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Converse theorem for the Minkowski inequality

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ABSTRACT

Let (Ω, Σ, μ) a measure space such that $0 < \mu(A) < 1 < \mu(B) < \infty$ for some $A, B \in \Sigma$. Under some natural conditions on the bijective functions $\varphi, \varphi_1, \varphi_2, \psi, \psi_1, \psi_2 : (0,\infty) \to (0,\infty)$ we prove that if

$$\psi\left(\int\limits_{\Omega(\mathbf{x}+\mathbf{y})} \varphi \circ (\mathbf{x}+\mathbf{y})\, d\mu\right) \leqslant \psi_1\left(\int\limits_{\Omega(\mathbf{x})} \varphi_1 \circ \mathbf{x}\, d\mu\right) + \psi_2\left(\int\limits_{\Omega(\mathbf{y})} \varphi_2 \circ \mathbf{y}\, d\mu\right)$$

for all nonnegative μ-integrable simple functions $\mathbf{x}, \mathbf{y} : \Omega \to \mathbb{R}$ (where $\Omega(\mathbf{x})$ stands for the support of $\mathbf{x}$, then there exists a real $p \geqslant 1$ such that

$$\frac{\varphi(t)}{\varphi(1)} = \frac{\varphi_i(t)}{\varphi_i(1)} = t^p, \qquad \frac{\psi(t)}{\psi(1)} = \frac{\psi_i(t)}{\psi_i(1)} = t^{1/p}, \quad i = 1, 2.$$

Some generalizations and relevant results for the reversed inequality are also presented.

1. Introduction

Let (Ω, Σ, μ) be a measure space. Let $S = S(\Omega, \Sigma, \mu)$ denote the linear real space of all μ-integrable simple functions $\mathbf{x} : \Omega \to \mathbb{R}$ and $S_+ = \{\mathbf{x} \in S\colon\ \mathbf{x} \geqslant 0\}$. For $\mathbf{x} \in S$ put

$$\Omega(\mathbf{x}) := \{\omega \in \Omega\colon\ \mathbf{x}(\omega) \neq 0\}.$$

For two arbitrarily fixed bijections $\varphi, \psi : (0,\infty) \to (0,\infty)$, the functional $\mathbb{P}_{\varphi,\psi} : S \to [0,\infty)$ given by the formula

$$\mathbb{P}_{\varphi,\psi}(\mathbf{x}) := \begin{cases} \psi(\int_{\Omega(\mathbf{x})} \varphi \circ |\mathbf{x}|\, d\mu) & \text{if } d\mu(\Omega(\mathbf{x})) > 0, \\ 0 & \text{if } d\mu(\Omega(\mathbf{x})) = 0, \end{cases}$$

is correctly defined. Moreover $\mathbb{P}_{\varphi,\psi}$ becomes the L^p-norm if $\varphi(t) = \varphi(1)t^p$ and $\psi = \varphi^{-1}$ for some $p \geqslant 1$.

Note that a weak form of the Minkowski inequality can be written as the implication: *If* $\varphi(t) = \varphi(1)t^p$ *and* $\psi = \varphi^{-1}$ *for some* $p \geqslant 1$ *then*

$$\mathbb{P}_{\varphi,\varphi^{-1}}(\mathbf{x}+\mathbf{y}) \leqslant \mathbb{P}_{\varphi,\varphi^{-1}}(\mathbf{x}) + \mathbb{P}_{\varphi,\varphi^{-1}}(\mathbf{y}), \quad x, y \in S_+(\Omega, \Sigma, \mu). \tag{M}$$

Answering to a natural question, in our earlier paper [2], under additional assumptions that $\varphi(0) = 0$ and φ^{-1} is continuous at 0, we have shown that the converse implication holds if, and only if, there are $A, B \in \Sigma$ such that

$$0 < \mu(A) < 1 < \mu(B) < \infty. \tag{1}$$

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In the present paper we prove the following significantly stronger result (Theorem 1): *Suppose that there are* $A, B \in \Sigma$ *such that* (1) *holds true and* $\varphi, \varphi_1, \varphi_2, \psi, \psi_1, \psi_2 : (0,\infty) \to (0,\infty)$ *are bijective,* ψ *is increasing, and such that*

$$\psi_1 \circ \varphi_1(s) + \psi_2 \circ \varphi_2(t) \leqslant \psi \circ \varphi(s+t), \quad s,t > 0. \tag{2}$$

Then

$$\mathbb{P}_{\varphi,\psi}(\mathbf{x}+\mathbf{y}) \leqslant \mathbb{P}_{\varphi_1,\psi_1}(\mathbf{x}) + \mathbb{P}_{\varphi_2,\psi_2}(\mathbf{y}), \quad \mathbf{x},\mathbf{y} \in S_+, \tag{$*$}$$

if, and only if, there exists a real $p \geqslant 1$ *such that*

$$\frac{\varphi(t)}{\varphi(1)} = \frac{\varphi_i(t)}{\varphi_i(1)} = t^p, \qquad \frac{\psi(t)}{\psi(1)} = \frac{\psi_i(t)}{\psi_i(1)} = t^{1/p}, \quad i = 1,2, \ t > 0.$$

This converse of Minkowski's inequality theorem generalizes the result of [2] where a very special case $\varphi_1 = \varphi_2 = \varphi$, $\psi_1 = \psi_2 = \psi =: \varphi^{-1}$ (with only one unknown function) was considered. Note that in this case inequality (2) is satisfied with equality. The main result of [9], where the case $\varphi_1 = \varphi_2 = \varphi$, $\psi_1 = \psi_2 = \psi$ was examined, is also generalized.

