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A pair of functional inequalities characterizing polynomials and Bernoulli numbers

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Summary. We show that if a function $f : \mathbb{R} \to \mathbb{R}$ continuous at least at one point satisfies the pair of functional inequalities

$$f(x+a) \leq f(x) + \sum_{j=0}^{k} \alpha_j x^j,$$

$$f(x+b) \leq f(x) + \sum_{j=0}^{k} \beta_j x^j,$$

and the constants a, b, α_i, β_i $(i = 0, 1, \dots, k)$ fulfil some general algebraic conditions, then f must be a polynomial. An explicit formula for the solution, involving Bernoulli numbers, is given.

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1. Introduction

The celebrated results of Mazur–Orlicz [8] characterize polynomials of degree at most k (in $\mathbb{R}^n$), under mild regularity conditions, with the aid of the functional equation

$$\Delta_h \circ (\Delta_h^k f)(x) = 0, \qquad x \in \mathbb{R}^n, \tag{$*$}$$

for all $h \in \mathbb{R}^n$, where Δ_h stands for the difference operator with the span h (cf. also McKiernan [10], Hosszu [3], Djokovic [1], Székelyhidi [14] and Kuczma [7], Chapter XV, § 9).

It is natural to ask whether it is necessary to assume that $(*)$ holds for all $h \in \mathbb{R}^n$, and if, under some conditions, equality can be replaced by inequality. One could expect that the answer is positive because in this paper we show that every polynomial $f : \mathbb{R} \longrightarrow \mathbb{R}$ can be characterized by the system of only two functional inequalities

$$\Delta_a f(x) \leq P(x), \quad \Delta_b f(x) \leq Q(x), \qquad x \in \mathbb{R},$$

where the numbers $a, b \in \mathbb{R}$ and the polynomials $P, Q : \mathbb{R} \longrightarrow \mathbb{R}$ of degree at most k satisfy some special conditions. More precisely, we show that the simultaneous system of functional inequalities

$$f(x+a) \leq f(x) + \sum_{j=0}^{k} \alpha_j x^j, \quad f(x+b) \leq f(x) + \sum_{j=0}^{k} \beta_j x^j, \tag{1}$$

where $a, b, \alpha_0, \alpha_1, \ldots, \alpha_k, \beta_0, \beta_1, \ldots, \beta_k$ are fixed real numbers, satisfying some algebraic conditions, and f is a real function of a real variable allows to characterize the polynomials of degree $k+1$. Let us remark that in the case when f satisfies the corresponding system of equations with $\alpha_0 = \alpha_1 = \cdots = \alpha_k = \beta_0 = \beta_1 = \cdots = \beta_k = 0$ the problem was studied by Montel [11] and Popoviciu [12] (cf. also Kuczma [6], p. 228).

In Section 2, assuming the continuity of the function f at least at one point, the inequality

$$(-1)^k \left(\frac{\alpha_k}{a} - \frac{\beta_k}{b} \right) \geq 0,$$

and some simple additional algebraic conditions only on a and b, we show that

$$g(x) := f(x) - \frac{\alpha_k}{a(k+1)} x^{k+1}$$

satisfies an analogous system of inequalities with some polynomials of degree $k-1$, the coefficients of which are uniquely determined by $a, b, \alpha_0, \alpha_1, \ldots, \alpha_k, \beta_0, \beta_1, \ldots, \beta_k$ (Theorem 1). This reduction theorem implies that, under suitable conditions, the function $f : \mathbb{R} \longrightarrow \mathbb{R}$, satisfying system (1) has to be a uniquely determined polynomial of degree $k+1$. Moreover, an application of Theorem 1 in turn k-times, for $k, k-1, \ldots, 1$, allows to find the coefficients of this polynomial, and to establish the explicit form of the solution of system (1). A serious inconvenience of this method is the fact that the procedure requires rather complicated calculations. However, it turned out to be very helpful in finding a direct formula of the polynomial which is given in Section 3 (Theorem 2). It is a little surprising that the coefficients of the solution depend on Bernoulli numbers.

2. Reduction theorem and a characterization of polynomials

By $\mathbb{Q}$ we denote the set of rational numbers. Put $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$.

