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Lagrangian mean-type mappings for which the arithmetic mean is invariant

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Abstract

We determine the class of all pairs of the Lagrangian means forming mean-type mappings which are invariant with respect to the arithmetic mean.

Keywords: Lagrangian mean; Gauss composition of means; Invariant mean

1. Introduction

Let $I \subseteq \mathbb{R}$ be an interval. A function $M : I^2 \to I$ such that

$$\min(x, y) \leqslant M(x, y) \leqslant \max(x, y), \quad x, y \in I,$$

is called a *mean*. Every mean is *reflexive*, that is $M(x, x) = x$ for all $x \in I$. If for all $x, y \in I$, $x \neq y$, these inequalities are strict, M is said to be a *strict mean*. A mean M is called *symmetric if* $M(x, y) = M(y, x)$, for all $x, y \in I$. (For more information about means cf., for instance, [2,3].)

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A mean $M : I^2 \to I$ is called *Lagrangian* if there is a continuous and strictly monotonic function $f : I \to \mathbb{R}$, a *generator* of the mean, such that $M = L_f$, where

$$L_f(x, y) := \begin{cases} f^{-1}\left(\frac{1}{x-y}\int_x^y f(t)\,dt\right) & \text{for } x \neq y, \\ x & \text{for } x = y. \end{cases}$$

Let $M, N : I^2 \to I$ be means. A mean $K : I^2 \to I$ is called *(M, N)-invariant* if

$$K\big(M(x, y), N(x, y)\big) = K(x, y), \quad x, y \in I.$$

If the means M and N are continuous and strict, then there exists a unique continuous (M, N)-invariant mean K, called also the *Gauss composition* of M and N and, moreover, K is strict and the sequence of iterates of the mean-type mapping $(M, N) : I^2 \to I^2$, called the *Gauss-iteration*, converges to the mean-type mapping (K, K) (cf. J.M. Borwein and P.B. Borwein [2, Chapter Eight], also [5,7]).

Let $f, g : I \to \mathbb{R}$ be strictly monotonic continuous. In Section 3 we prove that *the arithmetic mean A is (L_f, L_g)-invariant iff there is a $p \in R$ such that $L_f = L_{[p]}$ and $L_g = L_{[-p]}$ where, for $p \neq 0$,*

$$L_{[p]}(x, y) := \begin{cases} \frac{1}{p}\log\frac{e^{px} - e^{py}}{x - y}, & x \neq y, \\ x, & x = y, \end{cases} \quad x, y \in I,$$

and

$$L_{[0]}(x, y) := \lim_{p \to 0} L_{[p]}(x, y) = \frac{x + y}{2}.$$

Let us mention that all twice differentiable pairs (M, N) of quasi-arithmetic means such that A is (M, N)-invariant have been determined in [6]. Then Z. Daróczy and Gy. Maksa [4] substantially weakened the regularity conditions. Finally, Z. Daróczy and Zs. Páles in their important paper [5] indicated the strict connections of some questions concerning the Gauss composition with the *fifth* of Hilbert's problems and gave a complete solution.

2. A necessary condition for (L_f, L_g)-invariance of the arithmetic mean

Let $A(x, y) := \frac{x+y}{2}$ for $x, y \in I$. The problem to determine all continuous and strictly monotonic functions $f, g : I \to \mathbb{R}$ such that A is (L_f, L_g)-invariant reduces to the functional equation

$$L_f(x, y) + L_g(x, y) = x + y, \quad x, y \in I,\ x \neq y. \tag{1}$$

We begin this section with the following proposition.

Proposition 1. *If $f, g : I \to \mathbb{R}$ are strictly monotonic, twice continuously differentiable in an open interval I, $f' \neq 0 \neq g'$, and A is (L_f, L_g)-invariant, then*

$$f'g' = C$$

for some constant $C \in \mathbb{R}\backslash\{0\}$.