Inequality $(*)$ is obtained from (M) by replacing in each place φ and its inverse φ^{-1} by an arbitrary bijection. Inequality $(*)$ can be called a "Pexiderization" of inequality (M) as, for the first time, an analogous procedure was applied in 1993 by J.V. Pexider [11] for the Cauchy functional equation. Similarly, inequality (2) can be referred to as the "Pexiderization" of the superadditivity condition.

The finite dimensional counterparts of $(*)$ and the reversed inequalities are also considered.

Condition (1) plays here a crucial role. If a measure space fails to satisfy this condition, then there are some broad classes of non-power functions φ for which even the functional $\mathbb{P}_{\varphi,\varphi^{-1}}$ satisfies the triangle inequality. Condition (2) is also indispensable.

2. Some lemmas

We need the following (cf. [5] and [8]):

Lemma 1. *Let* real *numbers* a, b *such that* $0 < a < 1 < a+b$ *be fixed. Then a function* $f : (0,\infty) \to \mathbb{R}$ *such that*

$$\limsup_{t\to 0+} f(t) \leqslant 0$$

satisfies the inequality

$$f(as+bt) \leqslant af(s) + bf(t), \quad s,t > 0,$$

if, and only if, $f(t) = f(1)t$ *for all* $t > 0$.

Applying this lemma we obtain (cf. [5]):

Lemma 2. *Let real numbers* a, b *such that* $0 < a < 1 < a+b$ *be fixed. If a function* $F : (0,\infty)^2 \to \mathbb{R}$ *satisfies the inequality*

$$F(ax_1+bx_2, ay_1+by_2) \leqslant aF(x_1,y_1) + bF(x_2,y_2), \quad x_1,x_2,y_1,y_2 > 0,$$

and the condition

$$\limsup_{t\to 0+} F(tx,ty) \leqslant 0, \quad x,y > 0,$$

then F *is positively homogeneous, i.e.*

$$F(tx,ty) = tF(x,y), \quad t,x,y > 0.$$

Proof. Let us fix $x, y > 0$ and define $f : (0,\infty) \to \mathbb{R}$ by $f(t) := F(tx,ty)$. From the assumed inequality we have

$$\begin{aligned} f(as+bt) &= F\big((as+bt)x, (as+bt)y\big) = F\big(a(sx)+b(tx), a(sy)+b(ty)\big) \\ &\leqslant aF(sx,sy) + bF(tx,ty) = af(s) + bf(t) \end{aligned}$$

for all $s, t > 0$, and the result follows from Lemma 1. □

Remark 1. A finite dimensional counterpart of lemma is also true (cf. [5, Theorem 2], and [8]). Let a positive integer $n \geqslant 2$ and real numbers a, b such that $0 < a < 1 < a+b$ be fixed. If $F : (0,\infty)^n \to \mathbb{R}$ satisfies the condition

$$\limsup_{t\to 0+} F(tx_1,\dots,t_n) \leqslant 0, \quad x_1,\dots,x_n > 0,$$

and, for all $x_1, \dots, x_n, y_1, \dots, y_n > 0$, we have

$$F(ax_1 + by_1, \dots, ax_n + by_n) \leqslant aF(x_1, \dots, x_n) + bF(y_1, \dots, y_n)$$

then

$$F(tx_1, \dots, tx_n) = tF(x_1, \dots, x_n), \quad t, x_1, \dots, x_n > 0.$$

Lemma 3. *(See [3,7].) If $\psi : (0,\infty) \to (0,\infty)$ is subadditive, one-to-one, and $\lim_{t\to 0} \psi(t) = 0$, then ψ is increasing and continuous.*

3. Main results

Denote by χ_A the characteristic function of a set $A \subset \Omega$.

Theorem 1. *Let (Ω, Σ, μ) be a measure space such that there are two sets $A, B \in \Sigma$ satisfying the condition*

$$0 < \mu(A) < 1 < \mu(B) < \infty. \tag{1}$$

Suppose that $\varphi, \varphi_1, \varphi_2, \psi, \psi_1, \psi_2 : (0,\infty) \to (0,\infty)$ are bijective functions, ψ is strictly increasing and

$$\psi_1 \circ \varphi_1(s) + \psi_2 \circ \varphi_2(t) \leqslant \psi \circ \varphi(s+t), \quad s, t > 0. \tag{2}$$

Then the following conditions are equivalent:

(i) *the functions $\varphi, \varphi_1, \varphi_2, \psi, \psi_1, \psi_2$ satisfy the inequality*

$$\mathbb{P}_{\varphi,\psi}(\mathbf{x} + \mathbf{y}) \leqslant \mathbb{P}_{\varphi_1,\psi_1}(\mathbf{x}) + \mathbb{P}_{\varphi_2,\psi_2}(\mathbf{y}), \quad \mathbf{x}, \mathbf{y} \in S_+(A, B), \tag{3}$$

where $S_+(A,B) := \{x_1\chi_A + x_2\chi_{B\setminus A} \in S_+ \colon x_1, x_2 > 0\}$;

