

## The Converse Theorem for Minkowski's Inequality

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## ABSTRACT

Let  $(\Omega, \Sigma, \mu)$  be a measure space and  $\varphi, \psi : (0, \infty) \rightarrow (0, \infty)$  some bijective functions. Suppose that the functional  $\mathbb{P}_{\varphi, \psi}$  defined on class of  $\mu$ -integrable simple functions  $x : \Omega \rightarrow [0, \infty)$ ,  $\mu(\{\varpi : x(\varpi) > 0\}) > 0$ , by the formula

$$\mathbb{P}_{\varphi, \psi}(x) = \psi \left( \int_{\{x>0\}} \varphi \circ x d\mu \right)$$

satisfies the triangle inequality. We prove that if there are  $A, B \in \Sigma$  such that  $0 < \mu(A) < 1 < \mu(B) < \infty$ , the function  $\psi \circ \varphi$  is superadditive, and  $\lim_{t \rightarrow 0} \psi(t) = 0$  then there is a  $p \geq 1$  such that

$$\varphi(t) = \varphi(1)t^p, \quad \psi(t) = \psi(1)t^{1/p}, \quad t > 0.$$

The assumption  $\lim_{t \rightarrow 0} \psi(t) = 0$  can be significantly weakened or, for some measure spaces, even omitted. The remaining assumptions are essential. In particular, in each of the cases: (i)  $A \in \Sigma \Rightarrow \mu(A) = 0$  or  $\mu(A) \geq 1$ ; (ii)  $A \in \Sigma \Rightarrow \mu(A) \leq 1$  or  $\mu(A) = \infty$ , some broad classes of pairs  $(\varphi, \psi)$  of non-power functions for which  $\mathbb{P}_{\varphi, \psi}$  is subadditive are indicated. These results give a solution of an open problem posed by W. Wnuk. The reversed triangle inequality is also considered.

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## 1. INTRODUCTION

For a measure space  $(\Omega, \Sigma, \mu)$  denote by  $S = S(\Omega, \Sigma, \mu)$  the linear real space of all  $\mu$ -integrable simple functions  $x : \Omega \rightarrow \mathbb{R}$  and, for  $x \in S$ , put

$$\Omega(x) := \{\omega \in \Omega : x(\omega) \neq 0\}.$$

Note that for two arbitrarily fixed bijections  $\varphi, \psi : (0, \infty) \rightarrow (0, \infty)$ , the functional  $\mathbb{P}_{\varphi, \psi} : S \rightarrow [0, \infty)$  given by the formula

$$\mathbb{P}_{\varphi, \psi}(x) := \begin{cases} \psi\left(\int_{\Omega(x)} \varphi \circ |x| d\mu\right) & \text{if } \mu(\Omega(x)) > 0, \\ 0 & \text{if } \mu(\Omega(x)) = 0, \end{cases}$$

is correctly defined. Moreover taking

$$\varphi(t) = \varphi(1)t^p, \quad \psi(t) = \psi(1)t^{1/p}, \quad t > 0,$$

where  $p \geq 1$  and  $\varphi(1), \psi(1) > 0$  are arbitrarily fixed, the functional  $\mathbb{P}_{\varphi, \psi}$  is equal to the  $L^p$ -norm up to a multiplicative constant; in particular, the following implication is an obvious consequence of the Minkowski inequality:

$$\left(\frac{\varphi(t)}{\varphi(1)} = t^p \wedge \frac{\psi(t)}{\psi(1)} = t^{1/p}\right) \Rightarrow \forall x, y \in S \quad \mathbb{P}_{\varphi, \psi}(x + y) \leq \mathbb{P}_{\varphi, \psi}(x) + \mathbb{P}_{\varphi, \psi}(y).$$

In the present paper, assuming superadditivity of the function  $\psi \circ \varphi$  and some weak regularity conditions of  $\psi$ , we show that if in the underlying measure space  $(\Omega, \Sigma, \mu)$  there are two sets  $A, B \in \Sigma$  such that

$$0 < \mu(A) < 1 < \mu(B) < \infty,$$

then the converse implication holds true. The existence of these two sets plays here a crucial role. If a measure space fails to satisfy this condition, then there are some broad classes of pairs  $(\varphi, \psi)$  of non-power functions for which the functional  $\mathbb{P}_{\varphi, \psi}$  satisfies the triangle inequality (Propositions 2 and 3). The superadditivity condition of the function  $\psi \circ \varphi$  is also indispensable. In Theorem 1 we assume that  $\lim_{t \rightarrow 0+} \psi(t) = 0$ . This regularity assumption can be either significantly weakened (Remark 5) or, if the range of the measure is sufficiently rich, even removed (Theorem 2). Therefore we conjecture that it is superfluous. The relevant results hold true for the functionals  $\mathbb{P}_{\varphi, \psi} : S_+ \rightarrow [0, \infty)$  satisfying the reversed inequality, where  $S_+ := \{x \in S : x \geq 0\}$ . In this case no regularity assumptions of the functions  $\psi$  and  $\varphi$  are required.

