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# Distributions of homogeneous means of uniform independent random variables and a characterization of means

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*Dedicated to Professor János Aczél on the occasion of his eightieth birthday*

**Summary.** Let $U, V$ be two independent random variables uniformly distributed in $(0, 1]$ and let $M$ be a symmetric increasing and homogeneous mean. A general formula for the distribution function $F_{M\circ(U,V)}$ of the random variable $M \circ (U, V)$ is presented. Under some regularity assumptions, applying this formula, the implication

$$F_{M\circ(U,V)} = F_{N\circ(U,V)} \quad \Rightarrow \quad M = N,$$

(a characterization of means) is proved. Given a distribution function $F$, the integro-functional equation $F_{M\circ(U,V)} = F$ is considered. An application is provided.

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## 1. Introduction

Let $I \subset \mathbb{R}$ be an interval. A function $M : I^2 \to I$ such that

$$\min(u, v) \leq M(u, v) \leq \max(u, v), \qquad u, v \in I,$$

is called a (two-variable) *mean in* $I$. If for all $u, v \in I$, $u \neq v$, these inequalities are strict, $M$ is called a *strict mean*. A mean $M$ is *symmetric* if

$$M(u, v) = M(v, u), \qquad u, v \in I,$$

and *increasing* if it is increasing with respect to both variables.

Every mean is *reflexive*, i.e., $M(u, u) = u$. Note that every reflexive and increasing function $M : I^2 \to \mathbb{R}$ is a mean.

Let $a > 0$ be fixed. A mean $M : (0, a]^2 \to (0, a]$ is referred to as *homogeneous in* $(0, a]$ if

$$M(tu, tv) = tM(u, v), \qquad t \in (0, 1],\ u, v \in (0, a].$$

In Section 1 we show (Proposition 1) that every homogeneous mean $M$ in $(0, a]$ has a unique extension to a mean $M^* : (0, \infty)^2 \to (0, \infty)$, *homogeneous in* $(0, \infty)$,

i.e. such that

$$M^*(tx, ty) = tM^*(x, y), \qquad t, x, y \in (0, \infty),$$

and, moreover, if $M$ is strict, symmetric or increasing, then, respectively, so is $M^*$.

Let $M$ be a continuous mean in $I = (0, 1]$. Taking two independent random variables $U, V$ uniformly distributed in $(0, 1]$, one can construct a new random variable $M \circ (U, V)$, in general (cf. Remark 6) not uniformly distributed in $I$. Denote by $F_{M\circ(U,V)}$ the cumulative distribution function of the random variable $M \circ (U, V)$. In the present paper we examine the following problem: Determine a class $\mathcal{M}$ of means $M$ in $(0, 1]$ such that for all means $M, N \in \mathcal{M}$,

$$F_{M\circ(U,V)} = F_{N\circ(U,V)} \quad \Rightarrow \quad M = N,$$

that is, that the mapping $\mathcal{M} \ni M \to F_{M\circ(U,V)}$ is one-to-one. (For $M$ from such a class $\mathcal{M}$ we can write $F_M$ instead of $F_{M\circ(U,V)}$.) It can be interpreted as a characterization of a mean $M \in \mathcal{M}$ with the aid of its distribution function $F_M$ (for other ways of characterizations of means cf. [2]).

If $M$ is not a symmetric mean, then, obviously, $F_{M\circ(U,V)} = F_{N\circ(U,V)}$ where $N(x, y) := M(y, x)$, and $M \neq N$. Therefore we confine our considerations to symmetric means.

In Section 2 we establish the formulas for the distribution functions for the class $\mathcal{M}$ of symmetric monotonic and homogeneous means $M$ with the aid of their *traces* $h := M(\cdot, 1)$ (cf. [1], p. 231, for the term *trace* of a mean). In Section 3, applying these formulas, we prove that the mapping $\mathcal{M}_d \ni M \to F_{M\circ(U,V)}$, where $\mathcal{M}_d \subset \mathcal{M}$ is the subclass of all differentiable means, is one-to-one. The proof of this fact reduces to the uniqueness problem for the integro-functional equation

$$\varphi(x) - x^2 \int_x^{\frac{x}{\varphi(x)}} t^{-3}\varphi(t)dt = f(x), \qquad \varphi(1) = 1.$$

In Section 4, making use of this equation, we show that there is no differentiable homogeneous symmetric and increasing mean $M$ such that $M \circ (U, V)$ is uniformly distributed. In Section 5 we give a characterization of the extremal means min and max in the class of all continuous and symmetric means by their distribution functions. A motivating application from meteorology is presented in Section 6.

