International Mathematical Forum, 2, 2007, no. 47, 2321 - 2331

When M-convexity of a Function Implies its N-convexity for Some Means M and N

Janusz Matkowski

Institute of Mathematics, University of Zielona Góra
PL-65-246 Zielona Góra, Poland
and
Institute of Mathematics, Silesian University
PL-40-007 Katowice, Poland
J.Matkowski@im.uz.zgora.pl

Devoted to Professor Boris Paneah on the occasion of his 70th birthday

Abstract. Let M be an arbitrary strict mean in an interval J and $M_p^{[\varphi]}$ be a weighted quasi-arithmetic mean of the weight $p \in (0;1)$ and a generator $\varphi : J \to \mathbb{R}$. We prove that, for all intervals $I \subset J$ and for all continuous functions $f : I \to J$, the condition f is $M_p^{[\varphi]}$-affine implies f is M-convex is satisfied iff M is a quasi-arithmetic mean. Some variants of this result are proved and an open problem is posed.

Mathematics Subject Classification: Primary 26B25, 26E60; Secondary 39B22

Keywords: mean, quasi-arithmetic mean, $M-$convex function (convex function with respect to a mean M), M-affine function

1. Introduction

Let $J \subset \mathbb{R}$ be an interval. A function $M : J^2 \to \mathbb{R}$ is called a *mean* in J if

$$\min(x, y) \le M(x, y) \le \max(x, y), \qquad x, y \in J.$$

If, for all $x, y \in J, x \ne y$, these inequalities are strict, M is called *strict*; and *symmetric,* if $M(x, y) = M(y, x)$. Every mean M in J is *reflexive*, i.e.

$$M(x, x) = x, \qquad x \in J;$$

consequently, $M(I^2) = I$ for every subinterval $I \subseteq J$ and M is a mean in I. Obviously, every reflexive function $M : J^2 \to \mathbb{R}$ which is increasing in each variable is a mean.

Let $I \subset J$ be an interval. A function $f : I \to J$ is said to be *M-convex*, *M-concave*, *M-affine* if, respectively, (cf. J. Aczél [1], G. Aumann [3], and J. Matkowski & J. Rätz [8]),

$$f\left(M(x,y)\right) \le M(f(x), f(y)), \qquad x, y \in I;$$

$$f\left(M(x,y)\right) \ge M(f(x), f(y)), \qquad x, y \in I;$$

$$f\left(M(x,y)\right) = M(f(x), f(y)), \qquad x, y \in I.$$

If $M = A$ where $A(x,y) := \frac{x+y}{2}$, then A-convexity coincides with the *Jensen*-convexity.

Let us mention that the convexity of a function with respect to a non arithmetic mean appears in some characterization of L^p-norm [7] and the Gamma function [5].

Let N and M be some means in an interval J. In the present paper we show that:

if N is a weighted quasi-arithmetic mean and for any subinterval $I \subset J$ and any continuous function $f : I \to J$,

f is N-affine implies that f is M-convex,

then M must be quasi-arithmetic too. If moreover both means are symmetric, then $M = N$.

We also show that assuming either some special condition on the generator of quasi-arithmetic mean N or the continuity of M, one can substantially weaken the assumptions in this result.

A simple example shows that the quasi-arithmeticity of the means is meaningful. An open problem is posed.

2. Auxiliary results

Recall that for every continuous and strictly monotonic function $\varphi : J \to \mathbb{R}$ and $p \in [0,1]$, the function $M_p^{[\varphi]} : J^2 \to J$,

$$M_p^{[\varphi]}(x,y) := \varphi^{-1}\left(p\varphi(x) + (1-p)\varphi(y)\right), \qquad x, y \in J,$$

is a mean. $M_p^{[\varphi]}$ is called a *weighted quasi-arithmetic mean;* the function φ is referred to as its *generator,* and the numbers p and $1-p$ its *weights.* For $p = \frac{1}{2}$ this mean is denoted by $M^{[\varphi]}$ and is called *quasi-arithmetic.*

Remark 1. *The mean $M_p^{[\varphi]}$ is strict iff $p \in (0,1)$. Moreover we have $M_0^{[\varphi]}(x,y) = y$ and $M_1^{[\varphi]}(x,y) = x$ for all $x, y \in I$, i.e. $M_0^{[\varphi]}$ and $M_1^{[\varphi]}$ are the projective means.*