(ii) *there is a real $p \geqslant 1$ such that*

$$\frac{\varphi(t)}{\varphi(1)} = \frac{\varphi_i(t)}{\varphi_i(1)} = t^p, \qquad \frac{\psi(t)}{\psi(1)} = \frac{\psi_i(t)}{\psi_i(1)} = t^{1/p}, \quad i = 1, 2,\ t > 0,$$

and

$$\psi(1)\big[\varphi(1)\big]^{1/p} \leqslant \psi_i(1)\big[\varphi_i(1)\big]^{1/p}, \quad i = 1, 2;$$

(iii) *there is a real $p \geqslant 1$ such that*

$$\frac{\mathbb{P}_{\varphi,\psi}(x)}{\psi(1)\varphi(1)} = \frac{\mathbb{P}_{\varphi_1,\psi_1}(x)}{\psi_1(1)\varphi_1(1)} = \frac{\mathbb{P}_{\varphi_2,\psi_2}(x)}{\psi_2(1)\varphi_2(1)} = \left(\int_\Omega |x|^p \, d\mu\right)^{1/p}, \quad x \in S,$$

and

$$\psi(1)\big[\varphi(1)\big]^{1/p} = \psi_i(1)\big[\varphi_i(1)\big]^{1/p}, \quad i = 1, 2;$$

(iv) *the functions $\varphi, \varphi_1, \varphi_2, \psi, \psi_1, \psi_2$ satisfy the triangle inequality*

$$\mathbb{P}_{\varphi,\psi}(x + y) \leqslant \mathbb{P}_{\varphi_1,\psi_1}(x) + \mathbb{P}_{\varphi_2,\psi_2}(y), \quad x, y \in S.$$

Proof. To show the implication (i) $\Rightarrow$ (ii) suppose that (i) holds true and put $a := \mu(A)$, $b := \mu(B \setminus A)$. Then, by (1),

$$0 < a < 1 < a + b.$$

Setting

$$\mathbf{x} := x_1\chi_A + x_2\chi_{B\setminus A}, \qquad \mathbf{y} := y_1\chi_A + y_2\chi_{B\setminus A}, \quad x_1, x_2, y_1, y_2 > 0,$$

in inequality (3) we get, for all $x_1, x_2, y_1, y_2 > 0$,

$$\psi\big(a\varphi(x_1 + y_1) + b\varphi(x_2 + y_2)\big) \leqslant \psi_1\big(a\varphi_1(x_1) + b\varphi_1(x_2)\big) + \psi_2\big(a\varphi_2(y_1) + b\varphi_2(y_2)\big). \tag{4}$$

Replacing x_i by $\varphi_1^{-1}(x_i)$, y_i by $\varphi_2^{-1}(y_i)$ for $i = 1, 2$, and making use of the strict increasing monotonicity of ψ, we obtain, for all $x_1, x_2, y_1, y_2 > 0$,

$$a\varphi\big(\varphi_1^{-1}(x_1) + \varphi_2^{-1}(y_1)\big) + b\varphi\big(\varphi_1^{-1}(x_2) + \varphi_2^{-1}(y_2)\big) \leqslant \psi^{-1}\big(\psi_1(ax_1 + bx_2) + \psi_2(ay_1 + by_2)\big). \tag{5}$$

From (2) we have

$$\psi_1\big(\varphi_1(s)\big) + \psi_2\big(\varphi_2(t)\big) \leqslant \psi\big(\varphi(s+t)\big), \quad s, t > 0.$$

Replacing s by $\varphi_1^{-1}(s)$, t by $\varphi_2^{-1}(t)$, and making use of the increasing monotonicity of ψ, we can write this inequality in the following equivalent form

$$\psi^{-1}\big(\psi_1(s)+\psi_2(t)\big) \leqslant \varphi\big(\varphi_1^{-1}(s)+\varphi_2^{-1}(t)\big), \quad s,t>0. \tag{6}$$

This inequality and (5) imply that, for all $x_1, x_2, y_1, y_2 > 0$,

$$a\psi^{-1}\big(\psi_1(x_1)+\psi_2(y_1)\big)+b\psi^{-1}\big(\psi_1(x_2)+\psi_2(y_2)\big) \leqslant \psi^{-1}\big(\psi_1(ax_1+bx_2)+\psi_2(ay_1+by_2)\big),$$

which proves that the function $F:(0,\infty)^2 \to (0,\infty)$ defined by

$$F(x,y) := \psi^{-1}\big(\psi_1(x)+\psi_2(y)\big), \quad x,y>0,$$

satisfies the inequality

$$aF(x_1,y_1)+bF(x_2,y_2) \leqslant F(ax_1+bx_2, ay_1+by_2), \quad x_1,x_2,y_1,y_2>0.$$

By Lemma 2 the function F is positively homogeneous, i.e.

$$\psi^{-1}\big(\psi_1(tx)+\psi_2(ty)\big) = t\psi^{-1}\big(\psi_1(x)+\psi_2(y)\big), \quad t,x,y>0.$$