Proof. Let $F, G : I \to \mathbb{R}$ denote some primitive functions of f and g, respectively. Then

$$L_f(x, y) = f^{-1}\left(\frac{F(x) - F(y)}{x - y}\right), \qquad L_g(x, y) = g^{-1}\left(\frac{G(x) - G(y)}{x - y}\right)$$

for all $x, y \in I$, $x \neq y$. Since

$$\begin{aligned}\frac{\partial^2 L_f}{\partial x^2}(x, y) &= \frac{1}{f'(L_f(x, y))} \frac{f'(x)(x - y)^2 - 2[f(x)(x - y) - F(x) + F(y)]}{(x - y)^3} \\ &\quad - \frac{f''(L_f(x, y))}{[f'(L_f(x, y))]^3}\left[\frac{f(x)(x - y) - F(x) + F(y)}{(x - y)^2}\right]^2\end{aligned}$$

for all $x, y \in I$, $y \neq x$, and

$$\lim_{y \to x} \frac{f'(x)(x - y)^2 - 2[f(x)(x - y) - F(x) + F(y)]}{(x - y)^3} = \frac{f''(x)}{3},$$

$$\lim_{y \to x} \frac{f(x)(x - y) - F(x) + F(y)}{(x - y)^2} = \frac{f'(x)}{2},$$

we obtain

$$\lim_{y \to x} \frac{\partial^2 L_f}{\partial x^2}(x, y) = \frac{1}{12} \frac{f''(x)}{f'(x)}, \quad x \in I.$$

Obviously, we also have

$$\lim_{y \to x} \frac{\partial^2 L_g}{\partial x^2}(x, y) = \frac{1}{12} \frac{g''(x)}{g'(x)}, \quad x \in I.$$

Since (1) is equivalent to the functional equation

$$f^{-1}\left(\frac{F(x) - F(y)}{x - y}\right) + g^{-1}\left(\frac{G(x) - G(y)}{x - y}\right) = x + y, \quad x, y \in I, \ x \neq y, \tag{2}$$

we hence get

$$\frac{f''(x)}{f'(x)} + \frac{g''(x)}{g'(x)} = 0, \quad x \in I, \tag{3}$$

which implies the existence of a constant $C \in \mathbb{R}$ such that $f'(x)g'(x) = C$ for all $x \in I$. Obviously $C \neq 0$. This completes the proof. □

3. A regularity theorem

Theorem 1. *Let $f, g : I \to \mathbb{R}$ be continuous and strictly monotonic in an open interval I, and F, G be the primitives of f and g, respectively. If the arithmetic mean A is (L_f, L_g)-invariant, then f and g are of the class of C^∞ in I except for a nowhere dense subset of I.*

Proof. Assume first that for every $x_0 \in I$ there is a $y_0 \in I$, $x_0 \neq y_0$, such that

$$\frac{G(x_0)-G(y_0)}{x_0-y_0} \neq \frac{g(x_0)+g(y_0)}{2}.$$

Let us fix an $x_0 \in I$, put

$$u_0 := \frac{F(x_0)-F(y_0)}{x_0-y_0}, \qquad \Delta := \{(x,x)\colon x \in I\},$$

and define the function $\Phi : (I^2 \backslash \Delta) \times \mathbb{R} \to \mathbb{R}$ by

$$\Phi(x,y,u) := \frac{F(x)-F(y)}{x-y} - u.$$

Note that the function Φ is of the class C^1,

$$\Phi(x_0,y_0,u_0) = 0,$$

and

$$\frac{\partial \Phi}{\partial y}(x_0,y_0,u_0) = \frac{f(y_0)(y_0-x_0)-F(y_0)+F(x_0)}{(y_0-x_0)^2} \neq 0.$$