## 2. Homogeneous, symmetric and increasing means of two independent uniformly distributed random variables

In the proof of the main result of this section we need the following

**Proposition 1.** *Let $a > 0$ be fixed and let $M : (0, a]^2 \to \mathbb{R}$ be a homogeneous function, i.e.*

$$M(tu, tv) = tM(u, v), \qquad t \in (0, 1],\ u, v \in (0, a].$$

*Then there is a unique function* $M^* : (0,\infty)^2 \to (0,\infty)$ *such that*

$$M^*(tx,ty) = tM^*(x,y), \qquad t,x,y \in (0,\infty).$$

*Moreover,*

1. *if* $M$ *is a symmetric function, then so is* $M^*$;
2. *if* $M$ *is (strictly) increasing with respect to one (or both) variables, then so is* $M^*$;
3. *if* $M$ *is a (strict) mean in* $(0,a]$ *then* $M^*$ *is a (strict) mean in* $(0,\infty)$.

*Proof.* To define $M^* : (0,\infty)^2 \to (0,\infty)$, take arbitrary $x,y > 0$, choose an $s \in (0,1]$ such that $sx, sy \in (0,a]$ and put

$$M^*(x,y) := \frac{1}{s}M(sx,sy).$$

Note that if $r,s \in (0,1]$ are such that $rx, ry, sx, sy \in (0,a]$ and, for instance, $s < r$, then $\frac{s}{r} \in (0,1)$ and, by the homogeneity of $M$,

$$\frac{1}{s}M(sx,sy) = \frac{1}{s}M\left(\left(\frac{s}{r}\right)rx, \left(\frac{s}{r}\right)ry\right) = \frac{1}{s}\left(\frac{s}{r}\right)M(rx,ry) = \frac{1}{r}M(rx,ry),$$

which shows that $M^*(x,y)$ does not depend on the choice of $s$. Thus the function $M^*$ is correctly defined.

Now take arbitrary $t,x,y > 0$ and choose an $r \in (0,1]$ such that

$$rtx, rty, rx, ry \in (0,a], \qquad rt \le 1,$$

and put $s := rt$. Then, by the definition of $M^*$ and the independence of the value $M^*(x,y) = \frac{1}{s}M(sx,sy)$ on the choice of $s \in (0,1]$, we have

$$\begin{aligned} M^*(tx,ty) &= \frac{1}{r}M(r(tx),r(ty)) = t\left(\frac{1}{rt}\right)M((rt)x,(rt)y) \\ &= t\left(\frac{1}{s}M(sx,sy)\right) = tM^*(x,y), \end{aligned}$$

which proves that $M^*$ is homogeneous in $(0,\infty)$.

Since the remaining statements are easy to show, we omit their proofs. □

Note that the counterpart of this proposition for the functions $M : [a,\infty)^2 \to \mathbb{R}$ such that

$$M(tu,tv) = tM(u,v), \qquad t \in [1,\infty),\ u,v \in [a,\infty)$$

is also true.

In the sequel we assume that all means $M$ are defined in $(0,\infty)^2$. In view of Proposition 1 we can do it without any loss of generality.

The main result of this section reads as follows.

**Theorem 1.** *Let $M : (0,\infty)^2 \to (0,\infty)$ be a symmetric homogeneous mean such that the function $h := M(\cdot,1)$ (that is $h(t) := M(t,1)$ for all $t > 0$) is strictly increasing. Let $U, V$ be independent random variables uniformly distributed in $(0,1]$ and let $F_{M\circ(U,V)}$ denote the distribution function of the random variable $M \circ (U,V)$. Then $M$ is continuous and*

1. *if $h(0+) := \lim_{t\to 0+} h(t) > 0$, then*

$$F_{M\circ(U,V)}(x) = \begin{cases} 0, & \text{for } x \le 0 \\ x^2 \int_{h(0+)}^{+\infty} t^{-3}h^{-1}(t)dt, & \text{for } x \in (0, h(0+)] \\ h^{-1}(x) + x^2 \int_x^{\frac{x}{h^{-1}(x)}} t^{-3}h^{-1}(t)dt, & \text{for } x \in (h(0+), 1] \\ 1, & \text{for } x > 1 \end{cases};$$

2. *if $h(0+) = 0$ then*

$$F_{M\circ(U,V)}(x) = \begin{cases} 0, & \text{for } x \le 0 \\ h^{-1}(x) + x^2 \int_x^{\frac{x}{h^{-1}(x)}} t^{-3}h^{-1}(t)dt, & \text{for } x \in (0,1] \\ 1, & \text{for } x > 1 \end{cases}.$$