Replacing x by $\psi_1^{-1}(u)$ and y by $\psi_2^{-1}(v)$, we hence get

$$\psi^{-1}\big(\psi_1\big(t\psi_1^{-1}(u)\big)+\psi_2\big(t\psi_2^{-1}(v)\big)\big) = t\psi^{-1}(u+v), \quad t,u,v>0,$$

whence

$$\psi_1\big(t\psi_1^{-1}(u)\big)+\psi_2\big(t\psi_2^{-1}(v)\big) = \psi\big(t\psi^{-1}(u+v)\big), \quad t,u,v>0.$$

Putting

$$f_t := \psi\circ\big(t\psi^{-1}\big), \qquad g_t := \psi_1\circ\big(t\psi_1^{-1}\big), \qquad h_t := \psi_2\circ\big(t\psi_2^{-1}\big), \quad t>0,$$

we hence get the following Pexider functional equation

$$f_t(u+v) = g_t(u)+h_t(v), \quad u,v,t>0.$$

According to a known result (cf. [1, p. 44, Theorem 12]), for each $t>0$, there are: An additive function $A_t:(0,\infty)\to\mathbb{R}$ and some real constants α_t and β_t such that

$$f_t(u) = A_t(u)+\alpha_t+\beta_t, \qquad g_t(u) = A_t(u)+\alpha_t, \qquad h_t(u) = A_t(u)+\beta_t$$

for all $u,v,t>0$. (Obviously, the functions A_t and α_t, $\beta_t \in \mathbb{R}$ are uniquely determined.) Since the functions f_t, g_t, h_t $(t>0)$ map bijectively $(0,\infty)$ onto itself, we infer that, for every $t>0$, the additive function A_t is continuous and $\alpha_t = \beta_t = 0$. Consequently, for every $t>0$, there is a uniquely determined $m(t)>0$ such that

$$f_t(u) = g_t(u) = h_t(u) = m(t)u, \quad u,t>0,$$

which, by the definitions of f_t, g_t, h_t means that

$$\psi\big(t\psi^{-1}(u)\big) = \psi_1\big(t\psi_1^{-1}(u)\big) = \psi_2\big(t\psi_2^{-1}(u)\big) = m(t)u, \quad u,t>0. \tag{7}$$

In particular we have

$$\psi\big(t\psi^{-1}(u)\big) = m(t)u, \quad u,t>0, \tag{8}$$

whence, for all $s,t,u>0$,

$$m(st)u = \psi\big(st\psi^{-1}(u)\big) = \big[\psi\circ\big(s\psi^{-1}\big)\big]\circ\psi\circ\big(t\psi^{-1}\big)(u) = m(s)m(t)u.$$

Setting here $u=1$ gives

$$m(st) = m(s)m(t), \quad s,t>0,$$

that is the function $m:(0,\infty)\to(0,\infty)$ is multiplicative. As ψ is an increasing homeomorphism of $(0,\infty)$ and, by (8),

$$m(t) = \psi\big(t\psi^{-1}(1)\big), \quad t>0,$$

we conclude that m is also an increasing homeomorphism of $(0,\infty)$. Consequently,

$$m(t) = t^q, \quad t>0, \tag{9}$$

for some $q>0$. Setting $u=\psi(1)$ in (8), we hence get

$$\psi(t) = \psi(1)t^q, \quad t>0.$$

From (7) and (9) we also get

$$\psi_1\big(t\psi_1^{-1}(u)\big) = t^q u, \qquad \psi_2\big(t\psi_2^{-1}(u)\big) = t^q u, \quad u, t > 0,$$

whence, setting respectively, $u = \psi_1(1)$ and $u = \psi_2(1)$, we obtain

$$\psi_1(t) = \psi_1(1)t^q, \qquad \psi_2(t) = \psi_2(1)t^q, \quad t > 0,$$

Now we determine the functions φ, φ_1 and φ_2. Setting $x_2 = x_1 := s$ and $y_2 = y_1 := t$ in (5) we obtain

$$(a+b)\varphi\big(\varphi_1^{-1}(s) + \varphi_2^{-1}(t)\big) \leqslant \psi^{-1}\big(\psi_1\big((a+b)s\big) + \psi_2\big((a+b)t\big)\big)$$

and, consequently,

$$\varphi\big(\varphi_1^{-1}(s) + \varphi_2^{-1}(t)\big) \leqslant \big(c_1 s^q + c_2 t^q\big)^{1/q}, \quad s, t > 0,$$

where

$$c_1 := \frac{\psi_1(1)}{\psi(1)}, \qquad c_2 := \frac{\psi_2(1)}{\psi(1)}.$$

On the other hand, from (6), we have

$$\big(c_1 s^q + c_2 t^q\big)^{1/q} \leqslant \varphi\big(\varphi_1^{-1}(s) + \varphi_2^{-1}(t)\big), \quad s, t > 0.$$

Thus

$$\big(c_1 s^q + c_2 t^q\big)^{1/q} = \varphi\big(\varphi_1^{-1}(s) + \varphi_2^{-1}(t)\big), \quad s, t > 0,$$

whence

$$\big[\varphi(s+t)\big]^q = c_1\big[\varphi_1(s)\big]^q + c_2\big[\varphi_2(t)\big]^q, \quad s, t > 0.$$

