



An invariance of geometric mean with respect to Lagrangian means

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Received 27 April 2006

Available online 23 October 2006

Submitted by S. Grace

Abstract

The invariance of the geometric mean G with respect to the Lagrangian mean-type mapping (L^f, L^g) , i.e. the equation $G \circ (L^f, L^g) = G$, is considered. We show that the functions f and g must be of high class regularity. This fact allows to reduce the problem to a differential equation and determine the second derivatives of the generators f and g .

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Keywords: Mean; Lagrangian mean; Invariant mean; Iterate; Functional equation

1. Introduction

By L^f denote the Lagrangian mean generated by a function f defined on an interval $I \subset \mathbb{R}$ and by G , the geometric mean. The problem of invariance of the geometric mean G with respect to the mean-type mapping (L^f, L^g) reduces to the functional equation

$$G \circ (L^f, L^g) = G. \quad (1)$$

In this paper we consider this functional equation under the natural assumption that f and g are continuously differentiable functions.

One of the consequences of the invariance is the convergence of the sequence of iterates of the mapping (L^f, L^g) satisfying this equation to the mean-type mapping (G, G) (cf. [2,4]).

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In Section 3 we prove that the unknown functions f and g satisfying Eq. (1) must be of high class regularity. Applying this, in Section 4, we reduce the problem to a linear differential equation of the second order. Solving the differential equation we prove that either

$$f(x) = a_1 x^{-1} + b_1 x + c_1, \quad g(x) = a_2 x^{-1} + b_2 x + c_2 \quad (x \in I),$$

for some $a_1, a_2, b_1, b_2, c_1, c_2 \in \mathbb{R}$, $a_1 a_2 \neq 0$, or

$$f''(x) = x^{-3} \exp(ax^{-\frac{4}{3}} + b), \quad g''(x) = cx^{-3} \exp(-ax^{-\frac{4}{3}} - b) \quad (x \in I),$$

for some $a, b, c \in \mathbb{R}$, $ac \neq 0$.

Let us mention that the problem of invariance of the arithmetic mean in class of Lagrangian mean-type mappings has been solved in [5]. All pairs (M, N) of Stolarsky's means such that G is (M, N) -invariant have been determined in [1].

2. Some definitions and motivation

Let $I \subset \mathbb{R}$ be an interval. A function $M: I^2 \rightarrow \mathbb{R}$ is said to be a *mean* on I if

$$\min(x, y) \leq M(x, y) \leq \max(x, y), \quad x, y \in I.$$

If moreover for all $x, y \in I$, $x \neq y$, these inequalities are sharp, the mean M is called *strict*, and M is called *symmetric*, if for all $x, y \in I$, $M(x, y) = M(y, x)$.

Note that if $M: I^2 \rightarrow \mathbb{R}$ is a mean, then M is *reflexive*, that is,

$$M(x, x) = x, \quad x \in I,$$

and, consequently, for every interval $J \subset I$ we have $M(J^2) = J$; in particular, $M(I^2) = I$.

Let $I \subset \mathbb{R}$ be an interval and let $f: I \rightarrow \mathbb{R}$ be a differentiable function such that f' is injective. Then the function $L^f: I^2 \rightarrow I$ given by

$$L^f(x, y) := \begin{cases} (f')^{-1}\left(\frac{f(x)-f(y)}{x-y}\right), & x \neq y, \\ x, & x = y, \end{cases}$$

is correctly defined and it is called a *Lagrangian mean* generated by f .

Remark 1. (Cf. [3].) Let $f: I \rightarrow \mathbb{R}$ and $g: I \rightarrow \mathbb{R}$ be differentiable functions, such that f' and g' are injective. Then

$$L^f = L^g$$

if, and only if, there exist $a, b, c \in \mathbb{R}$, $a \neq 0$, such that

$$g(x) = af(x) + bx + c, \quad x \in I.$$

Let $M: I^2 \rightarrow I$, $N: I^2 \rightarrow I$ be means. A mean $K: I^2 \rightarrow I$ is called *invariant with respect to the mean-type mapping* $(M, N): I^2 \rightarrow I^2$, shortly, (M, N) -invariant, if

$$K(M(x, y), N(x, y)) = K(x, y), \quad x, y \in I.$$

As a motivation for this paper let us quote the following

Proposition 1. (Cf. [4].) Let $I \subset \mathbb{R}$ be an interval. If $(M, N): I^2 \rightarrow I^2$ is a continuous mean-type mapping such that at most one of the coordinate means M and N is not strict, then:

- (1) there is a continuous mean $K: I^2 \rightarrow I$ such that the sequence of iterates $((M, N)^n)_{n=1}^\infty$ of the mapping (M, N) converges (pointwise) to a continuous mean-type mapping $(K, K): I^2 \rightarrow I^2$;
- (2) K is (M, N) -invariant;
- (3) a continuous (M, N) -invariant mean-type mapping is unique;
- (4) if M and N are strict means then so is K .

