Sitzungsber. Abt. II (2002) 211: 117–136

Sitzungsberichte
Mathematisch-naturwissenschaftliche Klasse Abt. II
Mathematische, Physikalische und Technische Wissenschaften

A Modified Gołab-Schinzel Equation on a Restricted Domain (with Applications to Meteorology and Fluid Mechanics)

By

Peter Kahlig and **Janusz Matkowski**

(Vorgelegt in der Sitzung der math.-nat. Klasse am 21. März 2002
durch das w. M. Ludwig Reich)

Abstract

A modified Gołab-Schinzel equation on a restricted domain is considered. A new result on monotonic solutions is proved. The continuous or monotonic solutions can be used to model some nonlinear processes of meteorology and fluid mechanics. Symmetries of corresponding nonlinear differential equations are described by the modified Gołab-Schinzel equation.

1991 Mathematics Subject Classifications: Primary 39B12, 76-99.
Key words and phrases: Functional equation, Gołab-Schinzel functional equation on restricted domain, continuous solutions, monotonic solutions, evaporation of cloud droplets, water clock, water discharging from a reservoir, semigroups of symmetries of nonlinear processes.

1. Introduction

The Gołab-Schinzel functional equation

$$f(x + yf(x)) = f(x)f(y), \quad x, y \in \mathbb{R},$$

which appears in connection with determining some subgroups or subsemigroups of the affine group (cf. for instance J. ACZÉL and

J. DHOMBRES [1], Ch. 19), was considered by many authors (cf. for instance [2, 5, 6, 13]; cf. also [1] for an overview of existing literature on this equation). It is known that the nontrivial continuous solution $f : \mathbb{R} \to \mathbb{R}$ is either of the form

$$f(x) = 1 + cx, \qquad \text{or} \qquad f(x) = \max\{1 + cx, 0\}, \quad x \in \mathbb{R},$$

for some real constant c.

Discussing some applications to fluid mechanics with J. SCHWAIGER, it became apparent that it would be useful to examine the solutions of the Gołab-Schinzel equation on a restricted domain $\{(x, y) : x \geq 0, y \geq 0\}$. This problem was then considered by J. ACZÉL and J. SCHWAIGER [3], by L. REICH [21], and by M. SABLIK [24] (who determined the general continuous solutions on an interval containing 0). In some recent papers, the topic has been reconsidered with different restrictions (cf. J. BRZDEK [7], ANNA MUREŃKO [19], L. REICH [22]).

In Section 2, making use of results of J. ACZÉL and J. SCHWAIGER [3], we formulate Theorem 1 for a (pexiderized) modified Gołab-Schinzel equation on a restricted domain. We prove a new result on monotonic solutions (Theorem 2) and point to an open problem concerning the Gołab-Schinzel functional equation.

In Section 3 we present some applications to nonlinear processes of meteorology and fluid mechanics: evaporation of cloud droplets, water discharging from a reservoir, and the physical basis of the clepsydra (water clock). The corresponding nonlinear differential equations exhibit some *symmetries* which are expressed by a modified Gołab-Schinzel functional equation. The *intrinsic scale* of (solutions of) a Gołab-Schinzel equation, often denoted as x_0 or t_0, plays the role of an extinction time in our illustrative nonlinear processes.

In Section 4 we consider the *correspondence* of some functional equations, differential equations, and their solutions.

A pexiderized analogue of the Gołab-Schinzel equation with three unknown functions, on an unrestricted domain, was given in [1] (p. 340).

A physical application to the problem of relating interval scales (e.g. centigrades and degrees Fahrenheit) was mentioned in [15] where also a pexiderized version of the Gołab-Schinzel equation appears:

$$F(x + s(x)y) = s(x)F(y), \quad x, y \in \mathbb{R},$$

for continuous $F, s : \mathbb{R} \to \mathbb{R}$, where both F and s are unknown functions (s acting as a *scaling factor*). The nontrivial continuous solution is either

- the "modernistic" version, treating DANIEL G. FAHRENHEIT (1686–1736) as a modern scientist (cf. e.g. [11]),

$$s(x) = 1 + cx \text{ with some real constant } c,$$

$$F(x) = F(0)\, s(x) = F(0)\,(1 + cx),$$

leading to the modern relation between centigrades x and degrees Fahrenheit F, namely $F(x) = 32\,(1 + \frac{9}{160}x) = 32 + \frac{9}{5}x$;

or

- the "traditionalistic" version, treating Fahrenheit as a scientist in the tradition of Newton, avoiding negative temperature values by fixing 0 degrees Fahrenheit as the lowest temperature possible (cf. e.g. [8, 14, 25, 27]),

$$s(x) = \max\{1 + cx, 0\} \text{ with some real constant } c,$$

$$F(x) = F(0)\, s(x) = F(0) \max\{1 + cx, 0\},$$

leading to the traditionalistic relation $F(x) = \max\{32 + \frac{9}{5}x, 0\}$.