Remark 2. *Suppose that $\varphi, \phi : J \to \mathbb{R}$ are continuous and strictly monotonic, and $p, q \in (0,1)$. Then $M_p^{[\varphi]} = M_q^{[\phi]}$ if, and only if, $q = p$ and there are $a, b \in \mathbb{R}$, $a \neq 0$, such that*

$$\varphi(x) = a\phi(x) + b, \qquad x \in J,$$

(cf. [2], Corollary 5, p. 246 where the case $p = q = \frac{1}{2}$*is considered).*

This remark allows to assume, without any loss of generality, that the generator φ of the mean $M_p^{[\varphi]}$ is increasing and, if it is convenient, that $0 \in \varphi(J)$.

Remark 3. *Let M be a mean in an interval J. Considering M-convex (M-affine) functions we can assume, without any loss of generality, that $0 \in \operatorname{int} J$ (or $0 \in J$).*

To show this take an arbitrary $x_0 \in J$, put $J - x_0 := \{x - x_0 : x \in J\}$, and define $N : (J - x_0)^2 \to \mathbb{R}$ by

$$N(u, v) := M(u + x_0, v + x_0) - x_0, \qquad u, v \in J - x_0.$$

It is easy to verify that N is a mean in $J - x_0$, and if $M = M_p^{[\varphi]}$ then $N = M_p^{[\phi]}$ where $\phi : (J - x_0) \to \mathbb{R}$ is given by $\phi(u) := \varphi(u + x_0)$. Moreover, if $f : I \to J$ is M-convex, then $g : (I - x_0) \to J - x_0$ defined by

$$g(u) := f(u + x_0) - x_0, \qquad u \in J - x_0,$$

is N-convex. Indeed, for all $u, v \in J - x_0$ we have

$$\begin{aligned} g(N(u, v)) &= f\left(N(u, v) + x_0\right) - x_0 = f(M(u + x_0, v + x_0)) - x_0 \\ &\le M(f(u + x_0), f(v + x_0)) - x_0 \\ &= M([f(u + x_0) - x_0] + x_0, [f(v + x_0)) - x_0] + x_0) - x_0 \\ &= M(g(u) + x_0, g(v) + x_0) - x_0 \\ &= N(g(u), g(v)). \end{aligned}$$

If $f : I \to J$ is M-affine, then a similar reasoning shows that g is N-affine.

Let $J \subset \mathbb{R}$ be an interval and $p \in (0, 1)$. In the sequel we say that a function $g : J \to \mathbb{R}$ is *p-convex* (resp., *p-concave*, *p-affine*) if g is convex (resp. concave, affine) with respect to the weighted arithmetic mean $A_p(x, y) := px + (1-p)y$. In particular, $\frac{1}{2}$-convexity ($\frac{1}{2}$-concavity, $\frac{1}{2}$-affinity) coincides with Jensen convexity (Jensen cocavity, Jensen affinity, respectively).

Remark 4. *It follows from the Daróczy-Páles identity*

$$\frac{x + y}{2} = p\left((1 - p)x + p\frac{x + y}{2}\right) + (1 - p)\left((1 - p)\frac{x + y}{2} + py\right)$$

that every p-convex (p-concave, p-affine) function is Jensen convex (resp., Jensen concave, Jensen affine) .

Lemma 1. *Let $J \subset \mathbb{R}$ be an interval, $\varphi : J \to \mathbb{R}$ continuous and strictly increasing and $p \in (0, 1)$. Then*

1. *$f : I \to J$ is $M_p^{[\varphi]}$-convex ($M_p^{[\varphi]}$-concave, $M_p^{[\varphi]}$-affine) iff the function $\varphi \circ f \circ \varphi^{-1}$ is p-convex (p-concave, p-affine) in $\varphi(I)$;*
2. *if $f : I \to J$ is $M_p^{[\varphi]}$-affine and continuous at least at one point, then there are $a, b \in \mathbb{R}$ such that $\varphi \circ f \circ \varphi^{-1}(u) = au + b$ for all $u \in \varphi(I)$.*

