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Uniformly continuous superposition operators in the Banach space of Hölder functions

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ABSTRACT

Let $I, J \subset \mathbb{R}$ be intervals. One of the main results says that if a superposition operator H generated by a two place $h: I \times J \to \mathbb{R}$,

$$H(\varphi)(x) := h\big(x, \varphi(x)\big),$$

maps the set $Lip^\alpha(I, J)$ of all Hölder functions $\varphi: I \to J$ into the Banach space $Lip^\alpha(I, \mathbb{R})$ and is uniformly continuous with respect to the Lip^α-norm, then

$$h(x, y) = a(x)y + b(x), \quad x \in I, \ y \in J,$$

for some $a, b \in Lip^\alpha(I, \mathbb{R})$.

1. Introduction

Let $I, J \subset \mathbb{R}$ be intervals. By J^I denote the set of all functions $\varphi: I \to J$. For a given function $h: I \times J \to \mathbb{R}$, the mapping $H: J^I \to \mathbb{R}^I$ defined by

$$H(\varphi)(x) := h\big(x, \varphi(x)\big), \quad \varphi \in J^I,$$

is called a superposition (or Nemytskij) operator of a generator h.

The superposition operators play important role in the theory of differential equations, integral equations and functional equations. It is known that every locally defined operator mapping the set of continuous functions $C(I, J)$ into $C(I, \mathbb{R})$ must be a superposition operator. Moreover H maps $C(I, J)$ into $C(I, \mathbb{R})$ if, and only if, its generator h is continuous. At this background it is surprising enough that there are discontinuous functions $h: I \times \mathbb{R} \to \mathbb{R}$ generating the superpositions operators H which map the space of continuously differentiable functions $C^1(I, \mathbb{R})$ into itself (cf. [2, p. 209]). In [4, (1982)] it has been proved that if a superposition operator maps the Banach space $Lip(I, \mathbb{R})$ into itself and is globally Lipschitzian with respect to Lip-norm then its generator h must be of the form

$$h(x, y) = a(x)y + b(x), \quad x \in I, \ y \in \mathbb{R}.$$

Then this result has been extended to some other function Banach spaces (cf. [1,5–7], cf. also [2]).

Let $\alpha \in (0, 1]$. Theorem 1 of this paper reads as follows: if the operator H mapping the set $Lip^\alpha(I, J)$ of Hölder functions $\varphi: I \to J$ into the Banach space $Lip^\alpha(I, \mathbb{R})$ satisfies the inequality

$$\big\|H(\varphi) - H(\psi)\big\|_{Lip^\alpha} \leqslant \gamma\big(\|\varphi - \psi\|_{Lip^\alpha}\big), \quad \varphi, \psi \in Lip^\alpha(I, J),$$

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where $\gamma : [0,\infty) \to [0,\infty)$ is continuous at 0 and such that $\gamma(0) = 0$, then

$$h(x,y) = a(x)y + b(x), \quad x \in I,\ y \in J. \tag{$*$}$$

Applying this we conclude the main result (Theorem 2) saying that the generator h of any superposition operator mapping $Lip^\alpha(I,J)$ into $Lip^\alpha(I,\mathbb{R})$ and uniformly continuous with respect to the norm $\|\cdot\|_{Lip^\alpha}$ must of the form $(*)$. In the case $J = \mathbb{R}$ the assumptions can be weakened.

2. Results

Let $I, J \subset \mathbb{R}$ be some intervals and let $x_0 \in I$ be arbitrarily fixed. For a given $\alpha \in (0,1]$, by $Lip^\alpha(I,\mathbb{R})$ denote the Banach space of all Hölder functions $\varphi : I \to \mathbb{R}$ with the norm

$$\|\varphi\|_{Lip^\alpha} := \left|\varphi(x_0)\right| + \sup_{x,y\in I,\, x\neq y} \frac{|\varphi(x)-\varphi(y)|}{|x-y|^\alpha},$$

and $Lip^\alpha(I,J)$ denotes the set of all $\varphi \in Lip^\alpha(I,\mathbb{R})$ such that $\varphi(I) \subset J$.