Theorem 1 with  $\psi := \varphi^{-1}$  gives an improvement of the main result of [3].

This paper gives a solution of a problem posed by W. Wnuk in [11].

## 2. SOME AUXILIARY RESULTS

We shall need the following result about subadditive functions

**Lemma 1.** ([4]) *If  $\psi : (0, \infty) \rightarrow (0, \infty)$  is subadditive, one-to-one, and  $\lim_{t \rightarrow 0} \psi(t) = 0$ , then  $\psi$  is an increasing and continuous.*

**Remark 1.** Note that if  $\psi : (0, \infty) \rightarrow \mathbb{R}$  is such that the function  $(0, \infty) \in t \rightarrow \frac{\psi(t)}{t}$  is decreasing (increasing), then  $\psi$  is subadditive (superadditive).

The next lemma is a special case of a more general result in [5].

**Lemma 2.** Let real numbers  $a, b$  such that  $0 < a < 1 < a + b$  be fixed. Then a function  $F : (0, \infty)^2 \rightarrow [0, \infty)$  satisfies the inequality

$$aF(x_1, x_2) + bF(y_1, y_2) \leq F(ax_1 + by_1, ax_2 + by_2), \quad x_1, x_2, y_1, y_2 > 0,$$

if, and only if, the function  $F$  is positively homogeneous, i.e.

$$F(tx_1, tx_2) = tF(x_1, x_2), \quad t, x_1, x_2 > 0.$$

### 3. MAIN RESULTS

The characteristic function of a set  $A \subset \Omega$  is denoted by  $\chi_A$ .

We begin with the following

**Proposition 1.** Let  $(\Omega, \Sigma, \mu)$  be a measure space such that there are two sets  $A, B \in \Sigma$  satisfying the condition

$$0 < \mu(A) < 1 < \mu(B) < \infty.$$

Suppose that  $\varphi, \psi : (0, \infty) \rightarrow (0, \infty)$  are bijective functions such that the function  $\psi \circ \varphi$  is superadditive and

$$\lim_{t \rightarrow 0} \psi(t) = 0.$$

Then the following conditions are equivalent:

(i) the functional  $\mathbb{P}_{\varphi, \psi}$  satisfies the inequality

$$\mathbb{P}_{\varphi, \psi}(x + y) \leq \mathbb{P}_{\varphi, \psi}(x) + \mathbb{P}_{\varphi, \psi}(y), \quad x, y \in S_+(A, B),$$

where  $S_+(A, B) := \{x_1\chi_A + x_2\chi_{B \setminus A} \in S : x_1, x_2 \geq 0\}$ ;

(ii) there is a real  $p \geq 1$  such that

$$\varphi(t) = \varphi(1)t^p, \quad \psi(t) = \psi(1)t^{1/p}, \quad t > 0;$$

(iii) there are some real  $p \geq 1$  and  $c > 0$  such that

$$\mathbb{P}_{\varphi, \psi}(x) = c \left( \int_{\Omega} |x|^p d\mu \right)^{1/p}, \quad x \in S;$$

(iv) the functional  $\mathbb{P}_{\varphi, \psi} : S \rightarrow [0, \infty)$  satisfies the triangle inequality

$$\mathbb{P}_{\varphi, \psi}(x + y) \leq \mathbb{P}_{\varphi, \psi}(x) + \mathbb{P}_{\varphi, \psi}(y), \quad x, y \in S.$$

**Proof.** To show the implication (i)  $\Rightarrow$  (ii) suppose that (i) holds true and put  $a := \mu(A)$ ,  $b := \mu(B \setminus A)$ . Then, obviously,

$$0 < a < 1 < a + b.$$

For all  $x_1, x_2 \geq 0$ , the functions  $x := x_1 \chi_A$ ,  $y := x_2 \chi_{B \setminus A}$  belong to  $S_+(A, B)$ . Therefore, in view of (i), for all  $x_1, x_2 > 0$ ,

$$\begin{aligned} \psi(a\varphi(x_1) + b\varphi(x_2)) &= \mathbb{P}_{\varphi, \psi}(x + y) \\ &\leq \mathbb{P}_{\varphi, \psi}(x) + \mathbb{P}_{\varphi, \psi}(y) = \psi(a\varphi(x_1)) + \psi(b\varphi(x_2)). \end{aligned}$$