*Proof.* By the homogeneity and symmetry of $M$ we have

$$M(u,v) = uh\left(\frac{v}{u}\right) = vh\left(\frac{u}{v}\right) = M(v,u), \qquad u,v > 0. \tag{1}$$

Taking here $u := 1$ and $v := t$ we get

$$h(t) = th\left(\frac{1}{t}\right), \qquad t > 0, \tag{2}$$

or, equivalently,

$$\frac{h(t)}{t} = h\left(\frac{1}{t}\right), \qquad t > 0.$$

Hence the strict increasing monotonicity of $h$ implies that

$$(0,\infty) \ni t \mapsto \frac{h(t)}{t} \quad \text{is strictly decreasing.}$$

By the increasing monotonicity of $h$,

$$h(t-) \le h(t+), \qquad t > 0,$$

where $h(t-)$, $h(t+)$ denote the left and right limit at $t > 0$, respectively. On the other hand, the strict decreasing monotonicity of the function $(0,\infty) \ni t \mapsto \frac{h(t)}{t}$ implies that the opposite inequality holds true. Thus $h$ and, consequently, $M$ are continuous.

For $x \in (0,1]$, using the relation

$$M(u,v) = uh\left(\frac{v}{u}\right), \qquad u,v > 0,$$

and that $h$ is strictly increasing, we have

$$\begin{aligned} F_{M\circ(U,V)}(x) &= \text{Prob}\,(M\circ(U,V)\le x)\\ &= \lambda_2\left(\left\{(u,v): uh\left(\frac{v}{u}\right)\le x,\ 0\le u\le 1,\ 0\le v\le 1\right\}\right)\\ &= \lambda_2\left(\left\{(u,v): v\le uh^{-1}\left(\frac{x}{u}\right),\ 0\le u\le 1,\ 0\le v\le 1\right\}\right)\\ &= \lambda_2\left(\left\{(u,v):\ 0\le u\le 1,\ 0\le v\le \min\left(uh^{-1}\left(\frac{x}{u}\right),1\right)\right\}\right), \end{aligned}$$

where $\lambda_2$ denotes the 2-dimensional Lebesgue measure. Consequently,

$$F_{M\circ(U,V)}(x) = \int_0^1 \gamma(x,u)du, \qquad x\in(0,1], \tag{3}$$

where

$$\gamma(x,u) := \max\left\{0, \min\left(uh^{-1}\left(\frac{x}{u}\right),1\right)\right\}.$$

Since $h$ is increasing ($h\nearrow$) and $(0,\infty)\ni t\mapsto \frac{h(t)}{t}$ is decreasing ($\frac{h(t)}{t}\searrow$), we have the following sequence of equivalences:

$$\frac{h(t)}{t}\searrow \iff \frac{t}{h(t)}\nearrow \iff \frac{h^{-1}(t)}{t}\nearrow \iff \frac{h^{-1}\left(\frac{x}{t}\right)}{\frac{x}{t}}\searrow$$

whence, for every fixed $x>0$, the function

$$(0,\infty)\ni t\mapsto th^{-1}\left(\frac{x}{t}\right) \quad \text{is decreasing.}$$

Suppose that $h(0+)>0$. Applying relation (2) we have

$$\begin{aligned} \lim_{t\to 0+} th^{-1}\left(\frac{x}{t}\right) &= x\lim_{t\to 0+}\frac{h^{-1}\left(\frac{x}{t}\right)}{\frac{x}{t}} = x\lim_{t\to+\infty}\frac{h^{-1}(t)}{t} = x\lim_{t\to+\infty}\frac{t}{h(t)}\\ &= x\lim_{t\to+\infty}\frac{t}{th(\frac{1}{t})} = x\lim_{t\to+\infty}\frac{1}{h(\frac{1}{t})} = \frac{x}{h(0+)}. \end{aligned}$$

Moreover, for some $s>0$,

$$\begin{aligned} \lim_{t\to s} th^{-1}\left(\frac{x}{t}\right)=0 &\iff \lim_{t\to s-} h^{-1}\left(\frac{x}{t}\right)=0 \iff \lim_{t\to s-}\frac{x}{t}=h(0+)\\ &\iff \lim_{t\to s-} t=\frac{x}{h(0+)} \iff s=\frac{x}{h(0+)}. \end{aligned}$$

Hence, if $0<x\le h(0+)$, then

$$\lim_{t\to 0+} th^{-1}\left(\frac{x}{t}\right)=\frac{x}{h(0+)}\le 1,$$

and, since the function $t\to th^{-1}\left(\frac{x}{t}\right)$ is decreasing, we get

$$\gamma(x,u)=\begin{cases} uh^{-1}\left(\frac{x}{u}\right), & \text{for } u\in(0,\frac{x}{h(0+)}]\\ 0, & \text{for } u\in(\frac{x}{h(0+)},1] \end{cases}.$$