Theorem 1. *Let $I, J \subset \mathbb{R}$ be intervals, $\alpha \in (0,1]$ and $h : I \times J \to \mathbb{R}$. Let $\gamma : [0,\infty) \to [0,\infty)$ be continuous at* 0 *and* $\gamma(0) = 0$. *Suppose the superposition operator H of the generator h maps the set $Lip^\alpha(I,J)$ into the Banach space $Lip^\alpha(I,\mathbb{R})$.*

If H satisfies the inequality

$$\left\|H(\varphi) - H(\psi)\right\|_{Lip^\alpha} \leqslant \gamma\left(\|\varphi-\psi\|_{Lip^\alpha}\right), \quad \varphi,\psi \in Lip^\alpha(I,J), \tag{1}$$

then there exist $a, b \in Lip^\alpha(I,\mathbb{R})$ such that

$$h(x,y) = a(x)y + b(x), \quad x \in I,\ y \in J.$$

Proof. Without any loss of generality we can assume that $I = [0,c)$ where $0 < c \leqslant +\infty$ and that

$$\|\varphi\|_{Lip^\alpha} := \left|\varphi(0)\right| + \sup_{s,t\in I,\, s\neq t} \frac{|\varphi(s)-\varphi(t)|}{|s-t|^\alpha}.$$

Note that for arbitrary $y \in J$ the constant function $\varphi(t) = y$, $(t \in I)$, belongs to $Lip^\alpha(I,J)$. Since H maps $Lip^\alpha(I,J)$ into $Lip^\alpha(I,\mathbb{R})$, the function $H(\varphi) = h(\cdot,y) \in Lip^\alpha(I,\mathbb{R})$ and, consequently, h is continuous with respect to the first variable.

For arbitrarily fixed $y, \bar{y} \in J$ take $\varphi, \psi : I \to J$ defined by

$$\varphi(t) = y, \qquad \psi(t) = \bar{y}, \quad t \in I.$$

Then, of course, $\varphi, \psi \in Lip^\alpha(I,J)$ and, by the assumption, the functions $H(\varphi) = h(\cdot,y)$, $H(\psi) = h(\cdot,\bar{y})$ belong to $Lip^\alpha(I,\mathbb{R})$ and

$$\|\varphi-\psi\|_{Lip^\alpha} = |y-\bar{y}|.$$

Hence, for all $x \in I$,

$$\begin{aligned}
\left|h(x,y) - h(x,\bar{y})\right| &\leqslant \left|h(0,y) - h(0,\bar{y})\right| + \left|h(x,y) - h(x,\bar{y}) - h(0,y) + h(0,\bar{y})\right| \\
&= \left|h(0,y) - h(0,\bar{y})\right| + \frac{|h(x,y) - h(x,\bar{y}) - h(0,y) + h(0,\bar{y})|}{|x-0|^\alpha}|x|^\alpha \\
&\leqslant \max\left(1,|x|^\alpha\right)\left\|H(\varphi) - H(\psi)\right\|_{Lip^\alpha} \leqslant \max\left(1,|x|^\alpha\right)\gamma\left(|y-\bar{y}|\right).
\end{aligned}$$

This inequality, the continuity of γ at 0 and the equality $\gamma(0) = 0$ imply that h is continuous with respect to the second variable for every fixed $x \in I$.