In the sequel we adopt the convention $\sum_{j=0}^{-1} a_j = 0$.

Theorem 1. *Let $k \in \mathbb{N}_0$ and $a, b, \alpha_0, \alpha_1, \dots, \alpha_k, \beta_0, \beta_1, \dots, \beta_k \in \mathbb{R}$ be fixed and such that*

$$a < 0 < b, \qquad \frac{b}{a} \notin \mathbb{Q}, \qquad (-1)^k\left(\frac{\alpha_k}{a} - \frac{\beta_k}{b}\right) \geq 0.$$

If a function $f : \mathbb{R} \longrightarrow \mathbb{R}$ continuous at least at one point satisfies the system of functional inequalities

$$f(x+a) \leq f(x) + \sum_{j=0}^{k} \alpha_j x^j, \quad f(x+b) \leq f(x) + \sum_{j=0}^{k} \beta_j x^j, \quad x \in \mathbb{R}, \tag{1}$$

then $\frac{\alpha_k}{a} = \frac{\beta_k}{b}$ and the function $g : \mathbb{R} \longrightarrow \mathbb{R}$ defined by

$$g(x) := f(x) - \frac{\alpha_k}{a(k+1)} x^{k+1}, \qquad x \in \mathbb{R},$$

satisfies the system of inequalities

$$g(x+a) \leq g(x) + \sum_{j=0}^{k-1} \alpha_j^* x^j, \qquad g(x+b) \leq g(x) + \sum_{j=0}^{k-1} \beta_j^* x^j, \qquad x \in \mathbb{R},$$

where

$$\alpha_j^* := \alpha_j - \frac{\alpha_k}{(k+1)}\binom{k+1}{j} a^{k-j}, \qquad \beta_j^* := \beta_j - \frac{\beta_k}{(k+1)}\binom{k+1}{j} b^{k-j}.$$

Proof. Suppose that the function $f : \mathbb{R} \longrightarrow \mathbb{R}$ satisfies system (1).

Using an induction argument, we will show that f satisfies, for all $m, n \in \mathbb{N}$, $x \in \mathbb{R}$, the inequalities

$$f(x+ma) \leq f(x) + \sum_{j=0}^{k}\left(\sum_{i=0}^{k-j}\binom{i+j}{i} S_i^m \alpha_{i+j} a^i\right) x^j, \tag{2}$$

$$f(x+nb) \leq f(x) + \sum_{j=0}^{k}\left(\sum_{i=0}^{k-j}\binom{i+j}{i} S_i^n \beta_{i+j} b^i\right) x^j, \tag{3}$$

where

$$S_i^m := \sum_{j=0}^{m-1} j^i,$$

(we adopt here the convention $\sum_{j=0}^{0} j^0 = 1$).

For $m = 1$ inequality (2) coincides with the first of the inequalities (1). Suppose (2) holds true for an $m \in \mathbb{N}$. Applying in turn the first of inequalities (1), inequality (2) for an $m \in \mathbb{N}$, a simple rule for double sums (Fubini's theorem), and finally, changing of the indices in the considered sums, we obtain

$$f(x+(m+1)a) = f((x+ma)+a)$$

$$
\begin{aligned}
&\leq f(x+ma) + \sum_{j=0}^{k} \alpha_j (x+ma)^j \\
&\leq f(x) + \sum_{j=0}^{k} \left(\sum_{i=0}^{k-j} \binom{i+j}{i} S_i^m \alpha_{i+j} a^i \right) x^j + \sum_{j=0}^{k} \alpha_j \sum_{l=0}^{j} \binom{j}{l} (ma)^{j-l} x^l \\
&= f(x) + \sum_{j=0}^{k} \left(\sum_{i=0}^{k-j} \binom{i+j}{i} S_i^m \alpha_{i+j} a^i \right) x^j + \sum_{l=0}^{k} \sum_{j=l}^{k} \alpha_j \binom{j}{l} (ma)^{j-l} x^l \\
&= f(x) + \sum_{j=0}^{k} \left(\sum_{i=0}^{k-j} \binom{i+j}{i} S_i^m \alpha_{i+j} a^i \right) x^j + \sum_{j=0}^{k} \sum_{i=j}^{k} \alpha_i \binom{i}{j} (ma)^{i-j} x^j \\
&= f(x) + \sum_{j=0}^{k} \left(\sum_{i=0}^{k-j} \binom{i+j}{i} S_i^m \alpha_{i+j} a^i + \sum_{i=0}^{k-j} \alpha_{i+j} \binom{i+j}{j} (ma)^i \right) x^j \\
&= f(x) + \sum_{j=0}^{k} \left(\sum_{i=0}^{k-j} \binom{i+j}{i} (S_i^m + m^i) \alpha_{i+j} a^i \right) x^j \\
&= f(x) + \sum_{j=0}^{k} \left(\sum_{i=0}^{k-j} \binom{i+j}{i} S_i^{m+1} \alpha_{i+j} a^i \right) x^j
\end{aligned}
$$