If the last relation was not true, we would have

$$\frac{F(x_0)-F(y_0)}{x_0-y_0} = f(y_0),$$

and, by the Lagrange mean value theorem,

$$\frac{F(x_0)-F(y_0)}{x_0-y_0} = f(\xi),$$

for some $\xi \neq y_0$, whence $f(y_0) = f(\xi)$. This is a contradiction, as f, being strictly monotonic, is one-to-one. By the implicit function theorem, there exist a neighbourhood $D = (x_0-\delta, x_0+\delta) \times (u_0-\delta, u_0+\delta)$ of the point (x_0,u_0) for some $\delta > 0$, and a unique function $\varphi : D \to I$ of the class C^1 in D and such that

$$\varphi(x_0,u_0) = y_0, \qquad \Phi\big(x, \varphi(x,u), u\big) = 0, \quad (x,u) \in D,$$

that is

$$\varphi(x_0,u_0) = y_0, \qquad \frac{F(x)-F(\varphi(x,u))}{x-\varphi(x,u)} = u, \quad (x,u) \in D.$$

Moreover, since $\frac{\partial \Phi}{\partial x} \neq 0$ and $\frac{\partial \Phi}{\partial u} = -1$, we have $\frac{\partial \varphi}{\partial x} \neq 0$, $\frac{\partial \varphi}{\partial u} \neq 0$ in D. Setting $y = \varphi(x,u)$ in (2), we obtain

$$f^{-1}(u) + g^{-1}\left(\frac{G(x)-G(\varphi(x,u))}{x-\varphi(x,u)}\right) = x + \varphi(x,u), \quad x,u \in D. \tag{4}$$

Put

$$v_0 := \frac{G(x_0)-G(\varphi(x_0,u_0))}{x_0-\varphi(x_0,u_0)} = \frac{G(x_0)-G(y_0)}{x_0-y_0},$$

and define $\Psi : D \times \mathbb{R} \to \mathbb{R}$ by the formula

$$\Psi(x,u,v) = \frac{G(x) - G(\varphi(x,u))}{x - \varphi(x,u)} - v, \quad (x,u) \in D,\ v \in \mathbb{R}. \tag{5}$$

The function Ψ is of the class C^1.

Suppose first that

$$\frac{\partial \Psi}{\partial x}(x_0, u_0, v_0) \neq 0.$$

By the implicit function theorem there exist a neighbourhood W of the point (u_0, v_0),

$$W = (u_0 - \rho, u_0 + \rho) \times (v_0 - \rho, v_0 + \rho),$$

for some $\rho > 0$, and a unique function $\psi : W \to I$ of the class C^1 in W such that

$$\psi(u_0, v_0) = x_0, \qquad \Psi\bigl(\psi(u,v), u, v\bigr) = 0, \quad (u,v) \in W,$$

that is

$$\psi(u_0, v_0) = x_0, \qquad \frac{G(\psi(u,v)) - G(\varphi(\psi(u,v),u))}{\psi(u,v) - \varphi(\psi(u,v),u)} = v, \quad (u,v) \in W.$$

Substituting $x = \psi(u,v)$ in (4), we obtain

$$f^{-1}(u) + g^{-1}(v) = \psi(u,v) + \varphi\bigl(\psi(u,v), u\bigr), \quad (u,v) \in W.$$

Since the right-hand side is a function of the class C^1 in W, we infer that f^{-1} and g^{-1} are of the class C^1 in the intervals $(u_0 - \rho, u_0 + \rho)$ and $(v_0 - \rho, v_0 + \rho)$, respectively. Since the sets

$$\bigl\{u\colon (f^{-1})'(u) = 0\bigr\}, \qquad \bigl\{u\colon (g^{-1})'(u) = 0\bigr\}$$

are nowhere dense, it follows that the functions f and g are of the class C^1 in an open nonempty subinterval contained in $(x_0 - \delta, x_0 + \delta)$.