Proof. Suppose that f is $M_p^{[\varphi]}$-convex in convex in I. Then, for all $x, y \in I$.

$$f\left(\varphi^{-1}\left(p\varphi(x)+(1-p)\varphi(y)\right)\right) \le \varphi^{-1}\left(\varphi^{-1}\left(p\varphi(f(x))+(1-p)\varphi(f(y))\right)\right).$$

For arbitrary $u, v \in \varphi(I)$, taking here $x := \varphi^{-1}(u)$, $y := \varphi^{-1}(v)$ and making use of the increasing monotonicity of φ, we hence get

$$\varphi \circ f \circ \varphi^{-1}\left(pu+(1-p)v\right) \le p\varphi \circ f \circ \varphi^{-1}(u) + (1-p)\varphi \circ f \circ \varphi^{-1}(u),$$

which proves that $\varphi \circ f \circ \varphi^{-1}$ is p-convex. In the same way we can show the remaining assertions of part 1.

Now the second part of the lemma is a consequence of the Daróczy-Páles identity lemma and the classical theory of Jensen convex (or affine) functions (cf. M. Kuczma p. [6]). □

3. Some results

Clearly, every M-affine function is M-convex. We begin with the following

Theorem 1. *Let $\varphi : J \to \mathbb{R}$ be continuous and strictly monotonic in an open interval $J \subset \mathbb{R}$ such that $\varphi(J) = \mathbb{R}$, and $p \in (0,1)$ a fixed number. Suppose that $M : J^2 \to J$ is a mean. If, for all continuous functions $f : J \to J$,*

$$f \text{ is } M_p^{[\varphi]}\text{-affine} \implies f \text{ is } M\text{-convex},$$

then $M = M_q^{[\varphi]}$ for some $q \in [0,1]$. If, moreover, M is a strict mean, then $q \in (0,1)$. Furthermore, if $p = \frac{1}{2}$ and M is strict and symmetric, then $M = M^{[\varphi]}$.

Proof. By Lemma 1, taking into account that $\varphi(J) = \mathbb{R}$, we infer that, for all $a, b \in \mathbb{R}$, the function $f : J \to J$ given by

$$f(x) = \varphi^{-1}(a\varphi(x)+b), \qquad x \in J,$$

is $M_p^{[\varphi]}$-affine. From the assumed implication we infer that this function f is M-convex, that is

$$\varphi^{-1}(a\varphi(M(x,y))+b) \le M\left(\varphi^{-1}(a\varphi(x)+b), \varphi^{-1}(a\varphi(y)+b)\right), \qquad x, y \in J,$$

for all $a, b \in \mathbb{R}$. By Remark 2, without any loss of generality, we can assume that φ is strictly increasing. Therefore, repleacing x by $\varphi^{-1}(u)$ and y by $\varphi^{-1}(v)$, we hence get

$$a\varphi(M(\varphi^{-1}(u),\varphi^{-1}(v)))+b \le \varphi\left(M\left(\varphi^{-1}(au+b),\varphi^{-1}(av+b)\right)\right), \quad u, v \in \mathbb{R},$$

for all $a, b \in \mathbb{R}$, that is

$$aM^*(u,v)+b \le M^*(au+b, av+b), \quad a, b, u, v \in \mathbb{R}, \tag{1}$$

where $M^* : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is defined by

$$M^*(u,v) := \varphi(M(\varphi^{-1}(u),\varphi^{-1}(v))).$$

For $b = 0$ we hence get

$$aM^*(u,v) \le M^*(au, av), \qquad a, u, v \in \mathbb{R},$$

which, of course, implies that

(2) $$aM^*(u,v) = M^*(au, av), \qquad a,u,v \in \mathbb{R},$$

that is M^* is homogeneous.