Let us fix $x, \bar{x} \in I$, $x < \bar{x}$; $y_1, y_2, \bar{y}_1, \bar{y}_2 \in J$ and define the functions $\varphi_i : I \to J$,

$$\varphi_i(t) := \begin{cases} y_i, & t < x,\ t \in I, \\ \frac{\bar{y}_i - y_i}{\bar{x}-x}(t-x) + y_i, & x \leqslant t \leqslant \bar{x}, \\ \bar{y}_i, & t > \bar{x},\ t \in I, \end{cases} \qquad i = 1,2.$$

Note that $\varphi_i \in Lip^\alpha(I,J)$ for $i = 1,2$ and

$$\|\varphi_1 - \varphi_2\|_{Lip^\alpha} = |y_1 - y_2| + \frac{|y_1 - y_2 - \bar{y}_1 + \bar{y}_2|}{|x-\bar{x}|^\alpha}.$$

Since

$$\frac{|h(x,y_1)-h(x,y_2)-h(\bar{x},\bar{y}_1)+h(\bar{x},\bar{y}_2)|}{|x-\bar{x}|^\alpha}=\frac{|[H(\varphi_1)-H(\varphi_2)](x)-[H(\varphi_1)-H(\varphi_2)](\bar{x})|}{|x-\bar{x}|^\alpha}\leqslant\|H(\varphi_1)-H(\varphi_2)\|_{Lip^\alpha},$$

applying (1) with $\varphi=\varphi_1$, $\psi=\varphi_2$ we get

$$\frac{|h(x,y_1)-h(x,y_2)-h(\bar{x},\bar{y}_1)+h(\bar{x},\bar{y}_2)|}{|x-\bar{x}|^\alpha}\leqslant\gamma\left(|y_1-y_2|+\frac{|y_1-y_2-\bar{y}_1+\bar{y}_2|}{|x-\bar{x}|^\alpha}\right)$$

for all $x,\bar{x}\in I$, $x<\bar{x}$; $y_1,y_2,\bar{y}_1,\bar{y}_2\in J$.

Taking arbitrary $u,v\in J$ and setting here

$$y_1:=\frac{u+v}{2},\qquad y_2:=u,\qquad \bar{y}_1:=v,\qquad \bar{y}_2:=\frac{u+v}{2}$$

we obtain

$$\frac{|h(x,\frac{u+v}{2})-h(x,u)-h(\bar{x},v)+h(\bar{x},\frac{u+v}{2})|}{|x-\bar{x}|^\alpha}\leqslant\gamma\left(\frac{|u-v|}{2}\right)$$

whence

$$\left|h\left(x,\frac{u+v}{2}\right)-h(x,u)-h(\bar{x},v)+h\left(\bar{x},\frac{u+v}{2}\right)\right|\leqslant|x-\bar{x}|^\alpha\gamma\left(\frac{|u-v|}{2}\right)$$

for all $x,\bar{x}\in I$, $x<\bar{x}$; $u,v\in J$.

Letting here $\bar{x}$ tend to x and making use of the continuity of h with respect to the first variable, we hence get

$$2h\left(x,\frac{u+v}{2}\right)=h(x,v)+h(x,u),\quad x\in I,\ u\in J,$$

which proves that for every fixed $x\in I$ the function $h(x,\cdot)$ satisfies the Jensen functional equation in the interval J. The continuity of h with respect to the second variable implies that (cf. M. Kuczma [3, Chapter XIII, Section 2: Jensen Equation, p. 315, Theorem 1]) for every $x\in I$ there exist $a(x),b(x)\in\mathbb{R}$ such that

$$h(x,y)=a(x)y+b(x),\quad x\in I,\ y\in J.$$

If J is a nontrivial interval, there are $y_1,y_2\in J$, $y_1\neq y_2$. By assumption the functions $h(\cdot,y_1)=a(\cdot)y_1+b(\cdot)$ and $h(\cdot,y_2)=a(\cdot)y_2+b(\cdot)$ belong to $Lip^\alpha(I,\mathbb{R})$. It follows that $a,b\in Lip^\alpha(I,\mathbb{R})$. If J is a trivial interval then h does not depend on y and the result is obvious. □

Taking $\gamma(t)=Lt$ $(t\geqslant 0)$, $J=\mathbb{R}$ and $\alpha=1$ we get the result proved in [4] (cf. also [2]).