Remark 2. This proposition improves a well-known result in which it is assumed that both means M and N are strict (cf. for instance [2]). A unique continuous (M, N) -invariant mean K is also called the Gauss composition of M and N . Moreover the sequence of iterates of the mean-type mapping $(M, N): I^2 \rightarrow I^2$ is called the Gauss-iteration (cf. [2]).

3. A regularity theorem

Throughout this paper we assume that $I \subset (0, \infty)$ is an open interval.

Theorem 1. Let $f: I \rightarrow \mathbb{R}$ and $g: I \rightarrow \mathbb{R}$ be functions of the class C^1 , such that f' and g' are injective. If the geometric mean G is (L^f, L^g) -invariant, i.e.

$$(f')^{-1} \left(\frac{f(x) - f(y)}{x - y} \right) (g')^{-1} \left(\frac{g(x) - g(y)}{x - y} \right) = xy, \quad x, y \in I, x \neq y, \quad (2)$$

then f and g are of the class C^∞ in I except for a nowhere dense subset of I .

Proof. Assume first that for every $x_0 \in I$ there exists $y_0 \in I$, $x_0 \neq y_0$, such that

$$\frac{g(x_0) - g(y_0)}{x_0 - y_0} \neq \frac{x_0 g'(x_0) + y_0 g'(y_0)}{x_0 + y_0}.$$

Let us fix $x_0 \in I$, put

$$u_0 := \frac{f(x_0) - f(y_0)}{x_0 - y_0}, \quad \Delta := \{(x, x): x \in I\},$$

and define the function $\Phi: (I^2 \setminus \Delta) \times \mathbb{R} \rightarrow \mathbb{R}$ by

$$\Phi(x, y, u) := \frac{f(x) - f(y)}{x - y} - u.$$

Note that the function Φ is of the class C^1 , $\Phi(x_0, y_0, u_0) = 0$, and

$$\frac{\partial \Phi}{\partial y}(x_0, y_0, u_0) = \frac{f'(y_0)(y_0 - x_0) - f(y_0) + f(x_0)}{(x_0 - y_0)^2} \neq 0.$$

Indeed, if the last relation were not true, we would have

$$\frac{f(x_0) - f(y_0)}{x_0 - y_0} = f'(y_0),$$

and, by the Lagrange mean value theorem,

$$\frac{f(x_0) - f(y_0)}{x_0 - y_0} = f'(\xi),$$

for some $\xi \neq y_0$, whence $f'(y_0) = f'(\xi)$. This is a contradiction as f' is one-to-one. By the implicit function theorem, there exist a neighbourhood D of the point (x_0, u_0) ,

$$D = (x_0 - \delta, x_0 + \delta) \times (u_0 - \delta, u_0 + \delta)$$

for some $\delta > 0$, and a unique function $\varphi: D \rightarrow I$ of the class C^1 in D and such that

$$\varphi(x_0, u_0) = y_0, \quad \Phi(x, \varphi(x, u), u) = 0, \quad (x, u) \in D,$$

that is

$$\varphi(x_0, u_0) = y_0, \quad \frac{f(x) - f(\varphi(x, u))}{x - \varphi(x, u)} = u, \quad (x, u) \in D.$$

Setting $y = \varphi(x, u)$ in (2), we obtain

$$(f')^{-1}(u)(g')^{-1}\left(\frac{g(x) - g(\varphi(x, u))}{x - \varphi(x, u)}\right) = x\varphi(x, u), \quad (x, u) \in D. \quad (3)$$

Put

$$v_0 := \frac{g(x_0) - g(\varphi(x_0, u_0))}{x_0 - \varphi(x_0, u_0)} = \frac{g(x_0) - g(y_0)}{x_0 - y_0},$$

and define the function $\Psi: D \times \mathbb{R} \rightarrow \mathbb{R}$ by

$$\Psi(x, u, v) := \frac{g(x) - g(\varphi(x, u))}{x - \varphi(x, u)} - v, \quad (x, u) \in D, \quad v \in \mathbb{R}.$$

Note that the function Ψ is of the class C^1 , $\Psi(x_0, u_0, v_0) = 0$, and

$$\begin{aligned} \frac{\partial \Psi}{\partial x}(x_0, u_0, v_0) &= \frac{[g(\varphi(x_0, u_0)) - g(x_0)][1 - \frac{\partial \varphi}{\partial x}(x_0, u_0)]}{[x_0 - \varphi(x_0, u_0)]^2} \\ &\quad + \frac{g'(x_0) - g'(\varphi(x_0, u_0)) \frac{\partial \varphi}{\partial x}(x_0, u_0)}{x_0 - \varphi(x_0, u_0)}. \end{aligned}$$