for all $x \in \mathbb{R}$. By induction, inequality (2) holds true for all $m \in \mathbb{N}$.

Since the proof of inequality (3) is analogous, we omit it.

Now, replacing x by $x + ma$ in (3) and then applying in turn inequality (3), inequality (2), Newton's binomial formula, changing the order of the sums and, finally, introducing a suitable change of indices in the sums, we get

$$
\begin{aligned}
f(x+ma+nb) &= f((x+ma)+nb) \\
&\leq f(x+ma) + \sum_{j=0}^{k} \left(\sum_{i=0}^{k-j} \binom{i+j}{i} S_i^n \beta_{i+j} b^i \right) (x+ma)^j \\
&\leq f(x) + \sum_{j=0}^{k} \left(\sum_{i=0}^{k-j} \binom{i+j}{i} S_i^m \alpha_{i+j} a^i \right) x^j \\
&\quad + \sum_{j=0}^{k} \left(\sum_{i=0}^{k-j} \binom{i+j}{i} S_i^n \beta_{i+j} b^i \right) (x+ma)^j \\
&= f(x) + \sum_{j=0}^{k} \left(\sum_{i=0}^{k-j} \binom{i+j}{i} S_i^m \alpha_{i+j} a^i \right) x^j
\end{aligned}
$$

$$+\sum_{j=0}^{k}\left(\sum_{i=0}^{k-j}\binom{i+j}{i}S_i^n\beta_{i+j}b^i\right)\sum_{r=0}^{j}\binom{j}{r}(ma)^{j-r}x^r$$

$$=f(x)+\sum_{j=0}^{k}\left(\sum_{i=0}^{k-j}\binom{i+j}{i}S_i^m\alpha_{i+j}a^i\right)x^j$$

$$+\sum_{j=0}^{k}\sum_{r=0}^{j}\left(\sum_{i=0}^{k-j}\binom{i+j}{i}S_i^n\beta_{i+j}b^i\right)\binom{j}{r}(ma)^{j-r}x^r$$

$$=f(x)+\sum_{j=0}^{k}\left(\sum_{i=0}^{k-j}\binom{i+j}{i}S_i^m\alpha_{i+j}a^i\right)x^j$$

$$+\sum_{r=0}^{k}\sum_{j=r}^{k}\left(\sum_{i=0}^{k-j}\binom{i+j}{i}S_i^n\beta_{i+j}b^i\right)\binom{j}{r}(ma)^{j-r}x^r$$

$$=f(x)+\sum_{j=0}^{k}\left(\sum_{i=0}^{k-j}\binom{i+j}{i}S_i^m\alpha_{i+j}a^i\right)x^j$$

$$+\sum_{j=0}^{k}\sum_{l=j}^{k}\left(\sum_{i=0}^{k-l}\binom{i+l}{i}S_i^n\beta_{i+l}b^i\right)\binom{l}{j}(ma)^{l-j}x^j$$

$$=f(x)+\sum_{j=0}^{k}\left(\sum_{i=0}^{k-j}\binom{i+j}{i}S_i^m\alpha_{i+j}a^i\right.$$

$$\left.+\sum_{l=j}^{k}\left(\sum_{i=0}^{k-l}\binom{i+l}{i}S_i^n\beta_{i+l}b^i\right)\binom{l}{j}(ma)^{l-j}\right)x^j,$$

that is, f satisfies the inequality

$$f(x+ma+nb)\le f(x)+\sum_{j=0}^{k}\left(\sum_{i=0}^{k-j}\binom{i+j}{i}S_i^m\alpha_{i+j}a^i\right.$$

$$\left.+\sum_{l=j}^{k}\left(\sum_{i=0}^{k-l}\binom{i+l}{i}S_i^n\beta_{i+l}b^i\right)\binom{l}{j}(ma)^{l-j}\right)x^j \tag{4}$$

for all $m,n\in\mathbb{N}$, $x\in\mathbb{R}$.