Remark 1. Theorem 1 remains valid if we replace the norm $\|\cdot\|_{Lip^\alpha}$ by the following one

$$\|\varphi\|:=\sup_{x\in I}|\varphi(x)|+\sup_{x,y\in I,x\neq y}\frac{|\varphi(x)-\varphi(y)|}{|x-y|^\alpha}.$$

Both these norms are equivalent if the interval I is bounded. Indeed, if d is the length of I then

$$\|\varphi\|_{Lip^\alpha}\leqslant\|\varphi\|\leqslant(1+d^\alpha)\|\varphi\|_{Lip^\alpha}.$$

It easy to check the following: if $I\subset\mathbb{R}$ is a compact interval and $a,b\in Lip^\alpha(I,\mathbb{R})$ then the superposition operator H of the generator $h(x,y)=a(x)y+b(x)$ $(x\in I,\ y\in\mathbb{R})$ maps the set $Lip^\alpha(I,\mathbb{R})$ into the Banach space into itself and

$$\|H(\varphi)-H(\psi)\|_{Lip^\alpha}\leqslant\|\alpha\|_{Lip^\alpha}\|\varphi-\psi\|_{Lip^\alpha},\quad\varphi,\psi\in Lip^\alpha(I,\mathbb{R}),$$

that is H is Lipschitzian.

Remark 2. The assumptions of γ allow to show that, without any loss of generality, the function γ in Theorem 1 can be assumed to be increasing. It follows that the norms in Theorem 1 can be replaced by the equivalent once.

Remark 3. Denote by S the set of all functions $\varphi\in Lip^\alpha(I,J)$ such that

$$\varphi(t)=\begin{cases}y, & t<x,\ t\in I,\\ \frac{\bar{y}-y}{\bar{x}-x}(t-x)+y, & x\leqslant t\leqslant\bar{x},\\ \bar{y}, & t>\bar{x},\ t\in I\end{cases}$$

for some $x,\bar{x}\in I$, $x<\bar{x}$, $y,\bar{y}\in J$. It follows from the argument used in the prof that Theorem 1 remains valid if inequality is postulated only for all φ,ψ from the set S.

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If h is defined on $I \times \mathbb{R}$ we have the following stronger result than Theorem 1.

Proposition 1. *Let $I \subset \mathbb{R}$ be a bounded interval, $\alpha \in (0,1]$ and $h : I \times \mathbb{R} \to \mathbb{R}$. Let $\gamma : [0,\infty) \to [0,\infty)$ be continuous at 0 and $\gamma(0) = 0$. Let $\mathcal{A}$ be the set of all functions $\varphi : I \to \mathbb{R}$ of the form*

$$\varphi(t) = \alpha t + \beta, \quad t \in I.$$

Suppose the superposition operator H of the generator h maps the set $\mathcal{A}$ into the Banach space $Lip^\alpha(I,\mathbb{R})$. If H satisfies the inequality

$$\left\| H(\varphi) - H(\psi) \right\|_{Lip^\alpha} \leqslant \gamma\left(\|\varphi - \psi\|_{Lip^\alpha}\right), \quad \varphi, \psi \in \mathcal{A},$$

then there exist $a, b \in Lip^\alpha(I,\mathbb{R})$ such that

$$h(x,y) = a(x)y + b(x), \quad x \in I,\ y \in \mathbb{R}.$$

Proof. Since all the constant functions belong to $\mathcal{A}$, in a similar way as in Theorem 1 one can show that h is continuous with respect to both variables. Take arbitrary $x, \bar{x} \in I$, $x < \bar{x}$, $p, q, k, l \in \mathbb{R}$ and

$$\varphi(t) = pt + k, \qquad \psi(t) = qt + l, \quad t \in I.$$

Of course

$$\|\varphi - \psi\|_{Lip^\alpha} = |k - l| + |p - q| \sup_{s,t \in I,\, s \neq t} |s - t|^{1-\alpha} = |k - l| + |p - q||I|.$$