Suppose first that

$$\frac{\partial \Psi}{\partial x}(x_0, u_0, v_0) = 0.$$

Then, we would get

$$\begin{aligned} &\left[g'(x_0) - g'(\varphi(x_0, u_0)) \frac{\partial \varphi}{\partial x}(x_0, u_0) \right] [x_0 - \varphi(x_0, u_0)] \\ &= [g(x_0) - g(\varphi(x_0, u_0))] \left[1 - \frac{\partial \varphi}{\partial x}(x_0, u_0) \right]. \end{aligned} \quad (4)$$

Differentiating with respect to x both sides of (3), after simple calculations, we get

$$0 = \varphi(x_0, u_0) + x_0 \frac{\partial \varphi}{\partial x}(x_0, u_0),$$

and, consequently,

$$\frac{\partial \varphi}{\partial x}(x_0, u_0) = -\frac{\varphi(x_0, u_0)}{x_0} = -\frac{y_0}{x_0}.$$

Hence, and by (4) we would have

$$\frac{g(x_0) - g(y_0)}{x_0 - y_0} = \frac{x_0 g'(x_0) + y_0 g'(y_0)}{x_0 + y_0},$$

which contradicts the assumption. Thus

$$\frac{\partial \Psi}{\partial x}(x_0, u_0, v_0) \neq 0.$$

By the implicit function theorem there exist a neighbourhood W of the point (u_0, v_0) ,

$$W = (u_0 - \rho, u_0 + \rho) \times (v_0 - \rho, v_0 + \rho),$$

for some $\rho > 0$, and a unique function $\psi: W \rightarrow I$ of the class C^1 in W such that

$$\psi(u_0, v_0) = x_0, \quad \Psi(\psi(u, v), u, v) = 0, \quad (u, v) \in W,$$

that is

$$\psi(u_0, v_0) = x_0, \quad \frac{g(\psi(u, v)) - g(\varphi(\psi(u, v), u))}{\psi(u, v) - \varphi(\psi(u, v), u)} = v, \quad (u, v) \in W.$$

Substituting $x = \psi(u, v)$ in (3), we obtain

$$(f')^{-1}(u)(g')^{-1}(v) = \psi(u, v)\varphi(\psi(u, v), u), \quad (u, v) \in W.$$

Since the right-hand side is a function of the class C^1 in W , we infer that $(f')^{-1}$ and $(g')^{-1}$ are of the class C^1 in the intervals $(u_0 - \rho, u_0 + \rho)$ and $(v_0 - \rho, v_0 + \rho)$, respectively. Since the sets

$$\{u: ((f')^{-1})'(u) = 0\}, \quad \{v: ((g')^{-1})'(v) = 0\}$$

are nowhere dense, it follows that the functions f' and g' are of the class C^1 in an open nonempty subinterval contained in $(x_0 - \delta, x_0 + \delta)$, and consequently the functions f and g are of the class C^2 in that subinterval.

To finish the proof assume that there exists $x_0 \in I$ such that for all $y \in I$, $y \neq x_0$,

$$\frac{g(x_0) - g(y)}{x_0 - y} = \frac{x_0 g'(x_0) + y g'(y)}{x_0 + y}.$$

Then

$$g'(y) = \frac{g(y) - g(x_0)}{y - x_0} \left(\frac{x_0}{y} + 1 \right) - \frac{x_0}{y} g'(x_0), \quad y \in I,$$

which implies that g is of the class C^2 in $I \setminus \{x_0\}$. From (2) we infer that so is f .

Now an obvious induction proves that functions f and g are of the class C^∞ in I except for some nowhere dense subset of I . \square

4. Some necessary conditions for (L^f, L^g) -invariance of the geometric mean

The problem to determine all continuously differentiable functions $f, g: I \rightarrow \mathbb{R}$ such that G is (L^f, L^g) -invariant reduces to the functional equation

$$L^f(x, y)L^g(x, y) = xy, \quad x, y \in I, \quad x \neq y. \quad (5)$$

We begin this section with the following

Theorem 2. Let $f: I \rightarrow \mathbb{R}$ and $g: I \rightarrow \mathbb{R}$ be functions of the class C^1 , such that f' and g' are injective. If the geometric mean G is (L^f, L^g) -invariant, then for every nonempty open interval $J \subset I$ there exist a nonempty open subinterval $I_0 \subset J$ and $c \neq 0$, $c = c(I_0)$, such that

$$f''(x)g''(x) = \frac{c}{x^6}, \quad x \in I_0.$$

Proof. Assume that G is (L^f, L^g) -invariant and take an arbitrary open interval $J \subset I$. Using Theorem 1, we first show that there exists a maximal, nonempty open subinterval $I_0 \subset J$ such that f and g are of the class C^3 in I_0 and

$$f''(x) \neq 0 \neq g''(x), \quad x \in I_0.$$

Let $J_0 \subset J$ be an arbitrary nonempty open interval such that f and g are of the class C^3 in J_0 and

$$f''(x) \neq 0 \neq g''(x), \quad x \in J_0.$$

Note that

$$\text{int}\{x \in J_0: f''(x) \neq 0\} \neq \emptyset \neq \text{int}\{x \in J_0: g''(x) \neq 0\}.$$

