppose that the functions $f, g : (0, \infty) \to (0, \infty)$ are continuous,*

$$\min(t, 1) \leq f(t) \leq \max(t, 1), \qquad \min(t, 1) \leq g(t) \leq \max(t, 1), \qquad t > 0, \quad (10)$$

and

$$\min(t, 1) + \max(f(t), g(t)) < \min(f(t), g(t)) + \max(t, 1), \qquad t > 0, \ t \neq 1. \quad (11)$$

Then

1. *the functional equation*

$$\varphi(t) = g(t)\varphi\left(\frac{f(t)}{g(t)}\right), \qquad t \in (0, \infty),$$

has a unique continuous solution $\varphi : (0, \infty) \to (0, \infty)$ such that

$$\min(t, 1) \leq \varphi(t) \leq \max(t, 1), \qquad t > 0.$$

Moreover

$$\varphi(t) = \lim_{n \to \infty} \varphi_n(t), \qquad t \in (0, \infty),$$

where $\varphi_1 = f$ and

$$\varphi_{n+1}(t) = g(t)\varphi_n\left(\frac{f(t)}{g(t)}\right), \qquad t \in (0, \infty), \ n \in \mathbb{N};$$

2. *the functional equation*

$$\psi(t) = f(t)\psi\left(\frac{g(t)}{f(t)}\right), \qquad t \in (0, \infty),$$

has a unique solution $\psi : (0, \infty) \to (0, \infty)$ such that

$$\min(t, 1) \leq \psi(t) \leq \max(t, 1), \qquad t > 0.$$

Moreover

$$\psi(t) = \lim_{n \to \infty} \psi_n(t), \qquad t \in (0, \infty),$$

where $\psi_1 = g$ and

$$\psi_{n+1}(t) = f(t)\psi_n\left(\frac{g(t)}{f(t)}\right), \qquad t \in (0, \infty), \ n \in \mathbb{N};$$

3. the functions φ and ψ are equal.

Proof. By assumption, $\min(t,1) \le f(t) \le \max(t,1)$ for all $t>0$, whence

$$\min(x,y) \le yf\left(\frac{x}{y}\right) \le \max(x,y), \quad x,y>0.$$

Thus the function $M(x,y) := yf\left(\frac{x}{y}\right)$ for all $x,y>0$ is a mean in $(0,\infty)$. In the same way we can show that $N(x,y) := yg\left(\frac{x}{y}\right)$ is a mean in $(0,\infty)$.

Take arbitrary $x,y>0$, $x \ne y$. Setting $t := \frac{x}{y}$ in the assumed inequality and then multiplying both sides by y we obtain

$$\min(x,y) + \max(M(x,y),N(x,y)) < \min(M(x,y),N(x,y))) + \max(x,y).$$

Since the mean-type mapping (M,N) satisfies the assumptions of Theorem 1, we can argue as in the proof Theorem 2. □

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