Taking $a = 1$ in (1) we get

$$M^*(u,v) + b \le M^*(u+b, v+b), \qquad b,u,v \in \mathbb{R}.$$

Repleacing here u by $u-b$ and v by $v-b$ we obtain

$$M^*(u-b, v-b) \le M^*(u,v) - b, \qquad b,u,v \in \mathbb{R},$$

whence

$$M^*(u+b, v+b) \le M^*(u,v) + b, \qquad b,u,v \in \mathbb{R}.$$

Consequently,

(3) $$M^*(u+b, v+b) = M^*(u,v) + b, \qquad b,u,v \in \mathbb{R}.$$

Applying in turn (3) and (2) we obtain, for all $u.v \in \mathbb{R}$,

$$\begin{aligned} M^*(u,v) &= M^*((u-v)+v, 0+v) = M^*(u-v, 0) + v \\ &= (u-v)M^*(1,0) + v = qu + (1-q)v, \end{aligned}$$

where

$$q := M^*(1,0).$$

Hence, by the definition of M^*,

$$M(x,y) = \varphi^{-1}(q\varphi(x) + (1-q)\varphi(y)), \qquad x,y \in J.$$

Since M^* is a mean, we have $q \in [0,1]$. This completes the proof. □

Denote by $\mathbb{Q}$ the set of rational numbers.

The continuity of the mean M in Theorem 1 (as well as in Corollary 1) allows to weaken the basic assumption significally.

Theorem 2. *Let $\varphi : J \to \mathbb{R}$ be continuous and strictly monotonic in an open interval $J \subset \mathbb{R}$, $\varphi(J) = \mathbb{R}$, and $p \in (0,1)$ a fixed number. Suppose that $M : J^2 \to J$ is a continuous mean.*

If there are $a,b,c,d \in \mathbb{R}\backslash\{0\}$, $0 < a < 1 < b$, $\frac{\log b}{\log a} \notin \mathbb{Q}$ such that the functions

$$\varphi^{-1} \circ (a\varphi), \quad \varphi^{-1} \circ (b\varphi), \quad \varphi^{-1} \circ (c\varphi + d) \text{ are } M\text{-convex}$$

and the function

$$\varphi^{-1} \circ (-\varphi) \text{ is } M\text{-affine,}$$

then $M = M_q^{[\varphi]}$ for some $q \in [0;1]$. If moreover M is strict, then $q \in (0;1)$.

Proof. We can assume that φ is strictly increasing. From the M-convexity of the function $\varphi^{-1}\circ(a\varphi)$ we have

$$\varphi^{-1}(a\varphi(M(x,y)))\leq M(\varphi^{-1}(a\varphi(x)),\varphi^{-1}(a\varphi(y))),\qquad x,y\in J.$$

Repleacing x by $\varphi^{-1}(u)$ and y by $\varphi^{-1}(v)$, we hence get

$$a\varphi(M(\varphi^{-1}(u),\varphi^{-1}(v)))\leq\varphi(M\left(\varphi^{-1}(au),\varphi^{-1}(av)\right)),\quad u,v\in\mathbb{R},$$

whence, by induction,

$$a^n\varphi(M(\varphi^{-1}(u),\varphi^{-1}(v)))\leq\varphi(M\left(\varphi^{-1}(a^nu),\varphi^{-1}(a^nv)\right)),\quad u,v\in\mathbb{R},\ n\in\mathbb{N}.$$

¿From the M-convexity of the function $\varphi^{-1}\circ(b\varphi)$, in the same way, we get

$$b^m\varphi(M(\varphi^{-1}(u),\varphi^{-1}(v)))\leq\varphi(M\left(\varphi^{-1}(b^mu),\varphi^{-1}(b^mv)\right)),\quad u,v\in\mathbb{R},\ n\in\mathbb{N}.$$

Making use both these inequalities we obtain

$$a^nb^m\varphi(M(\varphi^{-1}(u),\varphi^{-1}(v)))\leq\varphi(M\left(\varphi^{-1}(a^nb^mu),\varphi^{-1}(a^nb^mv)\right))$$

for all $u,v\in\mathbb{R},\ n\in\mathbb{N}$. Since, by assumption, $0<a<1<b$, $\frac{\log g}{\log a}\notin\mathbb{Q}$, the set

$$\{a^nb^m:n,m\in\mathbb{N}\}$$

is dense in $(0,\infty)$. The last inequality and the continuity of M imply that

$$t\varphi(M(\varphi^{-1}(u),\varphi^{-1}(v)))\leq\varphi(M\left(\varphi^{-1}(tu),\varphi^{-1}(tv)\right)),\qquad u,v\in\mathbb{R},\ t>0.$$

