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Elliptic Means and Their Generalizations

By

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Abstract

In this note we introduce a one-parameter family of homogeneous means strictly related to ellipses. Each member of the family is a weighted power mean, and only one of them is both symmetric and quasi-arithmetic. Geometric interpretations are given, and higher-dimensional counterparts of these means are defined. Iterations of some mean-type mappings and some functional equations are considered.

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1. Introduction

Let $I \subset \mathbb{R}$ be an interval and $k \in \mathbb{N}$, $k \geq 2$, fixed. A function $M: I^k \to \mathbb{R}$ is said to be a *mean* if

$$\min(x_1,\ldots,x_k) \leq M(x_1,\ldots,x_k) \leq \max(x_1,\ldots,x_k), \quad x_1,\ldots,x_k \in I.$$

A mean M is called *strict* if $\min(x_1,\ldots,x_k) < \max(x_1,\ldots,x_k)$ implies that the above inequalities are strict, and M is called *symmetric* if, for every permutation $\sigma: \{1,\ldots,k\} \to \{1,\ldots,k\}$,

$$M(x_1,\ldots,x_k) = M(x_{\sigma(1)},\ldots,x_{\sigma(k)}), \quad x_1,\ldots,x_k \in I.$$

For any continuous and strictly monotonic function $\varphi: I \to \mathbb{R}$ and a sequence $w = (w_1,\ldots,w_k)$, $w_1 > 0,\ldots,w_k > 0$, $w_1 + \cdots$

$+w_k = 1$, the function $M^{\varphi}_{k,w}: I^k \to \mathbb{R}$,

$$M^{\varphi}_{k,w}(x_1,\ldots,x_k) := \varphi^{-1}(w_1\varphi(x_1) + \cdots + w_k\varphi(x_k)), \quad x_1,\ldots,x_k \in I,$$

is a strict mean, and it is called a *weighted quasi-arithmetic mean*. The function φ is referred to as a generator of the mean $M^{\varphi}_{k,w}$ and the numbers $w_1,\ldots,w_k$ as its weights. $M^{\varphi}_{k,w}$ is symmetric iff $M^{\varphi}_{k,w} = M^{\varphi}_k$ where

$$M^{\varphi}_k(x_1,\ldots,x_k) := \varphi^{-1}\left(\frac{\varphi(x_1) + \cdots + \varphi(x_k)}{k}\right), \quad x_1,\ldots,x_k \in I,$$

and M^{φ}_k is called a *quasi-arithmetic mean*.

A mean $M: (0,\infty)^k \to (0,\infty)$ is called *homogeneous* if

$$M(tx_1,\ldots,tx_k) = tM(x_1,\ldots,x_k), \quad t, x_1,\ldots,x_k > 0.$$

It is well known (cf. B. Jessen [4], also G. H. Hardy, J. E. Littlewood and G. Pólya [2], p. 68) that a weighted quasi-arithmetic mean is homogeneous iff it is a weighted power mean, that is, there is an $r \in \mathbb{R}$ such that $M^{[\varphi]}_{k,w} = M^{[r]}_{k,w}$ where

$$M^{[r]}_{k,w}(x_1,\ldots,x_k) := \begin{cases} (w_1x_1^r + \cdots + w_kx_k^r)^{1/r} & \text{for} \quad r \neq 0, \\ x_1^{w_1}\cdots x_k^{w_k} & \text{for} \quad r = 0. \end{cases}$$

Note that $M^{[0]}_k = G_k$ where G_k denotes the geometric mean.

In this note we show that the means $M^{[-2]}_{2,w}$ are strictly related to an ellipse, $M^{[-2]}_{3,w}$ to an ellipsoid, and $M^{[-2]}_{k,w}$ to a k-dimensional ellipsoid. In the first four sections we distinguish them by suitable symbols and formulate some of their properties as Propositions. In Section 5 we apply these Propositions to find all continuous solutions of a functional equation involving these means, closely related to the iteration of mean-type mappings.

2. Elliptic Means

We begin this section with the following quickly verifiable

Remark 1. *Let $p > 0$ be fixed. Then the function $E_p: (0,\infty)^2 \to (0,\infty)$,*

$$E_p(a,b) := ab\sqrt{\frac{p^2+1}{p^2a^2+b^2}}$$

is a mean.