Taking here  $x_1 := \varphi^{-1}(\frac{s}{a})$ ,  $x_2 := \varphi^{-1}(\frac{t}{b})$ , where  $s, t > 0$ , gives

$$\psi(s + t) \leq \psi(s) + \psi(t), \quad s, t > 0.$$

Since  $\psi$  is one-to-one and  $\lim_{t \rightarrow 0} \psi(t) = 0$ , by Lemma 1, the function  $\psi$  is an increasing homeomorphism of  $(0, \infty)$ . Setting

$$x := x_1 \chi_A + x_2 \chi_{B \setminus A}, \quad y := y_1 \chi_A + y_2 \chi_{B \setminus A}, \quad x_1, x_2, y_1, y_2 > 0,$$

in the assumed inequality we get

$$(1) \quad \psi(a\varphi(x_1 + y_1) + b\varphi(x_2 + y_2)) \leq \psi(a\varphi(x_1) + b\varphi(x_2)) + \psi(a\varphi(y_1) + b\varphi(y_2))$$

for all  $x_1, x_2, y_1, y_2 > 0$ . Replacing  $x_i$  by  $\varphi^{-1}(x_i)$ ,  $y_i$  by  $\varphi^{-1}(y_i)$  for  $i = 1, 2$ , and making use of the strict monotonicity of  $\psi$ , we obtain

$$(2) \quad \begin{aligned} a\varphi(\varphi^{-1}(x_1) + \varphi^{-1}(y_1)) + b\varphi(\varphi^{-1}(x_2) + \varphi^{-1}(y_2)) \\ \leq \psi^{-1}(\psi(ax_1 + bx_2) + \psi(ay_1 + by_2)) \end{aligned}$$

for all  $x_1, x_2, y_1, y_2 > 0$ .

By assumption the function  $\psi \circ \varphi$  is superadditive which means that

$$\psi(\varphi(s + t)) \geq \psi(\varphi(s)) + \psi(\varphi(t)), \quad s, t > 0.$$

Replacing  $s$  by  $\varphi^{-1}(s)$ ,  $t$  by  $\varphi^{-1}(t)$ , and making use of the increasing monotonicity of  $\psi$ , we can write this inequality in the following equivalent form

$$(3) \quad \psi^{-1}(\psi(s) + \psi(t)) \leq \varphi(\varphi^{-1}(s) + \varphi^{-1}(t)), \quad s, t > 0.$$

This inequality and (2) imply that, for all  $x_1, x_2, y_1, y_2 > 0$ ,

$$a\psi^{-1}(\psi(x_1) + \psi(y_1)) + b\psi^{-1}(\psi(x_2) + \psi(y_2)) \leq \psi^{-1}(\psi(ax_1 + bx_2) + \psi(ay_1 + by_2))$$

Thus the function  $F : (0, \infty)^2 \rightarrow (0, \infty)$  defined by

$$F(x_1, x_2) := \psi^{-1}(\psi(x_1) + \psi(x_2)), \quad x_1, x_2 > 0,$$

satisfies the inequality

$$aF(x_1, x_2) + bF(y_1, y_2) \leq F(ax_1 + by_1, ax_2 + by_2) \quad x_1, x_2, y_1, y_2 > 0.$$

By Lemma 2 the function  $F$  is positively homogeneous, i.e.

$$\psi^{-1}(\psi(tx_1) + \psi(tx_2)) = t\psi^{-1}(\psi(x_1) + \psi(x_2)), \quad t, x_1, x_2 > 0.$$

Replacing  $x_1$  by  $\psi^{-1}(x_1)$ ,  $x_2$  by  $\psi^{-1}(x_2)$ , we hence get

$$\psi(t\psi^{-1}(x_1 + x_2)) = \psi(t\psi^{-1}(x_1)) + \psi(t\psi^{-1}(x_2)), \quad t, x_1, x_2 > 0,$$

which proves that, for every fixed  $t > 0$ , the function  $\psi \circ (t\psi^{-1})$  is additive. The continuity of  $\psi \circ (t\psi^{-1})$  implies that

$$(4) \quad \psi(t\psi^{-1}(u)) = m(t)u, \quad t, u > 0,$$

for a function  $m : (0, \infty) \rightarrow (0, \infty)$ . Hence

$$(5) \quad m(t) = \psi(t\psi^{-1}(1)), \quad t > 0,$$

and, consequently,  $m$  is an increasing homeomorphism of  $(0, \infty)$ . By (4), for all  $s, t, u > 0$ , we have

$$m(st)u = \psi(st\psi^{-1}(u)) = [\psi \circ (s(\psi^{-1}) \circ (\psi \circ (t\psi^{-1})))](u) = m(s)m(t)u.$$