Suppose that

$$\frac{\partial \Psi}{\partial x}(x_0, u_0, v_0) = 0.$$

Then, by the definition of Ψ,

$$\frac{\partial \Psi}{\partial x}(x_0, u_0, v) = 0.$$

If there is a point $(x_1, u_1) \in D$ such that

$$\frac{\partial \Psi}{\partial x}(x_1, u_1, v) \neq 0,$$

then, choosing a $\delta_1 > 0$ such that

$$D_1 := (x_1 - \delta_1, x_1 + \delta_1) \times (u_1 - \delta_1, u_1 + \delta_1) \subset D,$$
$$\frac{\partial \Psi}{\partial x}(x,u) \neq 0, \quad (x,u) \in D_1,$$

we could repeat the above reasoning with (x_0, u_0) and D replaced by (x_1, u_1) and D_1, respectively.

If there were no a point $(x_1, u_1) \in D$ such that $\frac{\partial \Psi}{\partial x}(x_1, u_1, v) \neq 0$, then

$$\frac{\partial \Psi}{\partial x}(x, u, v) = 0, \quad (x, u) \in D, \ v \in \mathbb{R}.$$

Hence, differentiating with respect to x both sides of (5), we would get

$$\begin{aligned}&\big\{G(x) - G\big(\varphi(x,u)\big) - g\big(\varphi(x,u)\big)\big[x - \varphi(x,u)\big]\big\}\frac{\partial \varphi}{\partial x}(x,u)\\ &\quad = G(x) - G\big(\varphi(x,u)\big) - g(x)\big[x - \varphi(x,u)\big]\end{aligned}$$

for all $(x, u) \in D$. As in this case the function on right-hand side of (4) does not depend on x,

$$\frac{\partial \varphi}{\partial x} = -1 \quad \text{in } D,$$

whence

$$\begin{aligned}&-\big\{G(x) - G\big(\varphi(x,u)\big) - g\big(\varphi(x,u)\big)\big[x - \varphi(x,u)\big]\big\}\\ &\quad = G(x) - G\big(\varphi(x,u)\big) - g(x)\big[x - \varphi(x,u)\big].\end{aligned}$$

Consequently, setting $y := \varphi(x, u)$, we would get

$$\big[g(x) + g(y)\big](x - y) = 2\big[G(x) - G(y)\big]$$

for all $x \in (x_0 - \delta, x_0 + \delta)$, $y \in (y_0 - \varepsilon, y_0 + \varepsilon)$, for some $\varepsilon > 0$. In particular,

$$\frac{G(x_0) - G(y_0)}{x_0 - y_0} = \frac{g(x_0) + g(y_0)}{2},$$

which contradicts to the assumption.

Now, an obvious induction proves that f and g are of the class C^∞ in an open nonempty subinterval contained in $(x_0 - \delta, x_0 + \delta)$.

To finish the proof assume that there exists an $x_0 \in I$ such that for all $y \in I$, $y \neq x_0$,

$$\frac{G(x_0) - G(y)}{x_0 - y} = \frac{g(x_0) + g(y)}{2}.$$

Then

$$g(y) = 2\frac{G(y) - G(x_0)}{y - x_0} - g(x_0), \quad y \in I,$$

which implies that g is of the class of C^∞ in $I \backslash \{x_0\}$. From (2) we infer that so is f. □

4. Main result

The main result of this paper reads as follows.

Theorem 2. *Let $I \subset \mathbb{R}$ be an open interval. Suppose that $f, g : I \to \mathbb{R}$ are continuous and strictly monotonic. Then the following conditions are equivalent:*

(i) *the arithmetic mean* A *is* (L_f, L_g)*-invariant*;
(ii) *there are* $a, c, p \in \mathbb{R} \setminus \{0\}$, $b, d \in \mathbb{R}$, *such that either*

$$f(x) = ae^{px} + b, \qquad g(x) = ce^{-px} + d, \quad x \in I,$$

or

$$f(x) = ax + b, \qquad g(x) = cx + d, \quad x \in I;$$

(iii) *there is a* $p \in \mathbb{R}$ *such that*

$$L_f(x, y) = L_{[p]}(x, y), \qquad L_g(x, y) = L_{[-p]}(x, y), \quad x, y \in I.$$

Proof. Suppose that A is (L_f, L_g)-invariant. Then the functions f and g satisfy Eq. (1). By Theorem 1, there exists a nonempty open and maximal subinterval $I_1 \subset I$ such that f and g are four times continuously differentiable and

$$f'(x) \neq 0 \neq g'(x), \quad x \in I_1.$$

It follows that the functions L_f and L_g are four-times continuously differentiable in $I_1 \times I_1$.