Since (cf. for instance H. Rademacher [13], p. 4)

$$S_i^m=\sum_{j=0}^{m-1}j^i=\frac{1}{i+1}\sum_{j=0}^{i}\binom{i+1}{j}B_j\cdot m^{i-j+1},$$

where

$$B_j := \sum_{k=0}^{j} \frac{1}{k+1} \sum_{i=0}^{k} (-1)^i \binom{k}{i} i^j$$

is the j-th Bernoulli number, S_i^m is a polynomial of degree $i+1$ with respect to m and the coefficient of the term m^{i+1} is equal to $\frac{1}{i+1}$.

Let us note that, for a fixed $x \in \mathbb{R}$, the expression on the right-hand side of (4) is a polynomial with respect to m and n and the sum of all summands of degree $k+1$ (which is the highest one) is of the form

$$\frac{1}{k+1}\alpha_k a^k m^{k+1} + \sum_{j=0}^{k} \binom{k}{j} \frac{1}{j+1} \beta_k a^{k-j} b^j m^{k-j} n^{j+1}. \tag{5}$$

Since $ab < 0$ and $\frac{b}{a} \notin \mathbb{Q}$, the Kronecker theorem implies that the set $A = \{ma + nb : \ m, n \in \mathbb{N}\}$ is dense in $\mathbb{R}$ (cf. P. Halmos [2], p. 69, Theorem C, and [9]). Thus there exist two sequences (m_r) and (n_r) of positive integers such that

$$\lim_{r\to\infty} (m_r a + n_r b) = 0.$$

Note that

$$\lim_{r\to\infty} m_r = \lim_{r\to\infty} n_r = \infty$$

(otherwise the number $\frac{b}{a}$ would be rational).

Obviously

$$\lim_{r\to\infty} \frac{m_r a + n_r b}{n_r} = 0,$$

and, consequently,

$$\lim_{r\to\infty} \frac{m_r}{n_r} = -\frac{b}{a}.$$

Let $x_0 \in \mathbb{R}$ be a point of continuity of f. Put in (4) the numbers x_0, m_r, n_r instead of x, m, n, respectively. Dividing both sides of the obtained inequality by n_r^{k+1}, then letting $r \to \infty$, and taking into account (5), we obtain the inequality

$$0 \le \frac{1}{k+1}(-1)^{k+1}\left(\frac{b}{a}\right)^{k+1} \alpha_k a^k + \beta_k b^k \sum_{j=0}^{k} \binom{k}{j} \frac{(-1)^{k-j}}{j+1}.$$

Since

$$\sum_{j=0}^{k} \binom{k}{j} \frac{(-1)^{k-j}}{j+1} = \frac{(-1)^k}{k+1}$$

(cf. Remark 1 below), we hence get the inequality

$$0 \ge \frac{1}{k+1}(-1)^k \left(\frac{\alpha_k}{a} - \frac{\beta_k}{b}\right),$$

which is just opposite to the assumed one. Thus we have shown that

$$\frac{\alpha_k}{a} = \frac{\beta_k}{b}.$$

From this equality we get

$$\begin{aligned}
g(x+a) &= f(x+a) - \frac{\alpha_k}{a(k+1)}(x+a)^{k+1} \\
&\le f(x) + \sum_{j=0}^{k} \alpha_j x^j - \frac{\alpha_k}{a(k+1)} \sum_{j=0}^{k+1} \binom{k+1}{j} a^{k+1-j} x^j \\
&= f(x) - \frac{\alpha_k}{a(k+1)} x^{k+1} + \sum_{j=0}^{k-1} \alpha_j x^j + \alpha_k x^k \\
&\quad - \frac{\alpha_k}{a(k+1)} \sum_{j=0}^{k-1} \binom{k+1}{j} a^{k+1-j} x^j - \frac{\alpha_k}{a(k+1)}(k+1) a x^k \\
&= g(x) + \sum_{j=0}^{k-1} \left(\alpha_j - \frac{\alpha_k}{(k+1)} \binom{k+1}{j} a^{k-j} \right) x^j,
\end{aligned}$$