Apparently, DANIEL G. FAHRENHEIT gradually changed his scientific views from traditionalistic to modernistic (cf. [4, 10]).

2. A Modified Gołab-Schinzel Functional Equation and Its Solutions

Now let us consider a (pexiderized) modified Gołab-Schinzel equation on a restricted domain,

$$F(x + y\, s(x)^{1/p}) = s(x)F(y), \quad x, y \geq 0, \tag{1}$$

with a given number $p > 0$ and with unknown functions $s : [0, \infty) \to [0, \infty)$ and $F : [0, \infty) \to \mathbb{R}$. *Note:* as usual, $s(x)^a$ stands for $[s(x)]^a$.

Remark 1. *Let $p \in \mathbb{R}$, $p > 0$, be fixed, and suppose that $s : [0, \infty) \to [0, \infty)$ and $F : [0, \infty) \to \mathbb{R}$.*

(i) *If $F(0) = 0$ then the functions F and s satisfy Eq.* (1) *iff $F = 0$ and s is arbitrary.*
(ii) *If $F(0) \neq 0$ and the functions F and s satisfy Eq.* (1) *then $s(0) = 1$.*

Proof. Setting $y = 0$ in (1) gives (i); setting $x = y = 0$ gives (ii). □

We have the following (cf. [3]):

Theorem 1. *Let $p \in \mathbb{R}$, $p > 0$, be fixed, and suppose that $s : [0, \infty) \to [0, \infty)$ is continuous, $F : [0, \infty) \to \mathbb{R}$, and $F(0) \neq 0$. Then F, s satisfy Eq.* (1) *iff*

$$s(x) = (\max\{1 + cx, 0\})^p, \quad x \geq 0,$$
$$F(x) = F(0)s(x), \quad x \geq 0,$$

where $c \in \mathbb{R}$ and $F(0) \neq 0$ are arbitrary constants.

Proof. Setting $y := 0$ in (1) gives

$$F(x) = F(0)s(x), \quad x \geq 0. \tag{2}$$

Hence, making use of (1) and the assumption that $F(0) \neq 0$, we infer that s satisfies the equation

$$s(x + y\, s(x)^{1/p}) = s(x)s(y), \quad x, y \geq 0, \tag{2a}$$

and, obviously, the function $f : [0, \infty) \to [0, \infty)$, defined by

$$f(x) := s(x)^{1/p}, \quad x \geq 0, \tag{3}$$

is continuous and satisfies the classical Gołab-Schinzel functional equation:

$$f(x + yf(x)) = f(x)f(y), \quad x, y \geq 0. \tag{4}$$

According to Aczél and Schwaiger [3], Corollary,

$$f(x) = \max\{1 + cx, 0\}, \quad x \geq 0,$$

and, consequently,

$$s(x) = (\max\{1 + cx, 0\})^p, \quad x \geq 0,$$

which completes the proof. □

The main result of this section is

Theorem 2. *Let $p \in \mathbb{R}$, $p > 0$, be fixed. Suppose that $s : [0, \infty) \to [0, \infty)$ is monotonic, and $F : [0, \infty) \to \mathbb{R}$, $F(0) \neq 0$. Then F and s satisfy Eq.* (1) *iff either*

$$s(x) = \begin{cases} 1, & x = 0 \\ 0, & x > 0; \end{cases}$$

or there is a $c \in \mathbb{R}$ such that

$$s(x) = (\max\{1 + cx, 0\})^p, \quad x \geq 0, \tag{5}$$

and, in both cases,

$$F(x) = F(0)s(x), \quad x \geq 0.$$

Proof. Suppose that $F : [0, \infty) \to \mathbb{R}$ and $s : [0, \infty) \to [0, \infty)$ satisfy Eq. (1). By the monotonicity of s, the function $f : [0, \infty) \to [0, \infty)$, defined by (3), is also monotonic. Similarly as in the proof of the above proposition we show that F must be of the form (2) and the function f satisfies the Gołab-Schinzel equation (4). By Remark 1, we have $s(0) = 1$ and, consequently, $f(0) = 1$. Put

$$A := \{x \geq 0 | f(x) > 0\}.$$

By the monotonicity of f, either $A = [0, \infty)$ or $A = [0, x_0)$ for some $x_0 \geq 0$.

If $x_0 = 0$ then

$$f(x) = \begin{cases} 1 & \text{for } x = 0 \\ 0 & \text{for } x > 0, \end{cases}$$

and, obviously, $s := f$.

If $x_0 > 0$ then A is a nontrivial interval.