Replacing u and v by u/t and v/t, respectively, we obtain the reversed inequality, whence

$$t\varphi(M(\varphi^{-1}(u),\varphi^{-1}(v)))=\varphi(M\left(\varphi^{-1}(tu),\varphi^{-1}(tv)\right)),\qquad u,v\in\mathbb{R},\ t>0.$$

This proves that the mean $M^*:\mathbb{R}^2\to\mathbb{R}$ defined by

$$M^*(u,v):=\varphi(M(\varphi^{-1}(u),\varphi^{-1}(v))),\qquad u,v\in\mathbb{R},\tag{4}$$

is positively homogeneous. Since, by assumption, $\varphi^{-1}\circ(-\varphi)$ is M-affine, we have

$$\varphi^{-1}(-\varphi(M(x,y)))=M(\varphi^{-1}(-\varphi(x)),\varphi^{-1}(-\varphi(y))),\qquad x,y\in J,$$

which can be written in the form

$$-\varphi(M(\varphi^{-1}(u),\varphi^{-1}(v)))=\varphi(M\left(\varphi^{-1}(-u),\varphi^{-1}(-v)\right)),\qquad u,v\in\mathbb{R},$$

whence, by the definition of M^*,

$$M^*(-u,-v):=-M^*(u,v),\qquad u,v\in\mathbb{R}.\tag{5}$$

This relation and the positively homogeneity of M^* imply that M^* is homogeneous, i.e.

$$M^*(tu,tv):=tM^*(u,v),\qquad t,u,v\in\mathbb{R}.\tag{6}$$

By assumption, there are real nonzero c,d such that the function $\varphi^{-1}\circ(c\varphi+d)$ is M-convex. Consequently,

$$\varphi^{-1}(c\varphi(M(x,y))+d)\leq M\left(\varphi^{-1}(c\varphi(x)+d),\varphi^{-1}(c\varphi(y)+d)\right),\qquad x,y\in J,$$

which can be written in the form

$$c\varphi(M(\varphi^{-1}(u),\varphi^{-1}(v)))+d\leq\varphi(M\left(\varphi^{-1}(cu+d),\varphi^{-1}(cv+d)\right)),\quad u,v\in\mathbb{R},$$

whence

$$cM^*(u,v)+d\leq M^*(cu+d,cv+d),\qquad u,v\in\mathbb{R}.$$

Hence, applying (6), the homogeneity of M^*, we get

$$M^*(cu,cv)+d\leq M^*(cu+d,cv+d),\qquad u,v\in\mathbb{R},$$

and, consequently,

$$M^*(u,v)+d\leq M^*(u+d,v+d),\qquad u,v\in\mathbb{R}. \tag{7}$$

Replacing here u by $u-d$ and v by $v-d$ we get

$$M^*(u-d,v-d)\leq M^*(u,v)-d,\qquad u,v\in\mathbb{R},$$

whence, by replacing u by $-u$ and v by $-v$,

$$M^*(-u-d,-v-d)\leq M^*(-u,-v)-d,\qquad u,v\in\mathbb{R}.$$

Making use of (5), we obtain

$$-M^*(u+d,v+d)\leq -M^*(u,v)-d,\qquad u,v\in\mathbb{R},$$

that is

$$M^*(u+d,v+d)\geq M^*(u,v)+d,\qquad u,v\in\mathbb{R}, \tag{8}$$

The inequalities (7) and (8) imply that

$$M^*(u+d,v+d)=M^*(u,v)+d,\qquad u,v\in\mathbb{R}.$$

Hence, by the homogeneity of M^*,

$$M^*(tu+td,tv+td)=M^*(tu,tv)+td,\qquad t,u,v\in\mathbb{R},$$

whence

$$M^*(u+td,v+td)=M^*(u,v)+td,\qquad t,u,v\in\mathbb{R}.$$

Since $t\in\mathbb{R}$ is arbitrary, we conclude that

$$M^*(u+w,v+w)=M^*(u,v)+w,\qquad u,v,w\in\mathbb{R}. \tag{9}$$

Now, similarly as in the proof of Theorem 1, applying (6) and (9) we obtain

$$\begin{aligned}M^*(u,v)&=M^*((u-v)+v,0+v)=M^*(u-v,0)+v\\&=(u-v)M^*(1,0)+v)=qu+(1-q)v\end{aligned}$$

for all $u,v\in\mathbb{R}$, and the result is a consequence of (4). □

In the case $\varphi(J)\subsetneqq\mathbb{R}$ the following result holds true:

Theorem 3. *Let $\varphi : J \to \mathbb{R}$ be continuous and strictly monotonic in an open interval $J \subset \mathbb{R}$ and let $p \in (0,1)$ be a fixed number. Suppose that $M : J^2 \to J$ is a mean. If for all compact intervals $I \subset J$ and for all continuous functions $f : I \to J$,*

$$f \text{ is } M_p^{[\varphi]}\text{-affine} \implies f \text{ is } M\text{-convex},$$

then $M = M_q^{[\varphi]}$ for some $q \in [0;1]$. If moreover M is a strict mean, then $q \in (0;1)$. Furthermore, if M is symmetric, then $M = M^{[\varphi]}$.

Proof. By Remark 2 we may assume that $0 \in \operatorname{int} \varphi(J)$. By Remark 3 we can also assume that $0 \in \operatorname{int} J$. Take a compact subinterval $I \subset J$ such that $0 \in \operatorname{int}(I)$ and a continuous function $f : I \to J$. By Lemma 1, if f is $M_p^{[\varphi]}$-affine, then there are $a, b \in \mathbb{R}$ such that

$$f(x) = \varphi^{-1}(a\varphi(x) + b), \qquad x \in I. \tag{4}$$

Conversely, for all $a, b \in \mathbb{R}$ such that

$$(a\varphi(I) + b) \subset \varphi(J),$$

the function f given by this formula is $M_p^{[\varphi]}$-affine. Since $I \subset J$ is compact, there are $\alpha = \alpha(I) > 1$ and $\beta = \beta(I) > 0$ such that this inclusion holds true for all $a, b \in \mathbb{R}$ such that $0 \le a < \alpha$ and $|b| < \beta$. Suppose that condition 1 is satisfied. Then, for all $a, b \in \mathbb{R}$, such that $0 \le a < \alpha$ and $|b| < \beta$, we have

$$\varphi^{-1}(a\varphi(M(x,y)) + b) \le M\left(\varphi^{-1}(a\varphi(x) + b), \varphi^{-1}(a\varphi(y) + b)\right), \qquad x, y \in I,$$

or equivalently, for all $a, b \in \mathbb{R}$, such that $0 \le a < \alpha$ and $|b| < \beta$,

$$a\varphi(M(\varphi^{-1}(u), \varphi^{-1}(v))) + b \le \varphi\left(M\left(\varphi^{-1}(au + b), \varphi^{-1}(av + b)\right)\right), \quad u, v \in \varphi(I).$$

Thus, for all $a, b \in \mathbb{R}$, such that $0 \le a < \alpha$ and $|b| < \beta$,

$$aM_1(u,v) + b \le M_1(au + b, av + b), \quad u, v \in \varphi(I), \tag{5}$$

where $M_1 : \varphi(J) \times \varphi(J) \to \varphi(J)$ is defined by

$$M_1(u,v) := \varphi(M(\varphi^{-1}(u), \varphi^{-1}(v))), \qquad u, v \in \varphi(J).$$

Taking $b = 0$ we hence get

$$aM_1(u,v) \le M_1(au, av), \qquad u, v \in \varphi(I),$$

for all $a \in \mathbb{R}$ such that $0 \le a < \alpha$, which implies that

$$aM_1(u,v) = M_1(au, av), \qquad u, v \in \varphi(I), \tag{6}$$

for all $a \in \mathbb{R}$ such that $0 \le a < \alpha$.