whence, by the definition of α_j^*,

$$g(x+a) \le g(x) + \sum_{j=0}^{k-1} \alpha_j^* x^j, \qquad x \in \mathbb{R}.$$

We omit the same proof of the remaining inequality.

Remark 1. Integration of both sides of the identity

$$(t-1)^k = \sum_{j=0}^{k} \binom{k}{j} t^j (-1)^{k-j}, \qquad k \in \mathbb{N}, t \in \mathbb{R},$$

over the interval with endpoints 0 and x leads to the equality

$$\frac{(x-1)^{k+1}}{k+1} - \frac{(-1)^{k+1}}{k+1} = \sum_{j=0}^{k} \binom{k}{j} (-1)^{k-j} \frac{x^{j+1}}{j+1}.$$

Putting here $x = 1$ gives

$$\sum_{j=0}^{k} \binom{k}{j} \frac{(-1)^{k-j}}{j+1} = \frac{(-1)^k}{k+1}.$$

For a characterization of polynomials we shall need

Lemma 1 ([5]). *Let $a, b, \alpha_0, \beta_0 \in \mathbb{R}$ be fixed. Suppose that*

$$a < 0 < b, \qquad \frac{b}{a} \notin \mathbb{Q}, \qquad \frac{\alpha_0}{a} \geq \frac{\beta_0}{b},$$

and the function $f : \mathbb{R} \longrightarrow \mathbb{R}$ is continuous at least at one point.

If f satisfies the pair of functional inequalities

$$f(a+x) \leq f(x) + \alpha_0, \quad f(b+x) \leq f(x) + \beta_0, \qquad x \in \mathbb{R},$$

then

$$f(x) = px + f(0), \quad x \in \mathbb{R}, \qquad \textit{where} \quad p := \frac{\alpha_0}{a}.$$

Theorem 1 together with Lemma 1 suggests a procedure to find the explicit form of the polynomial being the solution of the system (1). We illustrate the results of this procedure in a few examples.

Let $k \in \mathbb{N}$ and $a, b, \alpha_0, \alpha_1, \ldots, \alpha_k, \beta_0, \beta_1, \ldots, \beta_k \in \mathbb{R}$ be fixed and such that

$$a < 0 < b, \qquad \frac{b}{a} \notin \mathbb{Q}, \quad \text{and} \quad (-1)^k \left(\frac{\alpha_k}{a} - \frac{\beta_k}{b} \right) \geq 0.$$

Suppose that the real function f is continuous at least at one point and satisfies (1).

Example 1. If $k = 1$ and

$$\frac{\alpha_0 - \frac{\alpha_1 a}{2}}{a} \geq \frac{\beta_0 - \frac{\beta_1 b}{2}}{b},$$

then, for all $x \in \mathbb{R}$,

$$f(x) = \frac{\alpha_1}{2a} x^2 + \frac{\alpha_0 - \frac{\alpha_1 a}{2}}{a} x + f(0).$$

Example 2. If $k = 2$ and

$$\frac{\alpha_1 - \alpha_2 a}{a} \leq \frac{\beta_1 - \beta_2 b}{b}, \tag{6}$$

$$\frac{\alpha_0 - \frac{\alpha_1 a}{2} + \frac{\alpha_2 a^2}{6}}{a} \geq \frac{\beta_0 - \frac{\beta_1 b}{2} + \frac{\beta_2 b^2}{6}}{b}, \tag{7}$$

then, for all $x \in \mathbb{R}$,

$$f(x) = \frac{\alpha_2}{3a} x^3 + \frac{\alpha_1 - \alpha_2 a}{2a} x^2 + \frac{\alpha_0 - \frac{\alpha_1 a}{2} + \frac{\alpha_2 a^2}{6}}{a} x + f(0).$$