Putting

$$f(t) := \big[\varphi(t)\big]^q, \qquad g(t) := c_1\big[\varphi_1(t)\big]^q, \qquad h(t) := c_2\big[\varphi_2(t)\big]^q, \quad t > 0, \tag{10}$$

we can write this equation in the form

$$f(s+t) = g(s) + h(t), \quad s, t > 0.$$

Applying again Theorem 12 from [1, p. 44], we infer that there exist: an additive function $A : \mathbb{R} \to \mathbb{R}$ and some real constant α and β such that

$$f(t) = A(t) + \alpha + \beta, \qquad g(t) = A(t) + \alpha, \qquad h(t) = A(t) + \beta, \quad t > 0.$$

The functions f, g, being bijections of $(0, \infty)$, are bounded below. It follows that the additive function A must be linear, that is

$$A(t) = mt, \quad t > 0,$$

for some $m > 0$ and, obviously, $\alpha = \beta = 0$. Thus

$$f(t) = g(t) = h(t) = mt, \quad t > 0.$$

Therefore, by the definitions (10) of the functions f, g and h we get

$$\varphi(t) = \varphi(1)t^p, \qquad \varphi_1(t) = \varphi_1(t)t^p, \qquad \varphi_2(t) = \varphi_2(1)t^p, \quad t > 0,$$

where

$$p := \frac{1}{q}.$$

Substituting the obtained functions in (4) and then making some obvious changes of variables, we get

$$\psi(1)\big[\varphi(1)\big]^{1/p}\big[(x_1+y_1)^p + (x_2+y_2)^p\big]^{1/p} \leqslant \psi_1(1)\big[\varphi_1(1)\big]^{1/p}\big(x_1^p + x_2^p\big)^{1/p} + \psi_2(1)\big[\varphi_2(1)\big]^{1/p}\big(y_1^p + y_2^p\big)^{1/p}$$

for all $x_1, x_2, y_1, y_2 > 0$. Hence, letting $y_1 \to 0$ and $y_2 \to 0$, we conclude that

$$\psi(1)\big[\varphi(1)\big]^{1/p} \leqslant \psi_1(1)\big[\varphi_1(1)\big]^{1/p},$$

and, letting $x_1 \to 0$ and $x_2 \to 0$, we infer that

$$\psi(1)\big[\varphi(1)\big]^{1/p} \leqslant \psi_2(1)\big[\varphi_2(1)\big]^{1/p}.$$

On the other hand, using the forms of $\varphi, \varphi_1, \varphi_2, \psi, \psi_1, \psi_2$ obtained above and assumption (2), we get

$$\psi_1(1)\big[\varphi_1(1)s^p\big]^{1/p} + \psi_2(1)\big[\varphi_2(1)t^p\big]^{1/p} \leqslant \psi(1)\big[\varphi(1)(s+t)^p\big]^{1/p}, \quad s,t>0,$$

and letting $t \to 0$ and then $s \to 0$ results

$$\psi_i(1)\big[\varphi_i(1)\big]^{1/p} \leqslant \psi(1)\big[\varphi(1)\big]^{1/p} \quad \text{for } i=1,2,$$

respectively. This completes the proof of the implication (i) ⇒ (ii).

The implications (ii)⇒ (iii) and (iv) ⇒ (i) are obvious and (iii) ⇒ (iv) follows from the Minkowski inequality. □

As an immediate consequence of Theorem 1 we obtain

Corollary 1. *Let (Ω, Σ, μ) be a measure space with two sets $A, B \in \Sigma$ satisfying condition* (1). *Suppose that $\varphi, \varphi_1, \varphi_2, \psi, \psi_1, \psi_2 : (0,\infty) \to (0,\infty)$ are bijective, ψ is strictly increasing and condition* (2) *is satisfied. If*

$$\mathbb{P}_{\varphi,\psi}(\mathbf{x}+\mathbf{y}) \leqslant \mathbb{P}_{\varphi_1,\psi_1}(\mathbf{x}) + \mathbb{P}_{\varphi_2,\psi_2}(\mathbf{y}), \quad \mathbf{x},\mathbf{y} \in S_+,$$

then there is a real $p \geqslant 1$ such that

$$\frac{\varphi(t)}{\varphi(1)} = \frac{\varphi_i(t)}{\varphi_i(1)} = t^p, \qquad \frac{\psi(t)}{\psi(1)} = \frac{\psi_i(t)}{\psi_i(1)} = t^{1/p}, \quad i=1,2,\ t>0.$$

Applying this corollary we prove the following

Theorem 2. *Let (Ω, Σ, μ) be a measure space with two sets $A, B \in \Sigma$ such that*

$$0 < \mu(A) < 1 < \mu(B) < \infty.$$

Suppose that $\varphi, \varphi_1, \varphi_2 : (0,\infty) \to (0,\infty)$ are bijective functions and φ is strictly increasing. Then

$$\mathbb{P}_{\varphi,\varphi^{-1}}(\mathbf{x}+\mathbf{y}) \leqslant \mathbb{P}_{\varphi_1,\varphi_1^{-1}}(\mathbf{x}) + \mathbb{P}_{\varphi_2,\varphi_2^{-1}}(\mathbf{y}), \quad \mathbf{x},\mathbf{y} \in S_+,$$

if, and only if, there is a real $p \geqslant 1$ such that

$$\varphi(t) = \varphi(1)t^p, \qquad \varphi_1(t) = \varphi_1(1)t^p, \qquad \varphi_2(t) = \varphi_2(1)t^p, \quad t>0. \tag{11}$$