We call the means E_p *elliptic* which is justified by the following

Geometric Interpretation. Consider an ellipse given by the equation

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

where $a, b>0$, and take an arbitrary $p>0$. It may be shown that the length $|OP|$, where O is the center of the ellipse and P is the intersection point of the ellipse and the half-line

$$y = px, \quad x \geq 0,$$

is given by $|OP| = E_p(a, b)$.

Proposition 1.

1. *For every $p>0$, E_p is a weighted power mean $M_{2,w}^{[-2]}$ with weights*

$$w = \left(\frac{1}{p^2+1}, \frac{p^2}{p^2+1}\right);$$

in particular, it is homogeneous, and the function $\varphi(t) = t^{-2}$ $(t>0)$ is a generator of this mean.

2. *E_p is symmetric iff $p = 1$;*
3. *E_p is quasi-arithmetic iff $p = 1$; moreover,*

$$E_1 = M_2^{[-2]}.$$

4. *For every $p>0$, E_p is G_2-conjugate to the weighted square-root mean $M_{2,w*}^{[2]}$ with weights*

$$w^* = \left(\frac{p^2}{p^2+1}, \frac{1}{p^2+1}\right),$$

that is

$$G_2 \circ (E_p, M_{2,w*}^{[2]}) = G_2,$$

where G_2 denotes the geometric mean. G_2 is the unique continuous mean which is invariant with respect to the mean-type mapping $(E_p, M_{2,w*}^{[2]})$: $(0,\infty)^2 \to (0,\infty)^2$; moreover, the sequence of iterates of the mean-type mapping $(E_p, M_{2,w*}^{[2]})$ converges to the mean-type mapping (G_2, G_2) in $(0,\infty)^2$.

5. *$\lim_{p\to 0} E_p(x, y) = x$ and $\lim_{p\to\infty} E_p(x, y) = y$ for all $x, y>0$.*

Proof. Parts 1–3 and 5 are not hard to verify. Part 4 is a consequence of some more general facts (the conjugate and invariant means were considered in [5] and [6], cf. also [1]). □

Remark 2. *It can be readily shown that the following commutation relation (involving a parameter transformation) holds:*

$$E_p(a,b) = E_{1/p}(b,a), \quad p,a,b>0.$$

3. Ellipsoidal Means

Let $p,q>0$ be fixed. Then the function $E_{p,q}$: $(0,\infty)^3 \to (0,\infty)$,

$$E_{p,q}(a,b,c) := abc\sqrt{\frac{p^2+q^2+1}{b^2c^2+p^2c^2a^2+q^2a^2b^2}}$$

is a mean. $E_{p,q}$ can be called an ellipsoidal mean because of the following

Geometric Interpretation. Consider an ellipsoid given by the equation

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$$

where $a,b,c>0$, and take the half-line determined by the equations

$$y=px, \quad z=qx, \quad x\geq 0,$$

for some arbitrary $p,q>0$. Calculations show that $E_{p,q}(a,b,c)$ is the length $|OP|$ where O is the center of the ellipsoid and P is the point of intersection of the ellipsoid and the half-line.

Proposition 2.

1. *For all $p,q>0$, $E_{p,q}$ is a weighted power mean $M_{3,w}^{[-2]}$ with weights*

$$w=\left(\frac{1}{p^2+q^2+1},\frac{p^2}{p^2+q^2+1},\frac{q^2}{p^2+q^2+1}\right);$$

   in particular, it is homogeneous, and the function $\varphi(t)=t^{-2}$ $(t>0)$ is a generator of this mean.
2. *$E_{p,q}$ is symmetric iff $p=q=1$;*
3. *$E_{p,q}$ is quasi-arithmetic iff $p=q=1$; moreover,*

$$E_{1,1}=M_3^{[-2]}.$$

4. *G_3 is the unique continuous mean which is invariant with respect to the mean-type mapping $\left(E_{p,q}, K_{p,q}, M^{[2]}_{3,w^*}\right)$: $(0,\infty)^3 \to (0,\infty)^3$ where the mean $K_{p,q}$: $(0,\infty)^3 \to (0,\infty)$ is defined by*

$$K_{p,q}(a,b,c) := \sqrt{\frac{p^2b^2c^2 + q^2a^2b^2 + c^2a^2}{p^2b^2 + q^2a^2 + c^2}},$$

$$w^* = \left(\frac{q^2}{p^2+q^2+1}, \frac{p^2}{p^2+q^2+1}, \frac{1}{p^2+q^2+1}\right)$$

that is

$$G_3 \circ \left(E_{p,q}, K_{p,q}, M^{[2]}_{3,w^*}\right) = G_3;$$

the sequence of iterates of the mean-type mapping $\left(E_{p,q}, K_{p,q}, M^{[2]}_{3,w^*}\right)$ converges to the mean-type mapping (G_3, G_3, G_3) in $(0,\infty)^3$.