Taking here  $u = 1$  we get that  $m$  is multiplicative:

$$m(st) = m(s)m(t), \quad s, t > 0.$$

The increasing monotonicity of  $m$  implies that there exists a  $q > 0$  such that

$$m(t) = t^q, \quad t > 0.$$

From (5) we infer that

$$\psi(t) = \psi(1)t^q, \quad t > 0,$$

whence

$$(\psi \circ \varphi)(t) = \psi(1)[\varphi(t)]^q, \quad t > 0.$$

The function  $\psi \circ \varphi$ , being superadditive and positive, is strictly increasing. The last relation implies that  $\varphi$  is also strictly increasing and, consequently,  $\varphi$  is an increasing homeomorphism of  $(0, \infty)$ . From (2) and (3) we have

$$\begin{aligned} & a\varphi(\varphi^{-1}(x_1) + \varphi^{-1}(y_1)) + b\varphi(\varphi^{-1}(x_2) + \varphi^{-1}(y_2)) \\ & \leq \varphi(\varphi^{-1}(ax_1 + bx_2) + \varphi^{-1}(ay_1 + by_2)) \end{aligned}$$

for all  $x_1, x_2, y_1, y_2 > 0$ .

Now, in the same way as in the case of the function  $\psi$ , we can show that there is a  $p > 0$  such that

$$\varphi(t) = \varphi(1)t^p, \quad t > 0.$$

Since

$$\psi \circ \varphi(t) = [(\psi \circ \varphi)(1)]t^{pq}, \quad t > 0,$$

and the function  $\psi \circ \varphi$  is superadditive, applying Remark 1, we infer that  $pq \geq 1$ . On the other hand, substituting  $\varphi(t) = \varphi(1)t^p$  and  $\psi(t) = \psi(1)t^q$  for  $t > 0$ , we obtain, for all  $x_1, x_2, y_1, y_2 > 0$ ,

$$[a(x_1 + y_1)^p + b(x_2 + y_2)^p]^q \leq (ax_1^p + bx_2^p)^q + (ay_1^p + by_2^p)^q.$$

Replacing here  $x_1, y_1, x_2, y_2$ , respectively, by

$$a^{-\frac{1}{p}}x_1, \quad a^{-\frac{1}{p}}y_1, \quad b^{-\frac{1}{p}}x_2, \quad b^{-\frac{1}{p}}y_2,$$

we obtain the inequality

$$[(x_1 + y_1)^p + (x_2 + y_2)^p]^q \leq (x_1^p + x_2^p)^q + (y_1^p + y_2^p)^q, \quad x_1, x_2, y_1, y_2 > 0.$$

Setting here  $x_1 = x_2 = y_1 = y_2 = 1$  we get  $2^{pq+q} \leq 2^{1+q}$ , whence  $pq \leq 1$  and, consequently,  $q = \frac{1}{p}$ . The above inequality with  $q = \frac{1}{p}$  becomes the simplest version of the Minkowski inequality which is known to hold only if  $p \geq 1$ . This completes the proof of the implication (i)  $\Rightarrow$  (ii).

The remaining implications follow from the Minkowski inequality.  $\square$

For a measure space  $(\Omega, \Sigma, \mu)$  put

$$S_+ = S_+(\Omega, \Sigma, \mu) := \{x \in S : x \geq 0\}.$$

As an immediate consequence of Proposition 1 we obtain the following

**Theorem 1.** *Let  $(\Omega, \Sigma, \mu)$  be a measure space such that there are two sets  $A, B \in \Sigma$  satisfying the condition*

$$0 < \mu(A) < 1 < \mu(B) < \infty.$$

*Suppose that  $\varphi, \psi : (0, \infty) \rightarrow (0, \infty)$  are bijective functions such that  $\psi \circ \varphi$  is superadditive and  $\lim_{t \rightarrow 0} \psi(t) = 0$ . Then the inequality*

$$\mathbb{P}_{\varphi, \psi}(x + y) \leq \mathbb{P}_{\varphi, \psi}(x) + \mathbb{P}_{\varphi, \psi}(y), \quad x, y \in S_+(\Omega, \Sigma, \mu),$$

*holds true if, and only if, there is a  $p \geq 1$  such that*

$$\varphi(t) = \varphi(1)t^p, \quad \psi(t) = \psi(1)t^{1/p}, \quad t > 0.$$

Taking here  $\psi := \varphi^{-1}$  we obtain the following improvement of the main result of [3] where  $\varphi$  is assumed to be defined on the closed half-line  $[0, \infty)$ .