Denote by F a primitive function of f and put, for short, $L := L_f(x, y)$. Making some calculations, we obtain, for all $x, y \in I_1$, $x \neq y$,

$$\begin{aligned}
\frac{\partial^4 L_f}{\partial x^2 \partial y^2} = &-\frac{f'''(L)f'(L) - 3[f''(L)]^2}{[f'(L)]^5}\left[\alpha^2\beta + (\alpha^*)^2\beta^* + 4\alpha\alpha^*\eta\right] \\
&- \frac{f^{(4)}(L)f'(L) - 4f'''(L)f''(L)}{[f'(L)]^6}\alpha^2(\alpha^*)^2 \\
&+ 3\frac{2f'''(L)f''(L)f'(L) - 5[f''(L)]^3}{[f'(L)]^7}(\alpha^*)^2(\beta^*)^2 \\
&- \frac{f''(L)}{[f'(L)]^3}\left[\beta\beta^* + 2\alpha\gamma + 2\alpha^*\gamma^* + 2\eta^2\right] + \frac{\delta}{f'(L)},
\end{aligned}$$

where

$$\alpha(x, y) := \frac{f(y)(y - x) - F(y) + F(x)}{(x - y)^2}, \qquad \alpha^*(x, y) := \alpha(y, x),$$

$$\beta(x, y) := \frac{f'(x)(x - y)^2 - 2[f(x)(x - y) - F(x) + F(y)]}{(x - y)^3},$$

$$\beta^*(x, y) := \beta(y, x),$$

$$\gamma(x, y) := \frac{6[F(y) - F(x)] - 2(x - y)[2f(x) + f(y) - (x - y)^2 f'(x)]}{(x - y)^4},$$

$$\gamma^*(x, y) := \gamma(y, x),$$

$$\delta(x, y) := \frac{2\{12[F(x) - F(y)] - 6(x - y)[f(x) + f(y)] + (x - y)^2[f'(x) - f'(y)]\}}{(x - y)^5},$$

$$\eta(x, y) := \frac{[f(x) + f(y)](x - y) + 2[F(y) - F(x)]}{(x - y)^3}.$$

Since

$$\lim_{y\to x}\alpha(x,y)=\lim_{y\to x}\alpha^*(x,y)=\frac{f'(x)}{2},\qquad \lim_{y\to x}\beta(x,y)=\lim_{y\to x}\beta^*(x,y)=\frac{f''(x)}{3},$$

$$\lim_{y\to x}\gamma(x,y)=\lim_{y\to x}\gamma^*(x,y)=\frac{f'''(x)}{12},$$

$$\lim_{y\to x}\delta(x,y)=\frac{f^{(4)}(x)}{30},\qquad \lim_{y\to x}\eta(x,y)=\frac{f''(x)}{6},$$

and, obviously, $\lim_{y\to x}L(x,y)=x$, we hence get

$$\lim_{y\to x}\frac{\partial^4 L_f}{\partial x^2\partial y^2}(x,y)=\frac{1}{8}\frac{f'''(x)f''(x)}{[f'(x)]^2}+\frac{1}{144}\frac{[f''(x)]^3}{[f'(x)]^3}+\frac{13}{48}\frac{f^{(4)}(x)}{f'(x)}$$