Proof. Put $\psi := \varphi^{-1},\ \psi_1 := \varphi_1^{-1},\ \psi_2 := \varphi_2^{-1}$. Then ψ is continuous, strictly increasing, and

$$\psi_1 \circ \varphi_1(s) + \psi_2 \circ \varphi_2(t) = s+t = \psi \circ \varphi(s+t), \quad s,t>0,$$

that is condition (2) is satisfied. By Corollary 1, the functions $\varphi, \varphi_1, \varphi_2$ must be of the form (11). Moreover, we have

$$\psi(1)\big[\varphi(1)\big]^{1/p} = \psi_i(1)\big[\varphi_i(1)\big]^{1/p} = 1, \quad i=1,2.$$

Now the result follows from Theorem 1. □

Since the extension of Lemma 2 holds true for n variables (cf. Remark 1), the n-dimensional counterpart of Theorem 1 (as well as each of the above results) remains valid and the proof is analogous. In particular we have

Theorem 3. *Let (Ω, Σ, μ) be a measure space with two sets $A, B \in \Sigma$ such that*

$$0 < \mu(A) < 1 < \mu(B) < \infty.$$

Suppose that $\varphi, \varphi_i, \psi, \psi_i : (0,\infty) \to (0,\infty)$ for $i=1,\dots,n$, $n \geqslant 2$, are bijective functions, ψ is strictly increasing, and

$$\sum_{i=1}^{n} \psi_i \circ \varphi_i(t_i) \leqslant \psi \circ \varphi\left(\sum_{i=1}^{n} t_i\right), \quad t_1,\dots,t_n > 0. \tag{12}$$

Then

$$\mathbb{P}_{\varphi,\psi}\left(\sum_{i=1}^{n} \mathbf{x}_i\right) \leqslant \sum_{i=1}^{n} \mathbb{P}_{\varphi_i,\psi_i}(\mathbf{x}_i), \quad \mathbf{x}_i \in S_+,\ i=1,\dots,n,$$

if, and only if, there is a real $p \geqslant 1$ such that

$$\frac{\varphi(t)}{\varphi(1)} = \frac{\varphi_i(t)}{\varphi_i(1)} = t^p, \qquad \frac{\psi(t)}{\psi(1)} = \frac{\psi_i(t)}{\psi_i(1)} = t^{1/p}, \quad i=1,\dots,n,\ t>0,$$

and

$$\psi(1)\big[\varphi(1)\big]^{1/p} = \psi_i(1)\big[\varphi_i(1)\big]^{1/p}, \quad i=1,\dots,n.$$

Remark 2. If $\varphi, \varphi_i, \psi, \psi_i : [0,\infty) \to [0,\infty)$ and $\varphi(0) = \varphi_i(0) = \psi_i(0) = 0$ for $i = 1, \dots, n$, $n \geqslant 2$, then this result is a consequence of Theorem 1.

To show this take arbitrarily $j, k \in \{1, \dots, n\}$, $j \neq k$. Without any loss of generality we can assume that $j = 1$ and $k = 2$. Taking in (12) $s := t_j$, $t = t_k$, $t_i = 0$ for $j \neq i \neq k$, we get

$$\psi_j \circ \varphi_j(s) + \psi_k \circ \varphi_k(t) \leqslant \psi \circ \varphi(s+t), \quad s, t > 0.$$

Taking $\mathbf{x}_j := \mathbf{x}$, $\mathbf{x}_k := \mathbf{y}$ for $\mathbf{x}, \mathbf{y} \in S_+(A, B)$ and $\mathbf{x}_i = \mathbf{0}$ for $j \neq i \neq k$, we get

$$\mathbb{P}_{\varphi,\psi}(\mathbf{x} + \mathbf{y}) \leqslant \mathbb{P}_{\varphi_j,\psi_j}(\mathbf{x}) + \mathbb{P}_{\varphi_k,\psi_k}(\mathbf{y}), \quad \mathbf{x}, \mathbf{y} \in S_+(A, B).$$

By Theorem 1, there is a real $p \geqslant 1$ such that, for all $t > 0$,

$$\frac{\varphi(t)}{\varphi(1)} = \frac{\varphi_j(t)}{\varphi_j(1)} = \frac{\varphi_k(t)}{\varphi k(1)} = t^p, \qquad \frac{\psi(t)}{\psi(1)} = \frac{\psi_j(t)}{\psi_j(1)} = \frac{\psi_k(t)}{\psi_k(1)} = t^{1/p},$$

and

$$\psi(1)\big[\varphi(1)\big]^{1/p} = \psi_j(1)\big[\varphi_j(1)\big]^{1/p}, \qquad \psi(1)\big[\varphi(1)\big]^{1/p} = \psi_k(1)\big[\varphi_k(1)\big]^{1/p}.$$