**Corollary 1.** *Let  $(\Omega, \Sigma, \mu)$  be a measure space such that there are two sets  $A, B \in \Sigma$  satisfying the condition  $0 < \mu(A) < 1 < \mu(B) < \infty$ . Suppose that  $\varphi : (0, \infty) \rightarrow (0, \infty)$  is bijective and such that  $\lim_{t \rightarrow 0} \varphi^{-1}(t) = 0$ . If the functional  $\mathbb{P}_\varphi := \mathbb{P}_{\varphi, \varphi^{-1}}$  satisfies the inequality*

$$\mathbb{P}_\varphi(x + y) \leq \mathbb{P}_\varphi(x) + \mathbb{P}_\varphi(y), \quad x, y \in S_+(\Omega, \Sigma, \mu), \quad x, y \geq 0,$$

*then there exists a real  $p \geq 1$  such that*

$$\varphi(t) = \varphi(1)t^p, \quad t > 0.$$

**Proof.** Since  $\mathbb{P}_\varphi := \mathbb{P}_{\varphi, \psi}$  where  $\psi = \varphi^{-1}$ , the function  $\psi \circ \varphi = id|_{(0, \infty)}$  being additive, is superadditive. Moreover, by the assumption,  $\lim_{t \rightarrow 0} \psi(t) = 0$ . Now the result follows from Theorem 1.  $\square$

**Remark 2.** In Theorem 1 the assumption about the measure space  $(\Omega, \Sigma, \mu)$  is essential.

Note that it is not satisfied if, and only if, one of the following cases occurs:

- (i) for every  $A \in \Sigma$ , we have  $\mu(A) = 0$  or  $\mu(A) \geq 1$ ;
- (ii) for every  $A \in \Sigma$ , we have  $\mu(A) \leq 1$  or  $\mu(A) = \infty$ .

In each of these two cases we shall indicate some large classes of pairs  $(\varphi, \psi)$  of non-power functions for which  $\mathbb{P}_{\varphi, \psi}$  satisfies the Minkowski type inequality.

**Proposition 2.** Let  $(\Omega, \Sigma, \mu)$  be a measure space such that for every  $A \in \Sigma$

$$\mu(A) = 0 \quad \text{or} \quad \mu(A) \geq 1.$$

If  $\varphi, \psi : (0, \infty) \rightarrow (0, \infty)$  are increasing,  $\varphi$  is bijective and convex,  $\log \circ \varphi \circ \exp$  is convex and  $\psi \circ \varphi$  is subadditive, then

$$\mathbb{P}_{\varphi, \psi}(x + y) \leq \mathbb{P}_{\varphi, \psi}(x) + \mathbb{P}_{\varphi, \psi}(y), \quad x, y \in S(\Omega, \Sigma, \mu).$$

**Proof.** In view of the main result of [6], generalizing a well known Mulholland's inequality ([9]), the assumptions of  $\varphi$  imply that

$$(6) \quad \varphi^{-1} \left( \int_{\Omega(x+y)} \varphi \circ |x + y| d\mu \right) \leq \varphi^{-1} \left( \int_{\Omega(x)} \varphi \circ |x| d\mu \right) + \varphi^{-1} \left( \int_{\Omega(y)} \varphi \circ |y| d\mu \right)$$

for all  $x, y \in S$  such that  $\mu(\Omega(x)), \mu(\Omega(y)) > 0$ . Now the result easily follows from the increasing monotonicity and subadditivity of the function  $\psi \circ \varphi$ , and the definition of  $\mathbb{P}_{\varphi, \psi}$ .  $\square$

**Example 1.** The functions  $\varphi$  given by  $\varphi(t) := \exp(t) - 1$ ,  $t > 0$ , and arbitrary  $\psi$  such that the function

$$(0, \infty) \ni t \mapsto \frac{\psi(e^t - 1)}{t} \quad \text{is non-increasing,}$$

satisfy the assumptions of the above Proposition 2 (cf. Remark 1). One can take for instance  $\psi(t) = \log(t + 1)$ ,  $t > 0$ .