Moreover, $K_{p,q}$ is symmetric iff $p = q = 1$.

5. $\lim_{q\to 0} E_{p,q}(x,y,z) = E_p(x,y)$ *and* $\lim_{q\to\infty} E_{p,q}(x,y,z) = z$ *for all* $x, y, z > 0$.

Remark 3. *The following commutation relations (involving some parameter transformations) can be readily verified*:

$$\begin{aligned} E_{p,q}(a,b,c) &= E_{q,p}(a,c,b) = E_{1/p,q/p}(b,a,c) = E_{q/p,1/p}(b,c,a) \\ &= E_{p/q,1/q}(c,b,a) = E_{1/q,p/q}(c,a,b) \end{aligned}$$

for all $a, b, c, p, q > 0$.

4. The General k-Dimensional Case

Let $k \in \mathbb{N}$, $k \geq 2$, and $p_1, \ldots, p_{k-1} > 0$ be fixed. Then the function $E_{p_1,\ldots,p_{k-1}}$: $(0,\infty)^k \to (0,\infty)$ defined by

$$\begin{aligned} &E_{p_1,\ldots,p_{k-1}}(a_1,\ldots,a_k) \\ &\quad := a_1\cdots a_k \sqrt{\frac{p_1^2 + \cdots + p_{k-1}^2 + 1}{p_1^2 a_2^2 \cdots a_k^2 + \cdots + p_{k-1}^2 a_1^2 \cdots a_{k-2}^2 a_k^2 + a_1^2 \cdots a_{k-1}^2}} \\ &\times \left(E_{p_1,\ldots,p_{k-1}}(a_1,\ldots,a_k) := \left(\prod_{i=1}^k a_i\right) \sqrt{\frac{\sum_{i=1}^{k-1} p_i^2 + 1}{\sum_{j=1}^{k-1} p_j^2 \prod_{i\neq j}^k a_i^2 + \prod_{i=1}^{k-1} a_i^2}}\right) \end{aligned}$$

is a mean. This mean may be referred to as k-dimensional ellipsoidal mean by an analogous geometric interpretation as in the previous cases.

Proposition 3.

1. *For all* $p_1, \dots, p_{k-1} > 0$, $E_{p_1,\dots,p_{k-1}}$ *is a weighted power mean* $M_{k,w}^{[-2]}$ *with weights*

$$w = \left(\frac{1}{\sum_{i=1}^{k-1} p_i^2 + 1}, \frac{p_1^2}{\sum_{i=1}^{k-1} p_i^2 + 1}, \dots, \frac{p_{k-1}^2}{\sum_{i=1}^{k-1} p_i^2 + 1} \right);$$

   in particular, it is homogeneous, and the function $\varphi(t) = t^{-2}$ $(t > 0)$ *is a generator of this mean.*
2. $E_{p_1,\dots,p_{k-1}}$ *is symmetric iff* $p_1 = \dots = p_{k-1} = 1$;
3. $E_{p_1,\dots,p_{k-1}}$ *is quasi-arithmetic iff* $p_1 = \dots = p_{k-1} = 1$; *moreover,*

$$E_{1,\dots,1} = M_k^{[-2]}.$$

4. G_k *is the unique continuous mean which is invariant with respect to the mean-type mapping* $(E_{1,\dots,1}, K_1, \dots, K_{k-1})\colon (0,\infty)^k \to (0,\infty)^k$ *where the means* $K_l\colon (0,\infty)^k \to (0,\infty)$, $l = 1, \dots, k-1$, *are defined by*

$$K_l(a_1, \dots, a_k) := \sqrt{\frac{(k-l) \sum_{j_1 < \dots < j_l} \prod_{i \notin \{j_1,\dots,j_l\}} a_i^2}{(l+1) \sum_{j_1 < \dots < j_{l+1}} \prod_{i \notin \{j_1,\dots,j_{l+1}\}} a_i^2}},$$

   that is

$$G_k \circ (E_{1,\dots,1}, K_1, \dots, K_{k-1}) = G_k;$$

   moreover, $K_{k-1} = M_k^{[2]}$, *and the sequence of iterates of the mean-type mapping* $(E_{1,\dots,1}, K_1, \dots, K_{k-2}, M_k^{[2]})$ *converges to the mean-type mapping* $(G_k, \dots, G_k)$ *in* $(0,\infty)^k$.