Assuming (without any loss of generality) that $I = [0,1]$, we hence get

$$\|\varphi - \psi\|_{Lip^\alpha} = |k - l| + |p - q|.$$

From the definition of the norm $\|\cdot\|_{Lip^\alpha}$

$$\frac{|h(x, px + k) - h(x, qx + l) - h(\bar{x}, p\bar{x} + k) + h(\bar{x}, q\bar{x} + l)|}{|x - \bar{x}|^\alpha} = \frac{|[H(\varphi) - H(\psi)](x) - [H(\varphi) - H(\psi)](\bar{x})|}{|x - \bar{x}|^\alpha} \leqslant \left\| H(\varphi) - H(\psi) \right\|_{Lip^\alpha}$$

whence, by assumption,

$$\frac{|h(x, px + k) - h(x, qx + l) - h(\bar{x}, p\bar{x} + k) + h(\bar{x}, q\bar{x} + l)|}{|x - \bar{x}|^\alpha} \leqslant \gamma\left(|k - l| + |p - q|\right),$$

which can be written in the form

$$\left|h(x, px + k) - h(x, qx + l) - h(\bar{x}, p\bar{x} + k) + h(\bar{x}, q\bar{x} + l)\right| \leqslant |x - \bar{x}|^\alpha \gamma\left(|k - l| + |p - q|\right).$$

Take arbitrary $u, v \in \mathbb{R}$. Putting here

$$p = q = \frac{u - v}{2(x - \bar{x})}, \qquad k = \frac{(2x - \bar{x})v - \bar{x}u}{2(x - \bar{x})}, \qquad l = \frac{(x - 2\bar{x})u + xv}{2(x - \bar{x})}$$

we obtain

$$\left| h\left(x, \frac{u + v}{2}\right) - h(x,u) - h(\bar{x}, v) + h\left(\bar{x}, \frac{u + v}{2}\right) \right| \leqslant |x - \bar{x}|^\alpha \gamma\left(\left|\frac{u - v}{2}\right|\right)$$

for all $x, \bar{x} \in I$, $x \neq \bar{x}$, and $u, v \in \mathbb{R}$. Letting $\bar{x}$ tend to x, by the continuity of h, we hence get

$$2h\left(x, \frac{u + v}{2}\right) = h(x,u) + h(x,v), \quad x \in I,\ u, v \in \mathbb{R}.$$

Now the same argument as in Theorem 1 completes the proof. □

Remark 4. Note that in the case $\alpha = 1$ the norm of the function $\varphi \in \mathcal{A}$ does not depend on the length of the interval I. Therefore the boundedness of I in Proposition 1 can be omitted.

To show that the set $\mathcal{A}$ of all affine functions in Proposition 1 cannot be replaced by the set $\mathcal{L} \subset \mathcal{A}$ of all linear functions consider the following

Example 1. Let $\alpha = 1$. Denote by $\mathcal{L}$ the set of all functions $\varphi : \mathbb{R} \to \mathbb{R}$ of the form $\varphi(t) = pt$ $(t \in \mathbb{R})$. Take a function $h : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ defined by

$$h(x, y) := \frac{1}{y^2 + 1}, \quad x, y \in \mathbb{R}.$$

Note that

$$\left|(h, px) - h(x, qx) - h(\bar{x}, p\bar{x}) + h(\bar{x}, q\bar{x})\right| = \left|\frac{(px + p\bar{x} + qx + q\bar{x})(pqx\bar{x} - 1)(pqx\bar{x} + 1)}{(p^2x^2 + 1)(q^2x^2 + 1)(p^2\bar{x}^2 + 1)(q^2\bar{x}^2 + 1)}\right| |p - q||x - \bar{x}|.$$