Indeed, if

$$f''(x) = 0, \quad x \in J_0,$$

then there exists $a \in \mathbb{R}$ such that

$$f'(x) = a, \quad x \in J_0,$$

which contradicts the injectivity of f' . Analogously we can show that

$$\text{int}\{x \in J_0: g''(x) \neq 0\} \neq \emptyset.$$

Suppose that

$$\text{int}\{x \in J_0: f''(x) \neq 0\} \cap \text{int}\{x \in J_0: g''(x) \neq 0\} = \emptyset.$$

Since $\text{int}\{x \in J_0: g''(x) \neq 0\} \neq \emptyset$ is an open subset of $I \subset (0, \infty)$, there exists a nonempty open interval $J_1 \subset \text{int}\{x \in J_0: g''(x) \neq 0\}$. Then

$$J_1 \cap \text{int}\{x \in J_0: f''(x) \neq 0\} = \emptyset,$$

whence

$$f''(x) = 0, \quad x \in J_1,$$

and, consequently,

$$f'(x) = a, \quad x \in J_1,$$

for some $a \in \mathbb{R}$, which contradicts the injectivity of f' . Thus we have shown that

$$\text{int}\{x \in J_0: f''(x) \neq 0\} \cap \text{int}\{x \in J_0: g''(x) \neq 0\} \neq \emptyset,$$

and, consequently, there exists a maximal (in the sense of inclusion) interval $I_0 \subset J$ such that

$$f''(x) \neq 0 \neq g''(x), \quad x \in I_0.$$

Since f and g are of the class C^3 in I_0 , the functions L^f and L^g are two-times continuously differentiable in $I_0 \times I_0$.

From (5) we have

$$(f')^{-1}\left(\frac{f(x)-f(y)}{x-y}\right)(g')^{-1}\left(\frac{g(x)-g(y)}{x-y}\right)=xy, \quad x, y \in I_0, \quad x \neq y. \quad (6)$$

Define the functions $B_f: I_0^2 \rightarrow I_0$ and $B_g: I_0^2 \rightarrow I_0$ by the formulas

$$B_f(x, y) := \begin{cases} \frac{f(x)-f(y)}{x-y} & \text{for } x \neq y, \\ f'(x) & \text{for } x = y, \end{cases} \quad (7)$$

and

$$B_g(x, y) := \begin{cases} \frac{g(x)-g(y)}{x-y} & \text{for } x \neq y, \\ g'(x) & \text{for } x = y. \end{cases} \quad (8)$$

The functions B_f, B_g are of the class C^2 and Eq. (6) can be written in the form

$$(f')^{-1}(B_f(x, y))(g')^{-1}(B_g(x, y)) = xy, \quad x, y \in I_0, \quad x \neq y.$$

Differentiating twice with respect to x both sides of the above equation, we get

$$\begin{aligned} & \left(\frac{\frac{\partial^2 B_f}{\partial x^2}(x, y)}{f''(L^f(x, y))} - \frac{\left[\frac{\partial B_f}{\partial x}(x, y) \right]^2 f'''(L^f(x, y))}{[f''(L^f(x, y))]^3} \right) L^g(x, y) \\ & + \left(\frac{\frac{\partial^2 B_g}{\partial x^2}(x, y)}{g''(L^g(x, y))} - \frac{\left[\frac{\partial B_g}{\partial x}(x, y) \right]^2 g'''(L^g(x, y))}{[g''(L^g(x, y))]^3} \right) L^f(x, y) \\ & + 2 \frac{\frac{\partial B_f}{\partial x}(x, y)}{f''(L^f(x, y))} \frac{\frac{\partial B_g}{\partial x}(x, y)}{g''(L^g(x, y))} = 0, \end{aligned} \quad (9)$$

where

$$\frac{\partial B_f}{\partial x}(x, y) = \frac{(x-y)f'(x) - f(x) + f(y)}{(x-y)^2}, \quad (10)$$

$$\frac{\partial^2 B_f}{\partial x^2}(x, y) = \frac{(x-y)^2 f''(x) - 2(x-y)f'(x) + 2(f(x) - f(y))}{(x-y)^3}. \quad (11)$$

Replacing f by g in (10) and (11) we obtain, respectively, the formulas for $\frac{\partial B_g}{\partial x}(x, y)$ and $\frac{\partial^2 B_g}{\partial x^2}(x, y)$, for all $x, y \in I_0, x \neq y$.