If $h(0+) < x \leq 1 = h(1)$, then

$$\lim_{t\to 0+} th^{-1}\left(\frac{x}{t}\right) = \frac{x}{h(0+)} > 1.$$

By (2) we have

$$th^{-1}\left(\frac{x}{t}\right) = 1 \Longleftrightarrow x = th\left(\frac{1}{t}\right) \Longleftrightarrow x = h(t) \Longleftrightarrow t = h^{-1}(x),$$

and, as for $t = 1$ we have $th^{-1}\left(\frac{x}{t}\right) = h^{-1}(x) \geq 0$, and since the function $t \to th^{-1}\left(\frac{x}{t}\right)$ is decreasing, we infer that for all $x \in (h(0+), 1]$,

$$\gamma(x,u) = \begin{cases} 1, & \text{for } u \in (0, h^{-1}(x)] \\ uh^{-1}\left(\frac{x}{u}\right), & \text{for } u \in (h^{-1}(x), 1] \end{cases}.$$

From (3) we get

$$F_{M\circ(U,V)}(x) = \begin{cases} \int_0^{\frac{x}{h(0+)}} uh^{-1}\left(\frac{x}{u}\right) du, & \text{for } x \in (0, h(0+)] \\ h^{-1}(x) + \int_{h^{-1}(x)}^1 uh^{-1}\left(\frac{x}{u}\right) du, & \text{for } x \in (h(0+), 1] \end{cases}.$$

Setting here

$$t = \frac{x}{u}, \tag{4}$$

by obvious calculations we obtain the first part of the theorem.

Now suppose that $h(0+) = 0$. Applying (2) we have

$$\lim_{t\to 0+} th^{-1}\left(\frac{x}{t}\right) = x \lim_{t\to 0+} \frac{h^{-1}\left(\frac{x}{t}\right)}{\frac{x}{t}} = x \lim_{t\to+\infty} \frac{h^{-1}(t)}{t} = x \lim_{t\to+\infty} \frac{t}{h(t)}$$
$$= x \lim_{t\to+\infty} \frac{t}{th(\frac{1}{t})} = x \lim_{t\to+\infty} \frac{1}{h(\frac{1}{t})} = +\infty.$$

Since the function $(0,\infty) \ni t \to th^{-1}\left(\frac{x}{t}\right)$ is decreasing and continuous, we infer that

$$th^{-1}\left(\frac{x}{t}\right) = 1 \Longleftrightarrow th\left(\frac{1}{t}\right) = x \Longleftrightarrow h(t) = x,$$

and, using (3), we obtain

$$h^{-1}(x) + \int_{h^{-1}(x)}^1 uh^{-1}\left(\frac{x}{u}\right) du \qquad \text{for } x \in (0,1].$$

The substitution (4) gives the second part of the theorem □

**Remark 1.** Note that if $h(0+) = 0$, then the formula for $F_{M\circ(U,V)}$ given in the first part of Theorem 1 reduces to that given in the second one.

**Remark 2.** The trace functions for the extremal means min and max are not strictly increasing. However it turns out that the formulas given by Theorem 1 are applicable for these means.

Indeed, in the case $M = \min$ the trace function $h$ is of the form

$$h(t) = \begin{cases} t, & t \in (0,1) \\ 1, & t > 1 \end{cases}.$$

We have $h(0+) = 0$ and $h\big|_{(0,1]} = \mathrm{id}\big|_{(0,1]} = h\Big|^{-1}_{(0,1]}$. Further,

$$\frac{x}{h\Big|^{-1}_{(0,1]}(x)} = 1, \qquad x \in (0,1],$$

and the values of $h$ in $(1,\infty)$ play no role in the second part of Theorem 1. Therefore, for independent random variables $U, V$, uniformly distributed in $(0,1]$, we get

$$F_{\min\circ(U,V)}(x) = x - x^2 \int_x^1 t^{-2} dt = 1 - (1-x)^2$$

for all $x \in (0,1]$ and, consequently,

$$F_{\min\circ(U,V)}(x) = \begin{cases} 0, & \text{for} \quad x \le 0 \\ 1-(1-x)^2, & \text{for} \quad x \in (0,1] \\ 1, & \text{for} \quad x > 1 \end{cases}.$$

