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Remarks on some problems of Th. M. Rassias

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Summary. During the 40th ISFE meeting Th. M. Rassias posed some open problems. A complete solution of one of these problems, and a partial solution of an another one are presented.

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During the 40th ISFE meeting Themistokles M. Rassias posed the following (cf. [4]):

Problem 1. *Find all functions $f : (0, \infty) \to \mathbb{R}$ such that*

$$f\left(\sqrt{xy}\right) + f\left(\frac{x^2 + y^2}{x + y}\right) = f(x) + f(y). \tag{1}$$

Problem 2. *Find all functions $f : (0, \infty) \to \mathbb{R}$ such that*

$$f\left(\frac{x + y + xy}{2}\right) + f\left(\frac{2xy}{x + y + xy}\right) = f(x) + f(y) + f(xy). \tag{2}$$

In this note we give a complete solution of the second problem and a partial solution of the first one.

Theorem 1. *A function $f : (0, \infty) \to \mathbb{R}$ satisfies equation (2) if, and only if, $f = 0$.*

Proof. Setting

$$y := \frac{x}{x + 1}, \qquad x > 0,$$

in (2) we obtain

$$f\left(\frac{x^2}{x + 1}\right) = 0$$

for all $x > 0$. This completes the proof as the function $(0, \infty) \ni x \to \frac{x^2}{x+1}$ maps the positive reals onto itself. □

Theorem 2. *A continuous function $f : (0, \infty) \to \mathbb{R}$ satisfies equation (1) if, and only if, f is constant.*

Proof. We shall apply the iteration method called the "Gauss-composition" of means which is discussed by Z. Daróczy and Zs. Páles in [2].

Denote by $M, N : (0,\infty)^2 \to (0,\infty)$ the functions given by

$$M(x,y) := \sqrt{xy}, \qquad N(x,y) = \frac{x^2+y^2}{x+y}, \qquad x, y > 0,$$

and by (M_n, N_n) the n-th iterate of the mapping $(M,N) : (0,\infty)^2 \to (0,\infty)^2$.

Since M and N are strict and continuous means, there is a unique continuous mean $K : (0,\infty)^2 \to (0,\infty)$ such that

$$\lim_{n\to\infty} M_n = K, \qquad \lim_{n\to\infty} N_n = K \tag{3}$$

pointwise in $(0,\infty)^2$; moreover K is strict and homogeneous because so are M and N (cf. [3], Theorem 1).

Suppose that $f : (0,\infty) \to \mathbb{R}$ is continuous and satisfies equation (1), i.e. that

$$f(M(x,y)) + f(N(x,y)) = f(x) + f(y), \qquad x, y > 0.$$

Hence, by an induction, we get

$$f(M_n(x,y)) + f(N_n(x,y)) = f(x) + f(y), \qquad n \in \mathbb{N}; x, y > 0.$$

Letting here $n \to \infty$, by (3) and the continuity of f, we obtain

$$2f(K(x,y)) = f(x) + f(y), \qquad x, y > 0. \tag{4}$$

Suppose that f is not one-to-one. Thus $f(a) = f(b)$ for some $a, b \in (0,\infty)$, $a < b$. Put $C := \{x \in [a,b] : f(x) = f(a)\}$. We shall show that $C = [a,b]$. For an indirect argument suppose that $[a,b] \backslash C \neq \emptyset$. Since C is closed, there is a nonempty open maximal interval $(c,d) \subset [a,b]\backslash C$. Of course we have $c, d \in C$, i.e. $f(c) = f(d) = f(a)$, and $c < d$. Hence, by (4),

$$2f(K(c,d)) = f(c) + f(d) = 2f(a),$$

which means that $K(c,d) \in C$. This is a contradiction because $c < K(c,d)) < d$. Let $(\alpha,\beta) \subset (0,\infty)$ be a nonempty maximal open interval such that $f(x) = f(a)$ for all $x \in (\alpha,\beta)$ and suppose that $\beta < \infty$. Put $\gamma := \frac{\alpha+\beta}{2}$. Since

$$K(\gamma,\beta) < K(\beta,\beta) = \beta,$$

the continuity of K implies that there is an $u \in (\beta,\infty)$ such that

$$K(\gamma,u) < \beta.$$

Since $\gamma < K(\gamma,u)$, we have $K(\gamma,u) \in (\alpha,\beta)$. Hence, by (4), we have

$$2f(a) = 2f(K(\gamma,u)) = f(\gamma) + f(u) = f(a) + f(u)$$

and, consequently, $f(u) = f(a)$. According to the previous part of the proof, $f(x) = f(a)$ for all $x \in (\alpha, u)$ which contradicts to the assumed maximality of the interval (α, β) and proves that $\beta = \infty$. In a similar way we show that $\alpha = 0$.

Thus we have shown that every continuous solution of equation (4) which is not one-to-one must be constant.

If f is one-to-one then, by the continuity, it is strictly monotonic, and from (4) we get

$$K(x,y) = f^{-1}\left(\frac{f(x)+f(y)}{2}\right), \qquad x, y > 0,$$

which proves that K is quasi-arithmetic mean and f is a generator of K. Since K is homogeneous we infer that it must be a power mean (cf. [1], p. 249) and, consequently, either

$$f(x) = ax^p + b, \qquad x > 0 \tag{5}$$

for some $p, a, b \in \mathbb{R}$, $p \neq 0 \neq a$, or

$$f(x) = a \log x + b, \qquad x > 0, \tag{6}$$

for some $a, b \in \mathbb{R}$, $a \neq 0$.

It is easy to verify that the functions of the form (6) are not solutions of equation (1).

Suppose that there is a $p \neq 0$ such that f given by (5) satisfies equation (1). Substituting this function in equation (1) and taking $y = 1$, after obvious simplifications, we get

$$x^{p/2} + \left(\frac{x^2+1}{x+1}\right)^p - x^p - 1 = 0, \qquad x > 0. \tag{7}$$

Calculating the second derivative (with respect to x) of both sides and then substituting $x = 1$ gives

$$\frac{p(3-2p)}{4} = 0.$$

It follows that $p = \frac{3}{2}$. Since it is easy to check that equation (7) with $p = \frac{3}{2}$ is not satisfied, no function of the form (5) is a solution of (1). This completes the proof of the "only if" part of the theorem. Since the remaining part is obvious, the proof is completed. □

Note that in the first part of the proof we have proved the following:

Theorem 3. *Let $M, N : (0,\infty)^2 \to (0,\infty)$ be continuous, homogeneous and strict means. If a function $f : (0,\infty) \to \mathbb{R}$ satisfies the functional equation*

$$f(M(x,y)) + f(N(x,y)) = f(x) + f(y), \qquad x, y > 0,$$

then either there are $a, b, p \in \mathbb{R}$, $a \neq 0 \neq p$, such that $f(x) = ax^p + b$, $x > 0$, or there are $a, b \in \mathbb{R}$, $a \neq 0$, such that $f(x) = a \log x + b$, $x > 0$.

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