Since, for $i = 1, 2$ and all $x \in I_0$,

$$\lim_{y \rightarrow x} \frac{\partial^{(i)} B_f}{\partial x^{(i)}}(x, y) = \frac{f^{(i+1)}(x)}{i+1}, \quad \lim_{y \rightarrow x} \frac{\partial^{(i)} B_g}{\partial x^{(i)}}(x, y) = \frac{g^{(i+1)}(x)}{i+1},$$

letting $y \rightarrow x$ in (9), we obtain

$$\frac{f'''(x)}{f''(x)} + \frac{g'''(x)}{g''(x)} = -\frac{6}{x}, \quad x \in I_0,$$

whence, for some $c \in \mathbb{R} \setminus \{0\}$,

$$f''(x)g''(x) = \frac{c}{x^6}, \quad x \in I_0.$$

Thus we have proved that for every open nonempty interval $J \subset I$ there exist an open nonempty subinterval $I_0 \subset J$ and $c \neq 0$, $c = c(I_0)$, such that

$$f''(x)g''(x) = \frac{c}{x^6}, \quad x \in I_0. \quad \square$$

Applying this result we prove the following

Theorem 3. Let $f: I \rightarrow \mathbb{R}$ and $g: I \rightarrow \mathbb{R}$ be functions of the class C^1 , such that f' and g' are injective. If the geometric mean G is (L^f, L^g) -invariant, then for every nonempty open interval $J \subset I$ there exists a nonempty open subinterval $I_0 \subset J$ such that

(i) the function $w_f: I_0 \rightarrow \mathbb{R}$ defined by

$$w_f(x) := \frac{f'''(x)}{f''(x)}, \quad x \in I_0,$$

satisfies the equation

$$[3 + xw_f(x)][9x^2w'_f(x) + 13xw_f(x) + 12] = 0, \quad x \in I_0;$$

(ii) the function $w_g: I_0 \rightarrow \mathbb{R}$ defined by

$$w_g(x) := \frac{g'''(x)}{g''(x)}, \quad x \in I_0,$$

satisfies the equation

$$[3 + xw_g(x)][9x^2w'_g(x) + 13xw_g(x) + 12] = 0, \quad x \in I_0.$$

Proof. Assume that G is (L^f, L^g) -invariant and take an arbitrary nonempty open interval $J \subset I$. By Theorem 1 there exists a nonempty open and maximal (in the sense of inclusion) subinterval $I_0 \subset J$ such that f and g are of the class C^5 in I_0 and

$$f''(x) \neq 0 \neq g''(x), \quad x \in I_0.$$

It follows that the functions L^f and L^g are four-times continuously differentiable in $I_0 \times I_0$.

Differentiating four times with respect to x both sides of Eq. (5) and using the functions B_f and B_g defined in the proof of Theorem 2, we get for all $x, y \in I_0$, $x \neq y$,

$$\begin{aligned} & \frac{\partial^4 L^f}{\partial x^4}(x, y)L^g(x, y) + \frac{\partial^4 L^g}{\partial x^4}(x, y)L^f(x, y) + 6\frac{\partial^2 L^f}{\partial x^2}(x, y)\frac{\partial^2 L^g}{\partial x^2}(x, y) \\ & + 4\frac{\partial^3 L^f}{\partial x^3}(x, y)\frac{\partial L^g}{\partial x}(x, y) + 4\frac{\partial L^f}{\partial x}(x, y)\frac{\partial^3 L^g}{\partial x^3}(x, y) = 0, \end{aligned} \quad (12)$$

where

$$\begin{aligned} \frac{\partial L^f}{\partial x} &= \frac{\partial B_f}{\partial x} \frac{1}{f''(L^f)}, \\ \frac{\partial^2 L^f}{\partial x^2} &= \frac{\partial^2 B_f}{\partial x^2} \frac{1}{f''(L^f)} - \left[\frac{\partial B_f}{\partial x} \right]^2 \frac{f'''(L^f)}{[f''(L^f)]^3}, \end{aligned}$$