Corollary 2. *Let (Ω, Σ, μ) be a measure space with two sets $A, B \in \Sigma$ satisfying condition (1), and let $n \in N$, $n \geqslant 2$ be fixed. Suppose that $\varphi, \psi : (0,\infty) \to (0,\infty)$ are bijective functions, $\lim_{t\to 0+} \psi(t) = 0$, and $\psi \circ \varphi$ is superadditive, i.e.*

$$\psi\big(\varphi(s)\big) + \psi\big(\varphi(t)\big) \leqslant \psi\big(\varphi(s+t)\big), \quad s, t > 0.$$

Then

$$\mathbb{P}_{\varphi,\psi}\left(\sum_{i=1}^n \mathbf{x}_i\right) \leqslant \sum_{i=1}^n \mathbb{P}_{\varphi,\psi}(\mathbf{x}_i), \quad \mathbf{x}_i \in S_+,\ i = 1, \dots, n,$$

if, and only if, there is a real $p \geqslant 1$ such that

$$\varphi(t) = \varphi(1)t^p, \qquad \psi(t) = \psi(1)t^{1/p}, \quad t > 0.$$

Proof. For $x_1, x_2 > 0$, the functions $\mathbf{x} := x_1\chi_A$, $\mathbf{y} := x_2\chi_{B\setminus A}$ belong to S_+. Setting $\mathbf{x}_1 = \mathbf{x}$, $\mathbf{x}_2 = \mathbf{y}$, $\mathbf{x}_i = \mathbf{0}$ for $i = 3, \dots, n$, in the assumed inequality, we get

$$\begin{aligned}\psi\big(a\varphi(x_1) + b\varphi(x_2)\big) &= \mathbb{P}_{\varphi,\psi}(\mathbf{x} + \mathbf{y}) \\ &\leqslant \mathbb{P}_{\varphi,\psi}(\mathbf{x}) + \mathbb{P}_{\varphi,\psi}(\mathbf{y}) = \psi\big(a\varphi(x_1)\big) + \psi\big(b\varphi(x_2)\big)\end{aligned}$$

for all $x_1, x_2 > 0$. Taking here $x_1 := \varphi^{-1}(\frac{s}{a})$, $x_2 := \varphi^{-1}(\frac{t}{b})$, where $s, t > 0$, we obtain

$$\psi(s+t) \leqslant \psi(s) + \psi(t), \quad s, t > 0.$$

Since ψ is one-to-one and $\lim_{t\to 0} \psi(t) = 0$, Lemma 3 implies that ψ is an increasing homeomorphism of $(0,\infty)$. Now we can apply Theorem 1. □

4. The reversed inequality

Remark 3. The functional $\mathbb{P}_{\varphi,\psi}$ is superadditive on the linear space S, i.e.

$$\mathbb{P}_{\varphi,\psi}(\mathbf{x} + \mathbf{y}) \geqslant \mathbb{P}_{\varphi,\psi}(\mathbf{x}) + \mathbb{P}_{\varphi,\psi}(\mathbf{y}), \quad \mathbf{x}, \mathbf{y} \in S,$$

iff the underlying measure space (Ω, Σ, μ) satisfies the following condition:

for every $A \in \Sigma$, either $\mu(A) = 0$ or $\mu(A) = \infty$.

In fact, if there were a set $A \in \Sigma$ such that $0 < \mu(A) < \infty$, then for $x := \chi_A$ and $y := -x$ we would get

$$0 = \mathbb{P}_{\varphi,\psi}(0) = \mathbb{P}_{\varphi,\psi}(x + y) \geqslant \mathbb{P}_{\varphi,\psi}(x) + \mathbb{P}_{\varphi,\psi}(-x) > 0.$$

Thus the problem of the global superadditivity of $\mathbb{P}_{\varphi,\psi}$ trivializes. Nevertheless the superadditivity of $\mathbb{P}_{\varphi,\psi}$ in S_+ becomes interesting. The following converse of the accompanying Minkowski inequality holds true.

Theorem 4. *Let (Ω, Σ, μ) be a measure space with two sets $A, B \in \Sigma$ satisfying condition* (1)*. Suppose that $\varphi, \varphi_i, \psi, \psi_i$: $(0, \infty) \to (0, \infty)$ for $i = 1, \dots, n$, $n \geqslant 2$, are bijective functions, ψ is strictly increasing, and*

$$\sum_{i=1}^{n} \psi_i \circ \varphi_i(t_i) \geqslant \psi \circ \varphi\left(\sum_{i=1}^{n} t_i\right), \quad t_1, \dots, t_n > 0.$$

Then

$$\mathbb{P}_{\varphi,\psi}\left(\sum_{i=1}^{n} \mathbf{x}_i\right) \geqslant \sum_{i=1}^{n} \mathbb{P}_{\varphi_i,\psi_i}(\mathbf{x}_i), \quad \mathbf{x}_i \in S_+,\ i = 1, \dots, n,$$

if, and only if, there is a real $p \in (0, 1)$ such that

$$\frac{\varphi(t)}{\varphi(1)} = \frac{\varphi_i(t)}{\varphi_i(1)} = t^p, \qquad \frac{\psi(t)}{\psi(1)} = \frac{\psi_i(t)}{\psi_i(1)} = t^{1/p}, \quad i = 1, \dots, n,\ t > 0,$$

and

$$\psi(1)\big[\varphi(1)\big]^{1/p} = \psi_i(1)\big[\varphi_i(1)\big]^{1/p}, \quad i = 1, \dots, n.$$

Since the proof is analogous to that of Theorem 1, we omit it.