If $M = \max$ the trace function $h$ is of the form

$$h(t) = \begin{cases} 1, & t \in (0,1) \\ t, & t > 1 \end{cases}.$$

We have $h(0+) = 1$ and $h\big|_{(1,\infty)} = \mathrm{id}\big|_{(1,\infty)} = h\Big|^{-1}_{(1,\infty)}$. In this case the values of $h$ in $(0,1)$ play no role in the first part of Theorem 1. Therefore, for independent random variables $U, V$, uniformly distributed in $(0,1]$, we get

$$F_{\max\circ(U,V)}(x) = x^2 \int_1^{+\infty} t^{-2} dt = x^2$$

for all $x \in (0,1]$ and, consequently,

$$F_{\max\circ(U,V)}(x) = \begin{cases} 0, & \text{for} \quad x \le 0 \\ x^2, & \text{for} \quad x \in (0, h(0+)] \\ 1, & \text{for} \quad x > 1 \end{cases}.$$

**Remark 3.** If $M$ is a mean, then $h = M(\cdot, 1)$ cannot be decreasing.

**Example 1.** Let $M := A$ where $A$ denotes the arithmetic mean $A(x,y) := \frac{x+y}{2}$, $x, y > 0$. $A$ is symmetric, homogeneous, and $h(t) := M(t,1) = \frac{t+1}{2}$, $t > 0$, is strictly increasing. Let $U, V$ be independent random variables, uniformly distributed in $(0,1]$, and let $F_A$ denote the distribution function of the random variable $A\circ(U,V)$.

Since $h(0+) = \frac{1}{2}$, applying the first part of Theorem 1 we obtain

$$F_{A\circ(U,V)}(x) = \begin{cases} 0, & \text{for} \quad x \le 0 \\ 2x^2, & \text{for} \quad x \in \left(0, \frac{1}{2}\right] \\ -1 + 4x - 2x^2, & \text{for} \quad x \in \left(\frac{1}{2}, 1\right] \\ 1, & \text{for} \quad x > 1 \end{cases}.$$

**Example 2.** Let $M := G$ where $G$ denotes the geometric mean $G(x,y) := \sqrt{xy}$, $x, y > 0$. $G$ is symmetric, homogeneous, and $h(t) := G(t,1) = \sqrt{t}$, $t > 0$, is strictly increasing. Let $U, V$ be independent random variables uniformly distributed in $(0,1]$, and let $F_G$ denote the distribution function of the random variable $G\circ(U,V)$. Since $h(0+) = 0$, applying the second part of Theorem 1 we obtain

$$F_{G\circ(U,V)}(x) = \begin{cases} 0, & \text{for} \quad x \le 0 \\ x^2(1 - 2\log x), & \text{for} \quad x \in (0,1] \\ 1, & \text{for} \quad x > 1 \end{cases}.$$

**Example 3.** Let $M := H$ where $H$ denotes the harmonic mean $H(x,y) := \frac{2xy}{x+y}$, $x, y > 0$. $H$ is symmetric, homogeneous, and $h(t) := H(t,1) = \frac{2t}{t+1}$, $t > 0$, is strictly increasing. Let $U, V$ be independent random variables uniformly distributed in $(0,1]$, and let $F_H$ denote the distribution function of the random variable $H\circ(U,V)$. Since $h(0+) = 0$, applying the second part of Theorem 1, we obtain

$$F_{H\circ(U,V)}(x) = \begin{cases} 0, & \text{for} \quad x \le 0 \\ x + \frac{x^2}{2}\log\frac{2-x}{x}, & \text{for} \quad x \in (0,1] \\ 1, & \text{for} \quad x > 1 \end{cases}.$$

**Example 4.** Let $M := R$ where $R$ denotes the quadratic (root-mean-square) mean $R(x,y) := \sqrt{\frac{x^2+y^2}{2}}$, $x, y > 0$. $R$ is symmetric, homogeneous, and $h(t) := R(t,1) = \sqrt{\frac{t^2+1}{2}}$, $t > 0$, is strictly increasing. Let $U, V$ be independent random variables, uniformly distributed in $(0,1]$, and let $F_R$ denote the distribution function of the random variable $R \circ (U,V)$. Since $h(0+) = \frac{1}{\sqrt{2}}$, applying the first part of Theorem 1, we obtain

$$F_{R\circ(U,V)}(x) = \begin{cases} 0, & \text{for} \quad x \le 0 \\ \frac{\pi}{2}x^2, & \text{for} \quad x \in \left(0, \frac{1}{\sqrt{2}}\right] \\ x^2\left(\frac{\pi}{2} - 2\arccos\left(\frac{1}{x\sqrt{2}}\right)\right) + \sqrt{2x^2-1}, & \text{for} \quad x \in \left(\frac{1}{\sqrt{2}}, 1\right] \\ 1, & \text{for} \quad x > 1 \end{cases}.$$

## 3. An integro-functional equation and a new characterization of means

In connection with Theorem 1 a natural question arises which we formulate as

**Problem 1.** *Suppose that $M$ and $N$ satisfy the assumptions of Theorem 1. Does then the implication*

$$F_{M\circ(U,V)} = F_{N\circ(U,V)} \quad \Rightarrow \quad M = N$$

*hold true?*

We prove that, under some regularity conditions of the involved means, the answer is affirmative. Namely, the following uniqueness theorem holds true.