$$\begin{aligned}\frac{\partial^3 L^f}{\partial x^3} &= \frac{\partial^3 B_f}{\partial x^3} \frac{1}{f''(L^f)} - \frac{\partial B_f}{\partial x} \frac{\partial^2 B_f}{\partial x^2} \frac{3f'''(L^f)}{[f''(L^f)]^3} \\ &\quad + \left(\frac{\partial B_f}{\partial x} \right)^3 \left[3 \frac{[f'''(L^f)]^2}{[f''(L^f)]^5} - \frac{f^{(4)}(L^f)}{[f''(L^f)]^4} \right], \\ \frac{\partial^4 L^f}{\partial x^4} &= \frac{\partial^4 B_f}{\partial x^4} \frac{1}{f''(L^f)} - \left[3 \left(\frac{\partial^2 B_f}{\partial x^2} \right)^2 + 4 \frac{\partial B_f}{\partial x} \frac{\partial^3 B_f}{\partial x^3} \right] \frac{f'''(L^f)}{[f''(L^f)]^3} \\ &\quad + 6 \left(\frac{\partial B_f}{\partial x} \right)^2 \frac{\partial^2 B_f}{\partial x^2} \left[3 \frac{[f'''(L^f)]^2}{[f''(L^f)]^5} - \frac{f^{(4)}(L^f)}{[f''(L^f)]^4} \right] \\ &\quad + \left(\frac{\partial B_f}{\partial x} \right)^4 \left[10 \frac{f'''(L^f)f^{(4)}(L^f)}{[f''(L^f)]^6} - 15 \frac{[f'''(L^f)]^3}{[f''(L^f)]^7} - \frac{f^{(5)}(L^f)}{[f''(L^f)]^5} \right], \\ \frac{\partial^3 B_f}{\partial x^3}(x, y) &= \frac{f'''(x)}{(x-y)} - 3 \frac{f''(x)}{(x-y)^2} + 6 \frac{f'(x)}{(x-y)^3} - 6 \frac{f(x) - f(y)}{(x-y)^4}, \\ \frac{\partial^4 B_f}{\partial x^4}(x, y) &= \frac{f^{(4)}(x)}{(x-y)} - 4 \frac{f'''(x)}{(x-y)^2} + 12 \frac{f''(x)}{(x-y)^3} - 24 \frac{f'(x)}{(x-y)^4} + 24 \frac{f(x) - f(y)}{(x-y)^5},\end{aligned}$$

and $\frac{\partial B_f}{\partial x}, \frac{\partial^2 B_f}{\partial x^2}$ are given by formulas (10) and (11), respectively.

Substituting here f by g we obtain $\frac{\partial^{(i)} B_g}{\partial x^{(i)}}$ for all $i \in \{1, 2, 3, 4\}$.

Since

$$\lim_{y \rightarrow x} \frac{\partial^{(i)} B_f}{\partial x^{(i)}}(x, y) = \frac{f^{(i+1)}(x)}{i+1}, \quad \lim_{y \rightarrow x} \frac{\partial^{(i)} B_g}{\partial x^{(i)}}(x, y) = \frac{g^{(i+1)}(x)}{i+1},$$

for all $i \in \{1, 2, 3, 4\}$ and $x \in I_0$, letting $y \rightarrow x$ in (12), we get

$$\begin{aligned}x \left[\frac{11}{80} \left(\frac{f^{(5)}}{f''} + \frac{g^{(5)}}{g''} \right) + \frac{11}{48} \left(\left(\frac{f'''}{f''} \right)^3 + \left(\frac{g'''}{g''} \right)^3 \right) - \frac{3}{8} \left(\frac{f'''}{f''} \frac{f^{(4)}}{f''} + \frac{g'''}{g''} \frac{g^{(4)}}{g''} \right) \right] \\ + \frac{1}{4} \left[\frac{f^{(4)}}{f''} - \left(\frac{f'''}{f''} \right)^2 + \frac{g^{(4)}}{g''} - \left(\frac{g'''}{g''} \right)^2 \right] + \frac{1}{24} \frac{f'''}{f''} \frac{g'''}{g''} = 0,\end{aligned}\tag{13}$$

where

$$f^{(i)}, \quad g^{(i)},$$

stand, respectively, for

$$f^{(i)}(x), \quad g^{(i)}(x), \quad i \in \{2, 3, 4, 5\}, \quad x \in I_0.$$

Since $f, g \in C^5$, by Theorem 2 we obtain

$$g''(x) = \frac{c}{x^6 f''(x)}, \quad x \in I_0,$$

for some $c \neq 0$, whence

$$\begin{aligned}g'''(x) &= -c \frac{6f''(x) + xf'''(x)}{x^7 [f''(x)]^2}, \\ g^{(4)}(x) &= c \frac{42[f''(x)]^2 + 2x^2[f'''(x)]^2 + 12xf''(x)f'''(x) - x^2 f''(x)f^{(4)}(x)}{x^8 [f''(x)]^3},\end{aligned}$$

$$g^{(5)}(x) = -c \left\{ \frac{336[f''(x)]^3 + 6x^3[f'''(x)]^3 + 36x^2 f''(x)[f'''(x)]^2}{x^9[f''(x)]^4} \right. \\ \left. + \frac{x^3[f''(x)]^2 f^{(5)}(x) + 126x[f''(x)]^2 f'''(x)}{x^9[f''(x)]^4} \right. \\ \left. - \frac{6x^3 f''(x) f'''(x) f^{(4)}(x) + 18x^2[f''(x)]^2 f^{(4)}(x)}{x^9[f''(x)]^4} \right\},$$