5. An Application to a Functional Equation

Haruki and Rassias [3] (cf. also [8]) posed the following

Problem 1. *Is it true that every continuous function* $f\colon (0,\infty) \times (0,\infty) \to \mathbb{R}$ *satisfying the functional equation*

$$f\left(\frac{2xy}{x+y}, \frac{x+y}{2} \right) = f(x,y), \quad x, y > 0,$$

is of the form

$$f(x,y) = F(xy), \quad x, y > 0,$$

where $F\colon (0,\infty) \to \mathbb{R}$ *is a continuous function of a single variable?*

The affirmative answer has been given by the second author (cf. [7]). Functional equations of the form

$$f(M(x,y),N(x,y)) = f(x,y), \quad x,y \in I,$$

where $M,N\colon I^2 \to I$ are means in an interval I, play an essential role in some problems connected with iterations of means. For

$$M(x,y) := \frac{x+y}{2} \quad \text{and} \quad N(x,y) := \sqrt{xy},$$

this equation appears in connection with the *AGM* iteration of Gauss and elliptic integrals (cf. for instance [1]).

Applying our Proposition 1 we can prove the following

Theorem 1. *Let $p>0$ be arbitrarily fixed. Suppose that $f\colon (0,\infty)^2 \to \mathbb{R}$ is continuous on the diagonal $\Delta := \{(x,x)\colon x>0\}$. Then f satisfies the functional equation*

$$f\left(xy\sqrt{\frac{p^2+1}{p^2x^2+y^2}}, \sqrt{\frac{p^2x^2+y^2}{p^2+1}}\right) = f(x,y), \quad x,y>0, \tag{1}$$

if, and only if,

$$f(x,y) = F(xy), \quad x,y>0,$$

where $F\colon (0,\infty) \to \mathbb{R}$ is a continuous function of a single variable.

Proof. Suppose that $f\colon (0,\infty)\times(0,\infty) \to \mathbb{R}$ is continuous and satisfies equation (1). Since

$$E_p(x,y) = xy\sqrt{\frac{p^2+1}{p^2x^2+y^2}}, \qquad M^{[2]}_{2,w^*}(x,y) = xy\sqrt{\frac{p^2x^2+y^2}{p^2+1}}, \quad x,y>0,$$

we can write equation (1) in the form

$$f \circ [(E_p, M^{[2]}_{2,w^*})] = f.$$

Hence, by induction, we get

$$f \circ [(E_p, M^{[2]}_{2,w^*})^n] = f, \quad n \in \mathbb{N},$$

where $(E_p, M^{[2]}_{2,w^*})^n$ denotes the n-th iteration of the mean-type mapping $(E_p, M^{[2]}_{2,w^*})$. By Proposition 1, letting here $n \to \infty$ and making use of the continuity of f on Δ, we obtain

$$f \circ [(G_2, G_2)] = f,$$

that is

$$f(x,y) = f[(\sqrt{xy}, \sqrt{xy})], \quad x,y>0.$$

Setting

$$F(u) := f(\sqrt{u}, \sqrt{u}), \quad u>0,$$

we have

$$f(x,y) = F(xy), \quad x,y>0.$$

Since the converse implication requires only simple calculations, the proof is complete. □

Similarly, applying Proposition 2, we can prove

Theorem 2. *Let $p,q>0$ be fixed. Suppose that $f\colon (0,\infty)^3 \to \mathbb{R}$ is continuous on the diagonal $\Delta := \{(x,x,x)\colon x>0\}$. Then f satisfies the functional equation*

$$f \circ (E_{p,q}(x,y,z), K_{p,q}(x,y,z), M^{[2]}_{3,w^*}(x,y,z)) = f(x,y,z), \quad x,y,z>0,$$

if, and only if,

$$f(x,y,z) = F(xyz), \quad x,y,z>0,$$

where $F\colon (0,\infty) \to \mathbb{R}$ is a continuous function of a single variable.

Remark 4. *A k-dimensional counterpart of the above results is also true. Its version for the symmetric means results from Proposition* 3.

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