Since

$$\left|\frac{(px + p\bar{x} + qx + q\bar{x})(pqx\bar{x} - 1)(pqx\bar{x} + 1)}{(p^2x^2 + 1)(q^2x^2 + 1)(p^2\bar{x}^2 + 1)(q^2\bar{x}^2 + 1)}\right| \leqslant \frac{1}{2}, \tag{2}$$

for all $x, \bar{x}, p, q \in \mathbb{R}$, we have

$$\left|(h, px) - h(x, qx) - h(\bar{x}, p\bar{x}) + h(\bar{x}, q\bar{x})\right| \leqslant \frac{1}{2}|p - q||x - \bar{x}|, \quad x, \bar{x}, p, q \in \mathbb{R}.$$

It follows that $\|H(\varphi) - H(\psi)\|_{Lip} \leqslant \frac{1}{2}\|\varphi - \psi\|_{Lip}$, $\varphi, \psi \in \mathcal{L}$.

To prove inequality (2) it is enough to show that it holds true for all positive $p, q, x, \bar{x}$.

Replacing px by u, $p\bar{x}$ by v, qx by w, and $q\bar{x}$ by t, the expression under the modulus in (2) takes the form

$$\frac{(u + v + w + t)(ut - 1)(vw + 1)}{(u^2 + 1)(v^2 + 1)(w^2 + 1)(t^2 + 1)}.$$

Since $(px)(q\bar{x}) = (p\bar{x})(qx)$, we have $t = vw/u$, whence

$$\frac{(u + v + w + t)(ut - 1)(vw + 1)}{(u^2 + 1)(v^2 + 1)(w^2 + 1)(t^2 + 1)} = f(u, v, w),$$

where

$$f(u, v, w) := \frac{u(u^2 + uv + uw + vw)(vw - 1)(vw + 1)}{(u^2 + 1)(v^2 + 1)(w^2 + 1)(v^2w^2 + u^2)}.$$

We shall show that

$$\left|f(u, v, w)\right| \leqslant \frac{1}{2}, \quad u, v, w > 0.$$

Calculating $\frac{\partial f}{\partial v}$ and $\frac{\partial f}{\partial w}$ we obtain

$$\frac{\partial}{\partial v} f(u, v, w) = \frac{u(u + w)}{(u^2 + 1)(v^2 + 1)^2(w^2 + 1)(u^2 + v^2w^2)^2},$$
$$\frac{\partial}{\partial w} f(u, v, w) = \frac{u(u + v)}{(u^2 + 1)(v^2 + 1)(w^2 + 1)^2(u^2 + v^2w^2)^2},$$

which implies that the function $f(u, v, w)$ is increasing with respect to v and w. Since

$$\lim_{v \to \infty} f(u, v, w) = \lim_{w \to \infty} f(u, v, w) = 0$$

and, for all $u > 0$,

$$\lim_{v \to 0} \lim_{w \to 0} f(u, v, w) = \lim_{w \to 0} \lim_{v \to 0} f(u, v, w) = -\frac{u}{(u^2 + 1)} \geqslant -\frac{1}{2},$$

the desired inequality is proved.

Consider also the following simpler

Example 2. Let $\alpha = 1$. Denote by $\mathcal{L}_+$ the set of all functions $\varphi : [0, \infty) \to [0, \infty)$ of the form $\varphi(t) = pt$ $(t \geqslant 0)$. Take a function $h : [0, \infty)^2 \to \mathbb{R}$, defined by

$$h(x, y) := \frac{1}{y + 1}, \quad x, y \geqslant 0.$$

Since

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$$\left|(h, px) - h(x, qx) - h(\bar{x}, p\bar{x}) + h(\bar{x}, q\bar{x})\right| = \left|\frac{pqx\bar{x} - 1}{(px+1)(qx+1)(p\bar{x}+1)(q\bar{x}+1)}\right| |p-q||x-\bar{x}|$$

and, for all $x, \bar{x}, p, q \in [0, \infty)$,

$$\left|\frac{pqx\bar{x} - 1}{(px+1)(qx+1)(p\bar{x}+1)(q\bar{x}+1)}\right| \leqslant 1,$$

we have

$$\left|(h, px) - h(x, qx) - h(\bar{x}, p\bar{x}) + h(\bar{x}, q\bar{x})\right| \leqslant |p-q||x-\bar{x}|, \quad x, \bar{x}, p, q \in [0, \infty).$$