**Theorem 2.** *Let $M, N : (0,\infty)^2 \to (0,\infty)$ be symmetric homogeneous means such that the functions $h := M(\cdot,1)$ and $g := N(\cdot,1)$ are strictly increasing, differentiable and such that $h'(x) \neq 0 \neq g'(x)$ for every $x > 0$. Let $U, V$ be independent random variables, uniformly distributed in $(0,1]$. If $F_{M\circ(U,V)} = F_{N\circ(U,V)}$, then $M = N$.*

*Proof.* Suppose that $F_{M\circ(U,V)} = F_{N\circ(U,V)}$. The assumptions imply that the functions $\varphi := h^{-1}$ and $\psi := g^{-1}$ are differentiable. We may assume that $g(0+) \leq h(0+)$. By Theorem 1 we have

$$\varphi(x) + x^2 \int_x^{\frac{x}{\varphi(x)}} t^{-3}\varphi(t)dt = \psi(x) + x^2 \int_x^{\frac{x}{\psi(x)}} t^{-3}\psi(t)dt$$

for all $x \in (h(0+),1]$, or, equivalently,

$$\frac{\psi(x)-\varphi(x)}{x^2} = \int_x^{\frac{x}{\varphi(x)}} t^{-3}\varphi(t)dt - \int_x^{\frac{x}{\psi(x)}} t^{-3}\psi(t)dt$$

for all $x \in (h(0+),1]$. Differentiating both sides with respect to $x$ we get

$$\frac{[\psi'(x)-\varphi'(x)]x - 2[\psi(x)-\varphi(x)]}{x^3} =$$

$$\frac{\varphi\left(\frac{x}{\varphi(x)}\right)}{\left(\frac{x}{\varphi(x)}\right)^3}\frac{\varphi(x)-x\varphi'(x)}{[\varphi(x)]^2} - \frac{\varphi(x)}{x^3} - \frac{\psi\left(\frac{x}{\psi(x)}\right)}{\left(\frac{x}{\psi(x)}\right)^3}\frac{\psi(x)-x\psi'(x)}{[\psi(x)]^2} + \frac{\psi(x)}{x^3}$$

for all $x \in (h(0+),1]$. Since the relations

$$h(t) = th\left(\frac{1}{t}\right), \qquad g(t) = tg\left(\frac{1}{t}\right), \qquad t > 0, \tag{5}$$

imply that

$$\varphi\left(\frac{x}{\varphi(x)}\right) = \frac{1}{\varphi(x)}, \qquad \psi\left(\frac{x}{\psi(x)}\right) = \frac{1}{\psi(x)} \tag{6}$$

for all $x > h(0+)$, we hence get

$$\frac{\psi'(x) - \varphi'(x)}{x^2} - \frac{2[\psi(x) - \varphi(x)]}{x^3} = \frac{\psi'(x)}{x^2} - \frac{\varphi'(x)}{x^2}$$

for all $x > h(0+)$, and, consequently,

$$\psi(x) = \varphi(x), \qquad x \in (h(0+), 1].$$

We will show that $g(0+) = h(0+)$. For an indirect argument suppose that $g(0+) < h(0+)$. Setting

$$a := \int_{h(0+)}^{+\infty} t^{-3} h^{-1}(t) dt,$$

by Theorem 1, we would have

$$ax^2 = \psi(x) + x^2 \int_x^{\frac{x}{\psi(x)}} t^{-3} \psi(t) dt$$

for all $x \in (g(0+), h(0+))$, whence

$$a - \frac{\psi(x)}{x^2} = \int_x^{\frac{x}{\psi(x)}} t^{-3} \psi(t) dt, \qquad x \in (g(0+), h(0+)).$$

Differentiating both sides of this equality and applying the second of the formulas (6) we obtain $\psi(x) = 0$ for all $x \in (g(0+), h(0+))$. This contradicts the inequality

$$\psi(x) > 0, \qquad x > g(0+),$$

and proves that $g(0+) = h(0+)$. From the definitions of $\varphi$ and $\psi$ we get $g(x) = h(x)$ for all $x \in (0, 1]$. Now, by (5) (or by Proposition 1), $g = h$. This completes the proof. □

**Remark 4.** The above result can be viewed as a characterization of differentiable symmetric homogeneous means $M$ with the aid of the distribution function $F_{M \circ (U,V)}$, where $U$ and $V$ are independent random variables uniformly distributed in $(0, 1]$. In particular, within the class $\mathcal{M}_d$ of all differentiable symmetric homogeneous means the notation $F_M$ instead of $F_{M \circ (U,V)}$ is justified.