Example 3. If $k = 3$ and

$$\frac{\alpha_2 - \frac{3}{2}\alpha_3 a}{a} \geq \frac{\beta_2 - \frac{3}{2}\beta_3 b}{b},$$

$$\frac{\alpha_1-\alpha_2 a+\frac{1}{2}\alpha_3 a^2}{a}\le\frac{\beta_1-\beta_2 b+\frac{1}{2}\beta_3 b^2}{b},$$
$$\frac{\alpha_0-\frac{\alpha_1 a}{2}+\frac{\alpha_2 a^2}{6}}{a}\ge\frac{\beta_0-\frac{\beta_1 b}{2}+\frac{\beta_2 b^2}{6}}{b},$$

then, for all $x\in\mathbb{R}$,

$$f(x)=\frac{\alpha_3}{4a}x^4+\frac{\alpha_2-\frac{3}{2}\alpha_3 a}{3a}x^3+\frac{\alpha_1-\alpha_2 a+\frac{\alpha_3 a^2}{2}}{2a}x^2+\frac{\alpha_0-\frac{\alpha_1 a}{2}+\frac{\alpha_2 a^2}{6}}{a}x+f(0).$$

Proof for the case $k=2$. According to Theorem 1, the function

$$g(x):=f(x)-\frac{\alpha_2}{3a}x^3,\qquad x\in\mathbb{R},$$

satisfies the system of inequalities

$$g(x+a)\le g(x)+(\alpha_1-\alpha_2 a)x+\alpha_0-\frac{\alpha_2 a^2}{3}$$
$$g(x+b)\le g(x)+(\beta_1-\beta_2 b)x+\beta_0-\frac{\beta_2 b^2}{3},$$

for all $x\in\mathbb{R}$. Since (6) holds true, we can again apply Theorem 1. Thus

$$h(x):=g(x)-\frac{\alpha_1-\alpha_2 a}{2a}x^2,$$

satisfies, for all $x\in\mathbb{R}$, the system

$$h(x+a)\le h(x)+\alpha_0-\frac{\alpha_1 a}{2}+\frac{\alpha_2 a^2}{6}$$
$$h(x+b)\le h(x)+\beta_0-\frac{\beta_1 b}{2}+\frac{\beta_2 b^2}{6}.$$

Now, the assumed inequality (7) and Lemma 1 imply that

$$h(x)=\frac{\alpha_0-\frac{\alpha_1 a}{2}+\frac{\alpha_2 a^2}{6}}{a}x+h(0),\qquad x\in\mathbb{R},$$

whence, by the definition of h and g,

$$f(x)=\frac{\alpha_2}{3a}x^3+\frac{\alpha_1-\alpha_2 a}{2a}x^2+\frac{\alpha_0-\frac{\alpha_1 a}{2}+\frac{\alpha_2 a^2}{6}}{a}x+f(0),\qquad x\in\mathbb{R}.$$

Let us note that in Example 3, in the formula for f, unexpectedly, α_3 does not appear in the coefficient at x.

These examples had suggested that in the explicit form of the solution the Bernoulli numbers might play a crucial role.

Remark 2. Applying Theorem 1 and the Montel's result one can get another argument for Lemma 1.

3. The explicit form of the solution

Lemma 2. *If $k \in \mathbb{N}$ and $a \in \mathbb{R}$, then for each $j = 0, 1, \ldots, k-1$,*

$$B_{k-j}\binom{k}{j}a^{k-j} = -\frac{1}{k+1}\sum_{i=j}^{k-1} B_{i-j}\binom{i}{j}\binom{k+1}{i}a^{k-j}.$$

Proof. By the known equality for Bernoulli numbers (cf. [4], p. 282), we have

$$\sum_{i=0}^{k}\binom{k+1}{i}B_i = 0, \quad k \geq 1.$$

Using this equality with k replaced by $k-j$, we obtain

$$\sum_{i=0}^{k-j}\binom{k+1-j}{i}B_i = 0, \qquad j = 0, 1, \ldots, k-1.$$

Multiplying both sides of these equalities by $\binom{k+1}{j}$ we get

$$\sum_{i=0}^{k-j}\binom{j+i}{j}\binom{k+1}{j+i}B_i = 0,$$

that is

$$\sum_{i=j}^{k}\binom{i}{j}\binom{k+1}{i}B_{i-j} = 0,$$

whence, obviously,

$$\sum_{i=j}^{k} B_{i-j}\binom{i}{j}\binom{k+1}{i}\frac{1}{k+1}a^{k-j} = 0, \qquad j = 0, 1, \ldots, k-1,$$

which completes the proof.