It follows that

$$\left\|H(\varphi) - H(\psi)\right\|_{Lip} \leqslant \|\varphi - \psi\|_{Lip}, \quad \varphi, \psi \in \mathcal{L}_+.$$

The main result reads as follows:

Theorem 2. *Let $I, J \subset \mathbb{R}$ be intervals and let $h : I \times J \to \mathbb{R}$. Suppose that the superposition operator H of the generator h maps the set $Lip^\alpha(I, J)$ into the Banach space $Lip^\alpha(I, \mathbb{R})$. If H is uniformly continuous (with respect to $Lip^\alpha(I, \mathbb{R})$ norm), then there exist $a, b \in Lip^\alpha(I, \mathbb{R})$ such that*

$$h(x, y) = a(x)y + b(x), \quad x \in I, \ y \in J.$$

Proof. Suppose that H is uniformly continuous. Then for every $\varepsilon > 0$ there is a $\delta > 0$ such that for all $\varphi, \psi \in Lip^\alpha(I, J)$,

$$\|\varphi - \psi\|_{Lip^\alpha} \leqslant \delta \quad \Longrightarrow \quad \left\|H(\varphi) - H(\psi)\right\|_{Lip^\alpha} \leqslant \varepsilon.$$

It follows that the function $\gamma : [0, \infty) \to [0, \infty)$, the modulus continuity of H,

$$\gamma(t) := \sup\left\{\left\|H(\varphi) - H(\psi)\right\|_{Lip^\alpha} : \|\varphi - \psi\|_{Lip^\alpha} \leqslant t\right\}, \quad t \geqslant 0,$$

is correctly defined, γ is continuous at 0, $\gamma(0) = 0$ and

$$\left\|H(\varphi) - H(\psi)\right\|_{Lip^\alpha} \leqslant \gamma\left(\|\varphi - \psi\|_{Lip^\alpha}\right), \quad \varphi, \psi \in Lip^\alpha(I, J).$$

Now the result is a consequence of Theorem 1. □

Similarly, applying Proposition 1, we get

Corollary 1. *Let $I \subset \mathbb{R}$ be nontrivial bounded interval, $\alpha \in (0, 1]$ and $h : I \times J \to \mathbb{R}$. Let $\mathcal{A} \subset Lip^\alpha(I, J)$ be defined as in Proposition 2. If the superposition operator H of the generator h maps the set $\mathcal{A}$ into $Lip^\alpha(I, \mathbb{R})$ and H is uniformly continuous (with respect to the norms), then there exist $a, b \in Lip^\alpha(I, \mathbb{R})$ such that*

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$$h(x, y) = a(x)y + b(x), \quad x \in I, \ y \in J.$$

Remark 5. Let $\sigma : (0, \infty) \to (0, \infty)$ be an increasing, $\sigma(0+) := \lim_{t \to 0+} \sigma(t) = 0$ and such that the function

$$(0, \infty) \ni t \to \frac{\sigma(t)}{t} \text{ is decreasing.}$$

(Then σ is continuous everywhere and subadditive.) Theorem 1 and Corollary 1 (as well as Proposition 1 and Theorem 2) remain true for the generalized function Banach space $Lip_\sigma(I, \mathbb{R})$ with the norm

$$\|\varphi\| := \left|\varphi(x_0)\right| + \sup_{x, y \in I, x \neq y} \frac{|\varphi(x) - \varphi(y)|}{\sigma(|x - y|)}.$$

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