**Proposition 3.** Let  $(\Omega, \Sigma, \mu)$  be a measure space such that, for every  $A \in \Sigma$ ,

$$\mu(A) \leq 1 \quad \text{or} \quad \mu(A) = \infty.$$

If  $\varphi, \psi : (0, \infty) \rightarrow (0, \infty)$  are increasing,  $\varphi$  is bijective, twice continuously differentiable,  $\varphi'' > 0$ ,  $\frac{\varphi}{\varphi'}$  is superadditive and  $\psi \circ \varphi$  is subadditive, then

$$\mathbb{P}_{\varphi, \psi}(x + y) \leq \mathbb{P}_{\varphi, \psi}(x) + \mathbb{P}_{\varphi, \psi}(y), \quad x, y \in S(\Omega, \Sigma, \mu).$$

**Proof.** Making use of Theorem 3 in [3], for all  $x, y \in S$  such that  $\mu(\Omega(x)), \mu(\Omega(y)) > 0$  inequality (6) holds true. Now the result follows from the increasing monotonicity and subadditivity of the function  $\psi \circ \varphi$ .  $\square$



**Example 2.** The function  $\varphi$  given by  $\varphi(t) := \frac{t^2}{t+1}$ ,  $t > 0$ , and arbitrary bijection  $\psi : (0, \infty) \rightarrow (0, \infty)$  such that the function  $(0, \infty) \ni t \rightarrow t^{-1}\psi(\frac{t^2}{t+1})$  is nonincreasing satisfy the assumptions of Proposition 3. One can take, for instance,  $\psi = \varphi^{-1}$ .

**Remark 3.** In Theorem 1 the assumption of superadditivity of  $\varphi \circ \psi$  is essential. It is a consequence of the following

**Proposition 4.** Let  $(\Omega, \Sigma, \mu)$  be an arbitrary measure space. If  $\varphi(t) = t^p$  ( $t > 0$ ) for some fixed  $p \geq 1$ , and  $\psi : (0, \infty) \rightarrow (0, \infty)$  is an increasing homeomorphism of  $(0, \infty)$  such that the function  $\psi \circ \varphi$  is subadditive, then

$$\mathbb{P}_{\varphi, \psi}(x+y) \leq \mathbb{P}_{\varphi, \psi}(x) + \mathbb{P}_{\varphi, \psi}(y), \quad x, y \in S(\Omega, \Sigma, \mu).$$

**Proof.** For all  $x, y \in S(\Omega, \Sigma, \mu)$  such that  $\mu(\Omega(x)), \mu(\Omega(y)) > 0$ , by Minkowski's inequality, we have

$$\begin{aligned} \mathbb{P}_{\varphi, \psi}(x+y) &= \psi \left( \int_{\Omega(x+y)} |x+y|^p d\mu \right) = \psi \circ \varphi \left( \|x+y\|_p \right) \\ &\leq \psi \circ \varphi \left( \|x\|_p + \|y\|_p \right) \leq \psi \circ \varphi \left( \|x\|_p \right) + \psi \circ \varphi \left( \|y\|_p \right) \\ &= \psi \left( \int_{\Omega(x)} |x|^p d\mu \right) + \psi \left( \int_{\Omega(y)} |y|^p d\mu \right) = \mathbb{P}_{\varphi, \psi}(x) + \mathbb{P}_{\varphi, \psi}(y), \end{aligned}$$

(where  $\|\cdot\|_p$  denotes here the  $L^p$ -norm).  $\square$

**Remark 4.** The assumption that  $\lim_{t \rightarrow 0} \psi(t) = 0$  plays an important technical role in the proof of our basic Proposition 1. Namely, together with Lemma 1, it allows to conclude that the function  $\psi$  is increasing and continuous on  $(0, \infty)$ .

Note that this assumption can be significantly relaxed.

**Remark 5.** Suppose that  $\psi : (0, \infty) \rightarrow (0, \infty)$  is subadditive. If a set  $C \subset (0, \infty)$  satisfies the following condition:

$$(0, \delta) \subset \sum_{j=1}^k C \text{ for some } \delta > 0 \text{ and a positive integer } k,$$

then  $\lim_{t \rightarrow 0} \psi|_C(t) = 0$  implies that  $\lim_{t \rightarrow 0} \psi(t) = 0$  (here  $\psi|_C$  denotes the restriction of  $\psi$  to  $C$ ).

By Raikov's theorem [10], if  $C \subset (0, \infty)$  and 0 is a point of positive density of  $C$ , then this condition is satisfied. Note also that if  $C$  is the Cantor set (so a set of measure zero), then this condition holds with  $k = 2$  and  $\delta = 2$  (cf. [1], p. 50).

The next result shows that if the range of measure  $\mu$  is rich enough, then the assumption  $\lim_{t \rightarrow 0} \psi(t) = 0$  in Theorem 1 can be omitted.