Theorem 2. *Let $k \in N$ and $a, b, \alpha_0, \alpha_1, \ldots, \alpha_k, \beta_0, \beta_1, \ldots, \beta_k \in \mathbb{R}$ such that $a < 0 < b$, $\frac{b}{a} \notin \mathbb{Q}$ be fixed. Suppose that a function $f : \mathbb{R} \to \mathbb{R}$ continuous at least at one point satisfies the pair of functional inequalities* (1). *If*

$$(-1)^j\left(\frac{\sum_{i=j}^{k} B_{i-j}\binom{i}{j}\alpha_i a^{i-j}}{a} - \frac{\sum_{i=j}^{k} B_{i-j}\binom{i}{j}\beta_i b^{i-j}}{b}\right) \geq 0, \quad j = 0, 1, \ldots, k, \quad (8)$$

then

$$f(x) = \sum_{j=0}^{k}\sum_{i=j}^{k}\frac{B_{i-j}\binom{i}{j}a^{i-j}\alpha_i}{a(j+1)}x^{j+1} + f(0) = \sum_{j=0}^{k}\sum_{i=j}^{k}\frac{B_{i-j}\binom{i}{j}b^{i-j}\beta_i}{b(j+1)}x^{j+1} + f(0). \tag{9}$$

Proof. In view of Example 1, our theorem holds true for $k = 1$. For the inductive argument assume that the result holds true for all positive integers less than k. By Theorem 1 the function $g(x) := f(x) - \frac{\alpha_k}{a(k+1)} x^{k+1}$ satisfies the system of inequalities

$$g(x+a) \le g(x) + \sum_{j=0}^{k-1} \alpha_j^* x^j, \qquad g(x+b) \le g(x) + \sum_{j=0}^{k-1} \beta_j^* x^j$$

for all $x \in \mathbb{R}$. By the inductive assumption we have

$$\begin{aligned} g(x) &= \sum_{j=0}^{k-1} \sum_{i=j}^{k-1} \frac{B_{i-j} \binom{i}{j} a^{i-j} \left(\alpha_i - \frac{\alpha_k}{k+1} \binom{k+1}{i} a^{k-i}\right)}{a(j+1)} x^{j+1} + g(0) \\ &= \sum_{j=0}^{k-1} \left(\sum_{i=j}^{k-1} \frac{B_{i-j} \binom{i}{j} a^{i-j} \alpha_i}{a(j+1)} - \sum_{i=j}^{k-1} \frac{B_{i-j} \binom{i}{j} \binom{k+1}{i} a^{k-j} \frac{\alpha_k}{k+1}}{a(j+1)} \right) x^{j+1} + g(0) \end{aligned}$$

under the conditions

$$(-1)^j \left(\frac{\sum\limits_{i=j}^{k-1} B_{i-j} \binom{i}{j} \alpha_i a^{i-j} - \sum\limits_{i=j}^{k-1} B_{i-j} \binom{i}{j} \binom{k+1}{i} a^{k-j} \frac{\alpha_k}{k+1}}{a} \right.$$
$$\left. - \frac{\sum\limits_{i=j}^{k-1} B_{i-j} \binom{i}{j} \beta_i b^{i-j} - \sum\limits_{i=j}^{k-1} B_{i-j} \binom{i}{j} \binom{k+1}{i} b^{k-j} \frac{\beta_k}{k+1}}{b} \right) \ge 0$$

for $j = 0, 1, \ldots, k-2$, and

$$(-1)^{k-1} \left(\frac{\alpha_{k-1} - \frac{\alpha_k}{k+1} \binom{k+1}{k-1} a}{a} - \frac{\beta_{k-1} - \frac{\beta_k}{k+1} \binom{k+1}{k-1} b}{b} \right) \ge 0,$$