For an arbitrary $(u,v) \in \mathbb{R} \times \mathbb{R}$ take $t > 0$ such that $\frac{u}{t}, \frac{v}{t} \in \varphi(I)$ and put

$$M^*(u,v) := tM_1(\frac{u}{t}, \frac{v}{t}).$$

To show that $M^*(u,v)$ does not depend on the choice of t, take an $s>0$ such that $\frac{u}{s}, \frac{v}{s} \in \varphi(I)$. Without any loss of generality we can assume that $s>t$. Since $0<\frac{s}{t}<1$, from (6) we have

$$tM_1(\frac{u}{t},\frac{v}{t}) = s\frac{t}{s}M_1(\frac{u}{t},\frac{v}{t}) = sM_1(\frac{t}{s}\frac{u}{t},\frac{t}{s}\frac{v}{t}) = sM_1(\frac{u}{s},\frac{v}{s}).$$

This proves that the function $M^* : \mathbb{R}\times\mathbb{R}\to\mathbb{R}$ is correctly defined. Clearly, M^* is positively homogeneous, that is

$$M^*(tu,tv) = tM^*(u,v), \qquad u,v\in\mathbb{R},\ t\geq 0.$$

Taking $a=1$ in (1) we get, for all $b\in\mathbb{R}$, $|b|<\beta$,

$$M_1(u,v)+b \leq M_1(u+b,v+b), \qquad u,v\in\varphi(I).$$

Repleacing here u by $u-b$ and v by $v-b$ we obtain

$$M_1(u-b,v-b) \leq M_1(u,v)-b, \qquad u,v\in\varphi(I),$$

for all $b\in\mathbb{R}$, $|b|<\beta$. Therefore

$$M_1(u+b,v+b) = M_1(u,v)+b, \qquad |b|<\beta;\ u,v\in\varphi(I).$$

Take arbitrary $u,v\in\mathbb{R}$, $b\in\mathbb{R}$ and choose a $t>0$ such that $\frac{u}{u}, \frac{v}{t}\in\varphi(I)$ and $\left|\frac{b}{t}\right|<\beta$. Then, by the definition of M^* and its homogeneity, we have

$$\begin{aligned} M^*(u+b,v+b) &= tM_1\left(\frac{u}{t}+\frac{b}{t},\frac{v}{t}+\frac{b}{t}\right) = t\left[M_1\left(\frac{u}{t},\frac{v}{t}\right)+\frac{b}{t}\right] \\ &= tM_1\left(\frac{u}{t},\frac{v}{t}\right)+b = M^*(u,v)+b. \end{aligned}$$

Now the same resoning as in the proof of Theorem 1 shows that there is a $q\in[0,1]$ such that

$$M^*(u,v) = qu+(1-q)v, \qquad u,v\in\mathbb{R}.$$

Since the function M^* is uniquelly determined and the definition does not depend on the the choice of the compact interval I, we infer that

$$\varphi(M(\varphi^{-1}(u),\varphi^{-1}(v))) = M^*(u,v), \qquad u,v\in\varphi(J).$$

Consequently,

$$M(x,y) = \varphi^{-1}(q\varphi(u)+(1-q)\varphi(y)), \qquad x,y\in J.$$

Since remaining statements are obvious, the proof is complete. □

4. A remark and open problem

Remark 5. *In the basic supposition of the above results that "N-affinity of some f implies its M-convexity", we assume that N is a quasi-arithmetic mean. This assumption is essential because there are different non-quasi-arithemetic means with the same classes of affine functions. For instance, the logarithmic mean* $L : (0,\infty)^2 \to (0,\infty)$,

$$L(x,y) = \begin{cases} \frac{x-y}{\log x - \log y}, & x \neq y \\ x, & x = y \end{cases}$$

and the mean $M : (0,\infty)^2 \to (0,\infty)$,

$$M(x,y) := \frac{2}{3}\frac{x^2 + xy + y^2}{x+y},$$

are symmetric and have the same classes continuous L-affine functions and M-affine functions coincide (cf. [9], [10]).

We end up this paper with the following open

Problem 1. *Let M and N be strict, symmetric and continuous means in an interval J. Suppose that for all intervals* $I \subset J$ *and for all continuous functions* $f : I \to J$, *the implication*

$$f \text{ is } N\text{-convex} \implies f \text{ is } M\text{-convex}$$

holds true. Is then $M = N$?

Acknowledgement 1. *The research was supported by the Silesian University Mathematical Department (Iterative Functional Equations and Real Analysis program).*

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Received: November 29, 2006