**Remark 5.** In the proof of Theorem 1 we have shown the uniqueness of the differentiable solution $\varphi$ of the following integro-functional equation

$$\varphi(x) + x^2 \int_x^{\frac{x}{\varphi(x)}} t^{-3} \varphi(t) dt = f(x), \qquad \varphi(1) = 1,$$

in a neighbourhood of the point 1. If $\varphi$ is differentiable, then $f$ is also differentiable and, by (6), this equation reduces to the linear differential equation

$$2x\varphi'(x) - 2\varphi(x) = xf'(x) - 2f(x).$$

It is an open problem, whether the differentiability assumption is necessary.

## 4. No mean of the class $\mathcal{M}_d$ is uniformly distributed

Let $U$ and $V$ be two independent random variables uniformly distributed in $(0,1]$. The problem to decide if there is a homogeneous symmetric and increasing mean $M : (0,\infty)^2 \to (0,\infty)$ such that $M \circ (U,V)$ is uniformly distributed reduces to an integro-functional equation

$$F_{M\circ(U,V)} = F$$

where $F_{M\circ(U,V)}$ is given by Theorem 1, with the unknown function $h^{-1}$, and $F(x) = x$ for $x \in (0,1]$. Suppose, for instance, that there is a continuous symmetric increasing and homogeneous mean $M : (0,\infty)^2 \to (0,\infty)$ such that the random variable $M \circ (U,V)$ is uniformly distributed. Then we would have

$$F_{M\circ(U,V)}(x) = x, \qquad x \in (0,1],$$

and, by Theorem 1,

$$x = h^{-1}(x) + x^2 \int_x^{\frac{x}{h^{-1}(x)}} t^{-3} h^{-1}(t)\,dt \qquad \text{for} \quad x \in (0,1],$$

where $h = M(\cdot,1)$. Putting $\varphi := h^{-1}$ we obtain the integro-functional equation

$$x = \varphi(x) + x^2 \int_x^{\frac{x}{\varphi(x)}} t^{-3}\varphi(t)\,dt \qquad \text{for} \quad x \in (0,1].$$

Assume that $\varphi$ is differentiable. Differentiating both sides and applying (6) we get

$$2x \int_x^{\frac{x}{\varphi(x)}} t^{-3}\varphi(t)\,dt = 1 \qquad \text{for} \quad x \in (0,1].$$

Differentiation of both sides of this equation gives $\varphi' = \frac{1}{2}$, whence, $h(t) = 2t + c$ for $t \in (0,1]$ and $c$ a real constant. Since (for any $c$) this $h$ is not a trace function of any mean, we have proved

**Remark 6.** Let $U$ and $V$ be two independent random variables uniformly distributed in $(0,1]$. There is no differentiable homogeneous symmetric and increasing mean $M$ such that $M \circ (U,V)$ is uniformly distributed.

## 5. A characterization of extremal means

In this section we show that the extremal means max and min can be characterized by their distribution functions.

**Proposition 2.** *Let $M : (0,1]^2 \to (0,1]$ be a continuous mean, let $U, V$ be independent random variables uniformly distributed in $(0,1]$ and let $F_M$ denote the distribution function of the random variable $M \circ (U,V)$. Then*

$$F_M(x) = \begin{cases} 0, & \text{for} \quad x \leq 0 \\ x^2, & \text{for} \quad x \in (0,1] \\ 1, & \text{for} \quad x > 1 \end{cases}$$

*if, and only if, $M = \max$.*

*Proof.* The "if" part follows from Remark 2.