**Theorem 2.** *Theorem 1 remains true if the assumption  $\lim_{t \rightarrow 0} \psi(t) = 0$  is replaced by the following condition: there are two sets  $C, D \in \Sigma$  such that*

$$C \cap D = \emptyset, \quad 0 < \mu(C) < 1, \quad \mu(C) \in \mathbb{Q}, \quad \mu(C) + \mu(D) = 1,$$

where  $\mathbb{Q}$  denotes the set of rational numbers.

**Proof.** Suppose that  $\mathbb{P}_{\varphi, \psi}(x + y) \leq \mathbb{P}_{\varphi, \psi}(x) + \mathbb{P}_{\varphi, \psi}(y)$ , for all  $x, y \in S_+$ . For

$$x := x_1 \chi_C + x_2 \chi_D, \quad y := y_1 \chi_C + y_2 \chi_D, \quad x_1, x_2, y_1, y_2 > 0,$$

we obtain

$$\psi(c\varphi(x_1 + y_1) + d\varphi(x_2 + y_2)) \leq \psi(c\varphi(x_1) + d\varphi(x_2)) + \psi(c\varphi(y_1) + d\varphi(y_2)),$$

where  $c := \mu(C)$ ,  $d := \mu(D)$  and  $c + d = 1$ . Taking here  $x_1 = y_2 := \varphi^{-1}(s)$  and  $x_2 = y_1 := \varphi^{-1}(t)$ , we get

$$(\psi \circ \varphi)[\varphi^{-1}(s) + \varphi^{-1}(t)] \leq \psi(cs + dt) + \psi(ct + ds), \quad s, t > 0.$$

Now the superadditivity of the function  $\psi \circ \varphi$  implies that

$$\psi(s) + \psi(t) \leq \psi(cs + dt) + \psi(ct + ds), \quad s, t > 0,$$

or, equivalently,

$$\psi(s) + \psi(t) \leq \psi(cs + (1 - c)t) + \psi(ct + (1 - c)s), \quad s, t > 0,$$

which means that the function  $\psi$  is  $c$ -Wright concave in  $(0, \infty)$ . Since  $c \in (0, 1)$  is a rational number, in view of the main result of [2], the function  $\psi$  is Jensen concave in  $(0, \infty)$ . By the Bernstein-Doetsch theorem (cf. for instance [1], p. 145), being bounded below,  $\psi$  is concave and, consequently, continuous. Summarizing, we have shown that  $\psi$  is a concave homeomorphism of  $(0, \infty)$ . This implies that  $\psi$  must be increasing and, consequently,  $\lim_{t \rightarrow 0} \psi(t) = 0$ . The proof is completed.

We end this section with the following

**Conjecture 1.** *The assumption that  $\lim_{t \rightarrow 0} \psi(t) = 0$  in Theorem 1 is superfluous.*

#### 4. RESULTS FOR THE REVERSED INEQUALITY

In the case of the reversed triangle inequality for the functional  $\mathbb{P}_{\varphi, \psi}$  the suitable theory is much simpler. In particular, in the proof of the following "converse theorem" we do not need any regularity assumption on the function  $\psi$ .

**Theorem 3.** *Let  $(\Omega, \Sigma, \mu)$  be a measure space with two sets  $A, B \in \Sigma$  such that*

$$0 < \mu(A) < 1 < \mu(B) < \infty.$$

*Suppose that  $\varphi, \psi : (0, \infty) \rightarrow (0, \infty)$  are bijective functions such that the function  $\psi \circ \varphi$  is subadditive. Then*

$$\mathbb{P}_{\varphi,\psi}(x+y) \geq \mathbb{P}_{\varphi,\psi}(x) + \mathbb{P}_{\varphi,\psi}(y), \quad x, y \in S_+(\Omega, \Sigma, \mu),$$

if, and only if there is a  $p \in (0, 1]$  such that

$$\varphi(t) = \varphi(1)t^p, \quad \psi(t) = \psi(1)t^{1/p}, \quad t > 0;$$

**Proof.** Suppose that  $\mathbb{P}_{\varphi,\psi}$  is superadditive on  $S_+$ . Similar reasoning as in the proof of Proposition 1 proves that  $\psi$  is superadditive. Since  $\psi$  is positive it follows that it is increasing and, consequently,  $\psi$  is an increasing homeomorphism of  $(0, \infty)$ . Setting in the assumed inequality