for all $x\in I_1$. In the same way we obtain

$$\lim_{y\to x}\frac{\partial^4 L_g}{\partial x^2\partial y^2}(x,y)=\frac{1}{8}\frac{g'''(x)g''(x)}{[g'(x)]^2}+\frac{1}{144}\frac{[g''(x)]^3}{[g'(x)]^3}+\frac{13}{48}\frac{g^{(4)}(x)}{g'(x)}$$

for all $x\in I_1$. From (1) we have

$$\frac{\partial^4 L_f}{\partial x^2\partial y^2}(x,y)+\frac{\partial^4 L_g}{\partial x^2\partial y^2}(x,y)=0,\quad x,y\in I_1,$$

whence

$$\left(\frac{f'''(x)f''(x)}{[f'(x)]^2}+\frac{g'''(x)g''(x)}{[g'(x)]^2}\right)+\frac{1}{18}\left(\frac{[f''(x)]^3}{[f'(x)]^3}+\frac{[g''(x)]^3}{[g'(x)]^3}\right)$$
$$+\frac{13}{6}\left(\frac{f^{(4)}(x)}{f'(x)}+\frac{g^{(4)}(x)}{g'(x)}\right)=0,\ x\in I_1.$$

In view of Proposition 1, $f'g'$ is constant in I_1. It follows that (3) holds. Thus

$$\frac{[f''(x)]^3}{[f'(x)]^3}+\frac{[g''(x)]^3}{[g'(x)]^3}=0,\quad x\in I_1,$$

and, consequently,

$$\frac{1}{4}\left(\frac{f'''(x)f''(x)}{[f'(x)]^2}+\frac{g'''(x)g''(x)}{[g'(x)]^2}\right)+\frac{13}{3}\left(\frac{f^{(4)}(x)}{f'(x)}+\frac{g^{(4)}(x)}{g'(x)}\right)=0,\quad x\in I_1. \tag{6}$$

Since

$$g'(x)=\frac{1}{f'(x)},\quad x\in I_1,$$

we have

$$g''(x)=-\frac{f''(x)}{[f'(x)]^2},\qquad g'''(x)=\frac{2[f''(x)]^2-f'''(x)f'(x)}{[f'(x)]^3},$$
$$g^{(4)}(x)=\frac{6f'''(x)f''(x)f'(x)-6[f''(x)]^3-f^{(4)}(x)[f'(x)]^2}{[f'(x)]^4},\quad x\in I_1.$$

Setting these functions into Eq. (6), we obtain the differential equation

$$f'''(x)f'(x)-\left[f''(x)\right]^2=0, \quad x\in I_1.$$

Solving this differential equation, we infer that either

$$f(x)=ax+b, \quad x\in I_1, \tag{7}$$

for some $a,b\in\mathbb{R}$, $a\neq 0$, or

$$f(x)=ae^{px}+b, \quad x\in I_1, \tag{8}$$

for some $a,b,p\in\mathbb{R}$, $p\neq 0\neq a$. The relation $f'g'=C\in\mathbb{R}$ implies that if f is of the form (7) then, for some $c,d\in\mathbb{R}$, $c\neq 0$,

$$f(x)=cx+d, \quad x\in I_1;$$

and if f is given by (8) then, for some $c,d\in\mathbb{R}$, $c\neq 0$,

$$g(x)=ce^{-px}+d, \quad x\in I_1.$$

Since f,g are of the class C^∞ in $\mathbb{R}$ and $f'\neq 0$ and $g'\neq 0$ in $\mathbb{R}$, we infer that $I_1=I$. Thus we have proved the implication (i) $\Rightarrow$ (ii).

Since the remaining implications are obvious, the proof is completed. □

5. Final remarks

Remark 1. Let an interval $I\subset\mathbb{R}$ and a $p\in\mathbb{R}$ be arbitrarily fixed. Then

$$\lim_{n\to\infty}\left(L_{[p]},L_{[-p]}\right)^n(x,y)=\left(\frac{x+y}{2},\frac{x+y}{2}\right), \quad (x,y)\in I^2,$$

where $(L_{[p]},L_{[-p]})^n$ denotes the nth iterate of the mean-type mapping $(L_{[p]},L_{[-p]}):I^2\to I^2$ (cf. [2, Chapter Eight]).