Since $M \leq \max$ we have, for all $x \in (0,1]$,

$$\left\{(u,v) \in (0,1]^2 : \max(u,v) \leq x\right\} \subset \left\{(u,v) \in (0,1]^2 : M(u,v) \leq x\right\}.$$

If $M \neq \max$ then, by the continuity of $M$, for some $x \in (0,1]$,

$$\lambda_2\left\{(u,v) \in (0,1]^2 : \max(u,v) \leq x\right\} < \lambda_2\left\{(u,v) \in (0,1]^2 : M(u,v) \leq x\right\}.$$

Hence, for some $x \in (0,1]$ we would have

$$x^2 = F_{\max}(x) < F_M(x) = x^2.$$

This contradiction completes the proof. □

In a similar way we can prove

**Proposition 3.** *Let $M : (0,1]^2 \to (0,1]$ be a continuous mean, let $U, V$ be independent random variables uniformly distributed in $(0,1]$ and let $F_M$ denote the distribution function of the random variable $M \circ (U,V)$. Then*

$$F_M(x) = \begin{cases} 0, & \text{for} \quad x \leq 0 \\ 1-(1-x)^2, & \text{for} \quad x \in (0,1] \\ 1, & \text{for} \quad x > 1 \end{cases}$$

*if, and only if, $M = \min$.*

**Remark 7.** It is easy to observe that the finite-dimensional counterparts of the above propositions hold true.

**Remark 8.** Our Uniqueness Theorem 2 guarantees that a certain distribution function pertains to one and only one mean (of two independent random variables $U, V$, uniformly distributed in $(0,1]$). However, the same distribution function may, of course, hold for some other function (of $U$ and $V$). For example, it can be readily shown (cf. [3]) that the distribution function of $U^{1/2}$ coincides with that of $\max(U,V)$.

**Remark 9.** A fact like the one mentioned at the end of Remark 8 may be used to replace the numerical calculation of one distribution, e.g. that of $U^{1/2}$, by the

computationally faster calculation of an "equivalent distribution", here that of $\max(U, V)$. (The term "equivalent distribution" has been coined in [3].)

## 6. Application to measurements

We consider the following problems:

**Problem 2.** *A meteorological element, admitting non-negative values only (e.g. air pressure, wind speed, absolute temperature near the ground), is measured (in appropriate units) with high precision. The value communicated to the public is rounded to an integer. What is the distribution of the rounding error?*

**Problem 3.** *To reduce accidental errors, a meteorological element is measured twice, and the arithmetic mean of these two values is formed. Rounding to an integer (as in Problem 2), what is the distribution of the rounding error?*

**Solution of Problem 2.** According to the problem setting, the measured value $t$ is the sum of an integer part $T$ and a fractional part $\Delta$, i.e. $t = T + \Delta$. It is reasonable to assume that $\Delta$ is a random variable nearly uniformly distributed in $[0, 1)$.

Concerning the rounding error, we have two options of interpretation:

**a:** The decomposition $t = T + \Delta$ may be viewed as a particular rounding process, namely rounding off. The integer part is a *base value*, and the fractional part $\Delta$ is *suppressed detail.*

**b:** Since the average of the fractional part $\Delta$ is $1/2$, we may view the composition of $t$ also in the following way: the average of the measured value $t$ is $t_{ave} = T + 1/2$, yielding the alternative decomposition $t = t_{ave} + \delta$, where $\delta$ is the *round-off error*, nearly uniformly distributed in $[-1/2, 1/2)$. (This interpretation conforms to [4] Sect. 25.)

**Solution of Problem 3.** Measuring the element twice, we obtain two values, $t_1 = T + \Delta_1$ and $t_2 = T + \Delta_2$ (assuming that the base value $T$ does not change between the two measurements). Forming the arithmetic mean $t_{mean} = (t_1 + t_2)/2$ we get the interpretations

**a:** $t_{mean} = T + \Delta_{mean}$ where the *mean suppressed detail* $\Delta_{mean} = (\Delta_1 + \Delta_2)/2$ has approximately the distribution of Example 1 in $[0, 1)$.

**b:** $t_{mean} = t_{ave} + \delta_{mean}$ where the *mean round-off error* $\delta_{mean} = (\delta_1 + \delta_2)/2$ has approximately the distribution of Example 1 in $[-1/2, 1/2)$.

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## References

[1] J. M. Borwein and P. B. Borwein, *Pi and the AGM*, John Wiley & Sons, New York–Chichester–Brisbane–Toronto–Singapore, 1987.

[2] P. S. Bullen, D. S. Mitrinović and P. M. Vasić, *Means and Their Inequalities*, Mathematics and Its Applications, D. Reidel Publishing Company, Dordrecht–Boston–Lancaster–Tokyo, 1988. – (Revised edition:) P. S. Bullen, *Handbook of Means and Their Inequalities*, Mathematics and Its Applications, Kluwer Academic Publishers, Dordrecht–Boston–London, 2003.

[3] D. E. Knuth, *The Art of Computer Programming*, Vol. 2 Seminumerical Algorithms, Addison Wesley Longman, Reading, 1998.

[4] B. L. Van der Waerden, *Mathematische Statistik*, Springer, Berlin, 1971.

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