$$x := x_1\chi_A + x_2\chi_{B \setminus A}, \quad y := y_1\chi_A + y_2\chi_{B \setminus A}, \quad x_1, x_2, y_1, y_2 > 0,$$

then replacing  $x_i$  by  $\varphi^{-1}(x_i)$ ,  $y_i$  by  $\varphi^{-1}(y_i)$  for  $i = 1, 2$ , and making use of the strict monotonicity of  $\psi$ , we obtain

$$\begin{aligned} & a\varphi(\varphi^{-1}(x_1) + \varphi^{-1}(y_1)) + b\varphi(\varphi^{-1}(x_2) + \varphi^{-1}(y_2)) \\ & \geq \psi^{-1}(\psi(ax_1 + bx_2) + \psi(ay_1 + by_2)) \end{aligned}$$

for all  $x_1, x_2, y_1, y_2 > 0$ .

From the assumed subadditivity of the function  $\psi \circ \varphi$  we infer that

$$\psi^{-1}(\psi(s) + \psi(t)) \leq \varphi(\varphi^{-1}(s) + \varphi^{-1}(t)), \quad s, t > 0.$$

The last two inequalities imply that the function  $F : (0, \infty)^2 \rightarrow (0, \infty)$ ,

$$F(x_1, x_2) := \psi^{-1}(\psi(x_1) + \psi(x_2)), \quad x_1, x_2 > 0,$$

satisfies the inequality

$$aF(x_1, x_2) + bF(y_1, y_2) \geq F(ax_1 + by_1, ax_2 + by_2) \quad x_1, x_2, y_1, y_2 > 0.$$

Applying Corollary 1 in [8] we infer that  $F$  is positively homogeneous. We omit the remaining part of the proof as it is analogous to that of the implication (i)  $\Rightarrow$  (ii) of Proposition 1.  $\square$

**Remark 6.** The functional  $\mathbb{P}_{\varphi,\psi}$  is superadditive on the whole linear space  $S(\Omega, \Sigma, \mu)$ , i.e.

$$\mathbb{P}_{\varphi,\psi}(x+y) \geq \mathbb{P}_{\varphi,\psi}(x) + \mathbb{P}_{\varphi,\psi}(y), \quad x, y \in S(\Omega, \Sigma, \mu),$$

if the underlying measure space satisfies the following condition:

for every  $A \in \Sigma$  either  $\mu(A) = 0$  or  $\mu(A) = \infty$ .

In fact, if there were a set  $A \in \Sigma$  such that  $0 < \mu(A) < \infty$ , then for  $x := \chi_A$  and  $y := -x$  we would get

$$0 = \mathbb{P}_{\varphi,\psi}(0) = \mathbb{P}_{\varphi,\psi}(x+y) \geq \mathbb{P}_{\varphi,\psi}(x) + \mathbb{P}_{\varphi,\psi}(-x) > 0.$$

Thus the problem of the global superadditivity of  $\mathbb{P}_{\varphi,\psi}$  trivializes.

**Remark 7.** With some obvious changes, the counterparts of Propositions 2, 3 and 4 are also valid.

### Remarks on the definition of the functional $\mathbb{P}_{\varphi,\psi}$

Suppose that  $\varphi, \psi : [0, \infty) \rightarrow [0, \infty)$  are one-to-one, onto, and  $\varphi(0) = 0 = \psi(0)$ . Then the functional  $p_{\varphi,\psi} : S \rightarrow [0, \infty)$  given by (cf. [3])

$$p_{\varphi,\psi}(x) := \psi\left(\int_{\Omega} \varphi \circ |x| d\mu\right), \quad x \in S(\Omega, \Sigma, \mu),$$

is correctly defined. Obviously, the counterparts of all the results proved above remain true. Moreover, the possibility of using 0 in the Minkowski inequality for  $p_{\varphi,\psi}$  allows to simplify some steps of the proof (for instance, in Proposition 1).

However

1) the results for the functional  $\mathbb{P}_{\varphi,\psi}$  are formally more general (the value at 0 plays no role);

2) in the proof of Proposition 1 we show that the function  $\varphi$  and  $\psi$  are multiplicative; the interval  $(0, \infty)$  is more natural domain for multiplicative functions than  $[0, \infty)$ ;

3) in the counterpart of Proposition 2 instead of  $\log \circ \varphi \circ \exp$  we should take  $\log \circ \varphi|_{(0,\infty)} \circ \exp$  (cf. also Examples 1 and 2 where adding 0 to the domain of the function  $\varphi$  would be rather inconvenient).

### FINAL REMARK

Theorem 1, Propositions 1, 2 and 3 give a solution of a problem posed by W. Wnuk in [11].

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