It is interesting that in the counterpart of Corollary 2, the regularity of ψ can be completely omitted. Namely, we have the following

Corollary 3. *Let (Ω, Σ, μ) be a measure space with two sets $A, B \in \Sigma$ satisfying the condition* (1) *and let $n \in N$, $n \geqslant 2$ be fixed. Suppose that $\varphi, \psi : (0, \infty) \to (0, \infty)$ are bijective functions, and $\psi \circ \varphi$ is superadditive, i.e.*

$$\psi\big(\varphi(s)\big) + \psi\big(\varphi(t)\big) \leqslant \psi\big(\varphi(s + t)\big), \quad s, t > 0.$$

Then

$$\mathbb{P}_{\varphi,\psi}\left(\sum_{i=1}^{n} \mathbf{x}_i\right) \geqslant \sum_{i=1}^{n} \mathbb{P}_{\varphi,\psi}(\mathbf{x}_i), \quad \mathbf{x}_i \in S_+,\ i = 1, \dots, n,$$

if, and only if, there is a $p \in (0, 1)$ such that

$$\varphi(t) = \varphi(1)t^p, \qquad \psi(t) = \psi(1)t^{1/p}, \quad t > 0.$$

Proof. For $x_1, x_2 > 0$, the functions $\mathbf{x} := x_1\chi_A$, $\mathbf{y} := x_2\chi_{B\setminus A}$ belong to S_+. Setting $\mathbf{x}_1 = \mathbf{x}$, $\mathbf{x}_2 = \mathbf{y}$, $\mathbf{x}_i = \mathbf{0}$ for $i = 3, \dots, n$, in the assumed inequality, we get

$$\begin{aligned}\psi\big(a\varphi(x_1) + b\varphi(x_2)\big) &= \mathbb{P}_{\varphi,\psi}(\mathbf{x} + \mathbf{y}) \\ &\geqslant \mathbb{P}_{\varphi,\psi}(\mathbf{x}) + \mathbb{P}_{\varphi,\psi}(\mathbf{y}) = \psi\big(a\varphi(x_1)\big) + \psi\big(b\varphi(x_2)\big),\end{aligned}$$

for all $x_1, x_2 > 0$. Taking $x_1 := \varphi^{-1}(\frac{s}{a})$, $x_2 := \varphi^{-1}(\frac{t}{b})$ we obtain

$$\psi(s + t) \geqslant \psi(s) + \psi(t), \quad s, t > 0.$$

Thus ψ is superadditive and, consequently, it is strictly monotonic. Now the result follows from Theorem 4. □

5. Remark on the basic assumption

The assumption of the underlying measure space (Ω, Σ, μ) in all the results is indispensable because in each of the cases:

(I) *for every $A \in \Sigma$, we have $\mu(A) = 0$ or $\mu(A) \geqslant 1$;*
(II) *for every $A \in \Sigma$, we have $\mu(A) \leqslant 1$ or $\mu(A) = \infty$,*

there are large classes of non-power bijective functions $\varphi : (0, \infty) \to (0, \infty)$ such that $\mathbb{P}_{\varphi,\varphi^{-1}}$ is subadditive in S or superadditive in S_+ (cf. Mulholland [10], also [4,6]).

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References

[1] J. Aczél, J. Dhombres, Functional Equations in Several Variables, Encyclopedia Math. Appl., Cambridge University Press, 1989.
[2] J. Matkowski, The converse of the Minkowski inequality theorem and its generalization, Proc. Amer. Math. Soc. 109 (1990) 663–675.
[3] J. Matkowski, T. Świątkowski, Quasi-monotonicity, Subadditive bijections of $\mathbb{R}_+$, and a characterization of L^p-norm, J. Math. Anal. Appl. 154 (1991) 493–506.
[4] J. Matkowski, L^p-like paranorms, in: D. Gronau, L. Reich (Eds.), Selected Topics in Functional Equations and Iteration Theory, Proceedings of the Austrian–Polish Seminar, Universität Graz, October 24-26, 1991, in: Grazer Math. Ber., vol. 316, 1992, pp. 103–139.
[5] J. Matkowski, Functional inequality characterizing nonnegative concave functions in $(0,\infty)^k$, Aequationes Math. 43 (1992) 219–224.
[6] J. Matkowski, On a generalization of Mulholland's inequality, Abh. Math. Sem. Univ. Hamburg 63 (1993) 97–103.
[7] J. Matkowski, T. Świątkowski, On subadditive functions of $\mathbb{R}_+$, Proc. Amer. Math. Soc. 119 (1993) 187–197.
[8] J. Matkowski, M. Pycia, Convex-like inequality, homogeneity, subadditivity, and a characterization of L^p-norm, Ann. Polon. Math. LX.3 (1995) 221–230.
[9] J. Matkowski, The converse theorem for Minkowski's inequality, Indag. Math. (N.S.) 15 (1) (2004) 73–84.
[10] H.P. Mulholland, On generalization of Minkowski inequality in the form of a triangle inequality, Proc. London Math. Soc. 51 (1950) 294–307.
[11] J.V. Pexider, Notiz über Funktionaltheoreme, Monatsh. Math. Phys. 14 (1903) 293–301.