for all $x \in I_0$. Using this in Eq. (13), after some calculations, we obtain

$$\left(3 + x \frac{f'''(x)}{f''(x)}\right) \left(12 - 9x^2 \left[\frac{f'''(x)}{f''(x)}\right]^2 + 9x^2 \frac{f^{(4)}(x)}{f''(x)} + 13x \frac{f'''(x)}{f''(x)}\right) = 0, \quad (14)$$

for all $x \in I_0$.

Note that the function w_f is of the class of C^1 in I_0 , and

$$w'_f(x) = \frac{f^{(4)}(x)}{f''(x)} - \left[\frac{f'''(x)}{f''(x)}\right]^2, \quad x \in I_0,$$

whence, by the definition of w_f ,

$$\frac{f^{(4)}(x)}{f''(x)} = w'_f(x) + (w_f(x))^2, \quad x \in I_0.$$

Using the definition of w_f and this relation we can write Eq. (14) in the form

$$(3 + xw_f(x))(9x^2w'_f(x) + 13xw_f(x) + 12) = 0, \quad x \in I_0,$$

and the proof of (i) is completed. Since the proof of (ii) is analogous we omit it. \square

Proposition 2. Let $I \subset (0, \infty)$ be an interval. A continuously differentiable function $w : I \rightarrow \mathbb{R}$ satisfies the equation

$$(3 + xw(x))(9x^2w'(x) + 13xw(x) + 12) = 0, \quad x \in I, \quad (15)$$

if, and only if, there exists $c \in \mathbb{R}$, such that

$$w(x) = -\frac{3}{x} + cx^{-\frac{13}{9}}, \quad x \in I. \quad (16)$$

Proof. Assume that the function w satisfies Eq. (15). Note that, if

$$3 + xw(x) = 0, \quad x \in I,$$

then (16) holds true with $c = 0$.

Suppose that there exists $x_0 \in I$ such that

$$3 + x_0w(x_0) \neq 0.$$

Then

$$Z := \{x \in I : 3 + xw(x) \neq 0\}$$

is an open set such that $\text{int } Z \neq \emptyset$. Let $I_0 \subset Z$ be a maximal (in the sense of inclusion) nonempty interval. From (15) we infer that

$$9x^2w'(x) + 13xw(x) + 12 = 0, \quad x \in I_0,$$

whence, after simple calculations, we get

$$w(x) = -\frac{3}{x} + cx^{-\frac{13}{9}}, \quad x \in I_0,$$

for some $c \in \mathbb{R} \setminus \{0\}$.

Suppose that $a := \sup I_0 \in \text{int } I$. Then there exists a sequence $x_n \in I \setminus I_0$, $x_n \notin Z$, $x_n \rightarrow a$ ($x_n > a$), such that,

$$3 + x_n w(x_n) = 0, \quad n \in \mathbb{N}.$$

Letting $n \rightarrow \infty$, by continuity of w , we hence obtain

$$w(a) = -\frac{3}{a}.$$

On the other hand,

$$w(a) = \lim_{x \rightarrow a^-} w(x) = -\frac{3}{a} + ca^{-\frac{13}{9}},$$

which cannot occur since $c \neq 0$. It proves that the right ends of the intervals I_0 and I are the same. In a similar way we can show that the left ends of I_0 and I are the same and, consequently, for some $c \neq 0$,

$$w(x) = -\frac{3}{x} + cx^{-\frac{13}{9}}, \quad x \in I.$$

It is easy to check that the function w given by (16) satisfies Eq. (15). \square

The main result of this section reads as follows.

Theorem 4. Let $f: I \rightarrow \mathbb{R}$ and $g: I \rightarrow \mathbb{R}$ be functions of the class C^1 , such that f' and g' are injective. If the geometric mean G is (L^f, L^g) -invariant, then either

$$f(x) = a_1 x^{-1} + b_1 x + c_1, \quad g(x) = a_2 x^{-1} + b_2 x + c_2 \quad (x \in I), \quad (17)$$

for some $a_1, a_2, b_1, b_2, c_1, c_2 \in \mathbb{R}$, $a_1 a_2 \neq 0$, or

$$f''(x) = x^{-3} \exp(ax^{-\frac{4}{3}} + b), \quad g''(x) = cx^{-3} \exp(-ax^{-\frac{4}{3}} - b) \quad (x \in I), \quad (18)$$

for some $a, b, c \in \mathbb{R}$, $ac \neq 0$.