i.e. under the conditions

$$(-1)^j \left(\frac{\sum\limits_{i=j}^{k-1} B_{i-j} \binom{i}{j} \alpha_i a^{i-j} - \sum\limits_{i=j}^{k-1} B_{i-j} \binom{i}{j} \binom{k+1}{i} a^{k-j} \frac{\alpha_k}{k+1}}{a} \right.$$
$$\left. - \frac{\sum\limits_{i=j}^{k-1} B_{i-j} \binom{i}{j} \beta_i b^{i-j} - \sum\limits_{i=j}^{k-1} B_{i-j} \binom{i}{j} \binom{k+1}{i} b^{k-j} \frac{\beta_k}{k+1}}{b} \right) \ge 0$$

for $j = 0, 1, \ldots, k-1$.

From Lemma 2 we get

$$\sum_{i=j}^{k-1} \frac{B_{i-j}\binom{i}{j}a^{i-j}\alpha_i}{a(j+1)} - \sum_{i=j}^{k-1} \frac{B_{i-j}\binom{i}{j}\binom{k+1}{i}a^{k-j}\frac{\alpha_k}{k+1}}{a(j+1)} = \sum_{i=j}^{k} \frac{B_{i-j}\binom{i}{j}a^{i-j}\alpha_i}{a(j+1)}$$

for $j = 0, 1, \ldots, k-1$. Because $g(0) = f(0)$, we hence obtain

$$g(x) = \sum_{j=0}^{k-1}\sum_{i=j}^{k} \frac{B_{i-j}\binom{i}{j}a^{i-j}\alpha_i}{a(j+1)} x^{j+1} + f(0)$$

under the conditions

$$(-1)^j \left(\frac{\sum\limits_{i=j}^{k} B_{i-j}\binom{i}{j}\alpha_i a^{i-j}}{a} - \frac{\sum\limits_{i=j}^{k} B_{i-j}\binom{i}{j}\beta_i b^{i-j}}{b} \right) \geq 0$$

for $j = 0, 1, \ldots, k-1$. Taking into account the definition of g we obtain

$$\begin{aligned} f(x) &= \sum_{j=0}^{k-1}\sum_{i=j}^{k} \frac{B_{i-j}\binom{i}{j}a^{i-j}\alpha_i}{a(j+1)} x^{j+1} + \frac{\alpha_k}{a(k+1)} x^{k+1} + f(0) \\ &= \sum_{j=0}^{k}\sum_{i=j}^{k} \frac{B_{i-j}\binom{i}{j}a^{i-j}\alpha_i}{a(j+1)} x^{j+1} + f(0) \end{aligned}$$

under the conditions

$$(-1)^j \left(\frac{\sum\limits_{i=j}^{k} B_{i-j}\binom{i}{j}\alpha_i a^{i-j}}{a} - \frac{\sum\limits_{i=j}^{k} B_{i-j}\binom{i}{j}\beta_i b^{i-j}}{b} \right) \geq 0$$

for $j = 0, 1, \ldots, k-1$, which proves the desired inductive implication.

Remark 3. Let $k \in \mathbb{N}$ be fixed and $f : \mathbb{R} \longrightarrow \mathbb{R}$ be a polynomial of degree $k+1$. Given $a \in \mathbb{R}$, $a \neq 0$, one can determine uniquely the system of numbers α_i, $i = 0, 1, \ldots, k+1$, such that f is given by (9). Repeating this procedure for $b \in \mathbb{R}$, such that $a < 0 < b$, $\frac{b}{a} \notin \mathbb{Q}$, one can determine the numbers $\beta_i, i = 0, 1, \ldots, k+1$, such that the relevant formula (9) holds true. Since the suitable coefficients are equal, it is obvious that relations (8) (being the equalities) for $j = 0, 1, \ldots, k+1$ are satisfied. Thus every polynomial can be characterized by Theorem 2.

Remark 4. The assumption (8) is essential for the uniqueness of the solution of system (1). To see this fact take $k = 0$, and observe that if $\frac{\alpha_0}{a} < \frac{\beta_0}{b}$, then the function $f := \sin$ satisfies (1) for all $a, b \in \mathbb{R}$ and $\alpha_0, \beta_0 \geq 2$.

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