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MATHEMATICS
(OPERATOR THEORY)

Some Inequalities and a Generalization of Banach's Principle

by

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Summary. In this note we give a necessary and sufficient condition for a system of n linear inequalities to have positive solutions, and a fixed-point theorem which generalizes the Banach principle and the results in [1] and [2].

To a matrix $(c_{ik}^{(0)})$, $i, k = 1, \ldots, n$, we associate the sequence of $n-1$ matrices $(c_{ik}^{(j)})$ defined as follows

$$c_{ik}^{(j)} = \begin{cases} c_{11}^{(j-1)} c_{i+1,k+1}^{(j-1)} + c_{i+1,1}^{(j-1)} c_{1,k+1}^{(j-1)} & \text{for} \quad i \neq k, \\ c_{11}^{(j-1)} c_{i+1,k+1}^{(j-1)} - c_{i+1,1}^{(j-1)} c_{1,k+1}^{(j-1)} & \text{for} \quad i = k, \end{cases}$$

$j = 1, \ldots, n-1$; $i, k = 1, \ldots, n-j$.

LEMMA. *Let $c_{ik}^{(0)} > 0$, $i, k = 1, \ldots, n$. The system of inequalities*

$$\sum_{\substack{k=1 \\ k \neq i}}^{n} c_{ik}^{(0)} r_k < c_{ii}^{(0)} r_i, \quad i = 1, \ldots, n,$$

has a solution $r_i > 0$, $i = 1, \ldots, n$, if and only if the following inequalities hold:

$$c_{ii}^{(j)} > 0, \quad i = 1, \ldots, n-j, \; j = 1, \ldots, n-1.$$

Using this Lemma we can prove the following fixed-point theorem.

THEOREM. *Let (X_i, ρ_i), $i = 1, \ldots, n$ be complete metric spaces. Suppose that the transformations $T_i \colon X_1 \times \ldots \times X_n \to X_i$, $i = 1, \ldots, n$, fulfil the following conditions*

$$\rho_i\left(T_i(x_1, \ldots, x_n), T_i(y_1, \ldots, y_n)\right) \leq \sum_{k=1}^{n} a_{ik} \rho_k(x_k, y_k), \quad x_k, y_k \in X_k,$$

where $a_{ik} \geq 0$, $i, k = 1, \ldots, n$. If the numbers $c_{ik}^{(0)}$ defined by

$$c_{ik}^{(0)} = \begin{cases} a_{ik} & \text{for} \quad i \neq k \\ 1 - a_{ik} & \text{for} \quad i = k \end{cases}, \quad i, k = 1, \ldots, n,$$

fulfil the inequalities

$$c_{ii}^{(l)}>0, \quad l=0,1,\dots,n-1;\ i=0,1,\dots,n-l,$$

then the system of equations

$$x_i=T_i(x_1,\dots,x_n), \quad i=1,\dots,n,$$

has exactly one solution $x_i\in X_i$, $i=1,\dots,n$. *Moreover*

$$x_i=\lim_{m\to\infty} x_i^m, \quad i=1,\dots,n,$$

where $x_i^0\in X_i$, $i=1,\dots,n$, *are arbitrarily chosen and*

$$x_i^{m+1}=T_i(x_1^m,\dots,x_n^m), \quad i=1,\dots,n;\ m=0,1,\dots.$$

The proofs will be given later.

For $n=1$ we obtain the Banach principle and for $n=2$ the results of Păvăloiu [1] and Rus [2].

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REFERENCES

[1] I. Păvăloiu, *La résolution des systèmes d'équations opérationnelles à l'aide des méthodes itératives*, Mathematica (Cluj), **11** (34) (1969), 137—141.

[2] I. A. Rus, *Asupra punctelor fixe ale aplicatiilor definite pe un produs cartezian. II: spații metrice*, St. Cerc. Mat., **24** (1972), 897—904.

Я. Матковски, **Некоторые неравенства и обобщение теоремы Банаха**

Содержание. В настоящей работе приводится условие необходимое и достаточное для того, чтобы некоторая система n линейных неравенств имела положительное решение. Кроме того, дается теорема, которая является обобщением теоремы Банаха о неподвижной точке, а также теорем приведенных в работах [1] и [2].