Proof. Assume that G is (L^f, L^g) -invariant. By Theorem 3 and Proposition 2 we infer that there exist a nonempty open and maximal (in the sense of inclusion) subinterval $I_0 \subset I$ and $c \in \mathbb{R}$ such that

$$\frac{f'''(x)}{f''(x)} = -\frac{3}{x} + cx^{-\frac{13}{9}}, \quad x \in I_0. \quad (19)$$

First we assume that $c = 0$. Then

$$f''(x) = d_1 x^{-3}, \quad x \in I_0,$$

for some $d_1 \neq 0$. By Theorem 2 we get

$$g''(x) = d_2 x^{-3}, \quad x \in I_0,$$

for some $d_2 \neq 0$, whence, after simple calculations,

$$f(x) = a_1 x^{-1} + b_1 x + c_1, \quad g(x) = a_2 x^{-1} + b_2 x + c_2 \quad (x \in I_0),$$

for some $a_1, a_2, b_1, b_2, c_1, c_2 \in \mathbb{R}$, $a_1 a_2 \neq 0$.

Now consider the case $c \neq 0$. By (19), after some calculations, we get

$$f''(x) = x^{-3} \exp(ax^{-\frac{1}{2}} + b), \quad x \in I_0,$$

for some $a, b \in \mathbb{R}$, and by Theorem 2 we obtain

$$g''(x) = cx^{-3} \exp(-ax^{-\frac{1}{2}} - b), \quad x \in I_0,$$

for some $c \neq 0$.

Now we show that $I_0 = I$. For an indirect argument assume that $x_0 := \sup I_0 < \sup I$, and that f and g are, respectively, of the formulas (17) in I_0 . In view of Remark 1, and taking into account the continuity of f and g , we can assume that

$$f(x) = \frac{1}{x} = g(x), \quad x \in I_0 \cup \{x_0\}.$$

Taking $y_0 \in I_0$ we have

$$L^f(x_0, y_0) \in I_0 \quad \text{and} \quad L^g(x_0, y_0) \in I_0,$$

moreover, by the continuity of the Lagrangian means, there exists $\delta_0 > 0$, such that, for $x \in (x_0, x_0 + \delta_0) \cap I$,

$$L^f(x, y_0) \in I_0 \quad \text{and} \quad L^g(x, y_0) \in I_0.$$

Thus, by (2), we obtain

$$(f')^{-1} \left(\frac{f(x) - f(y_0)}{x - y_0} \right) (g')^{-1} \left(\frac{g(x) - g(y_0)}{x - y_0} \right) = xy_0, \quad x \in (x_0, x_0 + \delta_0) \cap I,$$

whence, after some calculations, we get

$$\frac{(x - y_0)^2}{(y_0 f(x) - 1)(y_0 g(x) - 1)} = x^2, \quad x \in (x_0, x_0 + \delta_0) \cap I,$$

and, consequently,

$$f(x) = \frac{y_0^2 - 2y_0 x + y_0 x^2 g(x)}{y_0^2 x^2 g(x) - y_0 x^2}, \quad x \in (x_0, x_0 + \delta_0) \cap I.$$

Similarly, taking $y_1 \in I_0$, $y_1 \neq y_0$, we have

$$g(x) = \frac{y_1^2 - 2y_1 x + y_1 x^2 f(x)}{y_1^2 x^2 f(x) - y_1 x^2}, \quad x \in (x_0, x_0 + \delta_1) \cap I,$$

for some $\delta_1 > 0$. Taking $\delta := \min\{\delta_0, \delta_1\}$ we hence obtain

$$(y_0 - y_1)(x^2[f(x)]^2 - 2xf(x) + 1) = 0, \quad x \in (x_0, x_0 + \delta) \cap I,$$

and, consequently,

$$f(x) = \frac{1}{x}, \quad x \in (x_0, x_0 + \delta) \cap I,$$

which proves that $\sup I_0 = \sup I$. Similarly we can show that $\inf I_0 = \inf I$. Thus we have proved that, if the functions f and g are of the form (17) in a nonempty open subinterval of I , then, necessarily, f and g must be of the form (17) on I . This fact excludes the coexistence of two disjoint nonempty open subintervals of I such that the functions f and g are of the form (17) on one of them, and of the form (18) on the remaining one. This completes the proof. \square

Remark 3. In case when $f: I \rightarrow \mathbb{R}$ and $g: I \rightarrow \mathbb{R}$ are of the form (17) then, it is easy to see, that

$$L^f(x, y) = L^g(x, y) = G(x, y), \quad x, y \in I.$$

In the remaining case the means L^f and L^g cannot be expressed by the elementary functions. This fact makes impossible to verify by a direct calculations if the geometric mean G is (L^f, L^g) -invariant. Thus, in this case, the question if the converse of Theorem 4 holds true remains open.

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