Remark 2. Assume that $f,g:I\to\mathbb{R}$ are strictly monotonic, three-times continuously differentiable, $f'f''\neq 0\neq g'g''$ in an open interval I, and A is (L_f,L_g)-invariant. Since, for all $x\in I$,

$$\lim_{y\to x}\frac{\partial^3 L_f}{\partial x^2\partial y}(x,y)=\frac{1}{24}\left[\left(\frac{f''(x)}{f'(x)}\right)^2-\frac{f'''(x)}{f'(x)}\right],$$

$$\lim_{y\to x}\frac{\partial^3 L_g}{\partial x^2\partial y}(x,y)=\frac{1}{24}\left[\left(\frac{g''(x)}{g'(x)}\right)^2-\frac{g'''(x)}{g'(x)}\right],$$

Eq. (4) implies that

$$\left(\frac{f''(x)}{f'(x)}\right)^2-\frac{f'''(x)}{f'(x)}+\left(\frac{g''(x)}{g'(x)}\right)^2-\frac{g'''(x)}{g'(x)}=0, \quad x\in I.$$

Simple calculations show that if f,g are such that $f'g'$ is a nonzero constant, then this differential equation is satisfied. This explains why in the proof of the above theorem the fourth derivative is used.

Remark 3. If the arithmetic mean A is (L_f, L_g)-invariant then, for all $x, y \in I$,

$$L_f(x,y) = f^{-1}\left(\frac{f\left(\frac{3x+y}{2} - L_g\left(x, \frac{x+y}{2}\right)\right) + f\left(\frac{x+3y}{2} - L_g\left(\frac{x+y}{2}, y\right)\right)}{2}\right),$$

that is

$$L_f = Q_f \circ \left(M_g, M_g^*\right),$$

where Q_f is a quasi-arithmetic mean of a generator f that is

$$Q_f(x,y) := f^{-1}\left(\frac{f(x)+f(y)}{2}\right), \quad x, y \in I,$$

and M_g, M_g^* are means defined by

$$M_g(x,y) := \frac{3x+y}{2} - L_g\left(x, \frac{x+y}{2}\right), \qquad M_g^*(x,y) := M_g(y,x).$$

In fact, if A is (L_f, L_g)-invariant, then

$$F(x) = F(s) + (x-s) f\left(x + s - L_g(x,s)\right), \quad x, s \in I,$$

where F denotes a primitive function of F. Replacing here x by y, we get

$$F(y) = F(s) + (x-s) f\left(y + s - L_g(y,s)\right), \quad x, s \in I.$$

Subtracting these two equations with $s := \frac{x+y}{2}$ gives the desired formula (cf. L.R. Berrone and J. Moro [1]).

References

[1] L.R. Berrone, J. Moro, Lagrangian means, Aequationes Math. 55 (1998) 217–226.
[2] J.M. Borwein, P.B. Borwein, Pi and the AGM, Wiley, New York, 1987.
[3] P.S. Bullen, D.S. Mitrinović, P.M. Vasić, Means and Their Inequalities, Math. Appl., Reidel, Dordrecht, 1988.
[4] Z. Daróczy, Gy. Maksa, On a problem of Matkowski, Colloq. Math. 82 (1999) 117–123.
[5] Z. Daróczy, Zs. Páles, Gauss-compositions of means and the solution of the Matkowski–Suto problem, Publ. Math. Debrecen 61 (2002) 157–218.
[6] J. Matkowski, Invariant and complementary quasi-arithmetic means, Aequationes Math. 57 (1999) 87–107.
[7] J. Matkowski, Iterations of mean-type mappings and invariant means, Ann. Math. Sil. 13 (1999) 211–226.