Suppose first that f is not strictly monotonic in A. Then there are $x_1, x_2 \in A$, $x_1 < x_2$ such that $f(x_1) = f(x_2)$. Then (cf. [1], p. 311), for all $x \geq x_1$,

$$\begin{aligned} f(x + (x_2 - x_1)) &= f\left(x_2 + \frac{x - x_1}{f(x_1)} f(x_1)\right) \\ &= f\left(x_2 + \frac{x - x_1}{f(x_1)} f(x_2)\right) = f(x_2) f\left(\frac{x - x_1}{f(x_1)}\right) \\ &= f(x_1) f\left(\frac{x - x_1}{f(x_1)}\right) = f\left(x_1 + \frac{x - x_1}{f(x_1)} f(x_1)\right) = f(x). \end{aligned}$$

which proves that f is a $(x_2 - x_1)$-periodic function. Thus $A = (0, \infty)$, and the monotonicity of f together with the relation $f(0) = 1$ imply that $f = 1$ on $(0, \infty)$. By (3), $s = f$ and, consequently, s is of the form (5) with $c = 0$.

Now consider the case when f is strictly monotonic in A. From (4), for all $x, y \in A$, we have

$$f(x + yf(x)) = f(x)f(y) = f(y + xf(y)),$$

whence $x + yf(x) = y + xf(y)$. It follows that

$$\frac{f(x) - 1}{x} = \frac{f(y) - 1}{y}, \quad x, y > 0.$$

Consequently, there is a $c \in \mathbb{R}$ such that $\frac{f(x)-1}{x} = c$ for all $x \in A, x \neq 0$. Since $f(0) = 1$, we infer that

$$f(x) = 1 + cx, \quad x \in A.$$

If f is (strictly) increasing then, obviously, $A = (0, \infty)$, so $c > 0$,

$$f(x) = 1 + cx, \quad x \geq 0,$$

and, by (3),

$$s(x) = (1 + cx)^p = (\max\{1 + cx, 0\})^p, \quad x \geq 0.$$

If f is strictly decreasing then $c < 0$ and $A = [0, x_0)$ for an $x_0 \in \mathbb{R}$. We shall show that

$$f(x_0-) := \lim_{x \to x_0-} f(x_0) = 0.$$

For an indirect argument suppose that $f(x_0-) > 0$. Taking $s, t \in (0, x_0)$ such that

$$s + tf(s) > x_0$$

we would get

$$f(s + tf(s)) = f(s)f(t) > 0,$$

which contradicts the definition of the point x_0. Thus

$$f(x_0-) = 0 = f(x_0),$$

and, by the monotonicity of f, we have $f(x) = 0$ for all $x \geq x_0$. From (3) we again get that

$$s(x) = (\max\{1 + cx, 0\})^p, \quad x \geq 0.$$

In each of these cases the function F is of the form (2). This completes the proof. $\square$

Remark 2. *Letting formally $p \to \infty$ in (1), there results a (pexiderized) Cauchy equation on a restricted domain*:

$$F(x + y) = s(x)F(y), \quad x, y \geq 0, \tag{6}$$

with unknown functions $F, s : [0, \infty) \to [0, \infty)$.

Apart from the trivial solution $F = 0$ and arbitrary s, under the assumption that s is measurable or continuous at least at a point (or the graph of s is not dense in $[0, \infty)^2$), the solutions of (6) are

$$s(x) = \exp(cx), \qquad F(x) = F(0)s(x), \quad x \geq 0,$$

where $c \in \mathbb{R}$ and $F(0) \geq 0$ are arbitrary constants (a corresponding result for $F : \mathbb{R} \to \mathbb{R}$ and $s = F$ was obtained in [16]) and, moreover, *the first solution in Theorem 2, namely* $s(0) = 1$, $s(x) = 0$ *for* $x > 0$, *and* $F(x) = F(0)s(x)$ *for* $x \geq 0$.

Let us note the following

Remark 3. *Suppose that* $f : \mathbb{R} \to \mathbb{R}$ *satisfies the Gołab-Schinzel functional equation*

$$f(x + yf(x)) = f(x)f(y), \quad x, y \in \mathbb{R}.$$

If there is a point $x_0 \in \mathbb{R}$ *such that* $f(x_0) \neq 0$ *and* f *is differentiable at* x_0, *then there exists a* $c \in \mathbb{R}$ *such that either*

$$f(x) = 1 + cx, \quad x \in \mathbb{R},$$

or

$$f(x) = \max\{1 + cx, 0\}, \quad x \in \mathbb{R}.$$

Proof. (Sketch.) Put $A := \{x \in \mathbb{R} | f(x) > 0\}$. Thus $x_0 \in A$, and it is easy to see that $0 \in A$ and $f(0) = 1$. Since

$$\frac{f(x + yf(x)) - f(x)}{yf(x)} = \frac{f(y) - 1}{y}, \quad x \in A, y \in \mathbb{R}\setminus\{0\},$$

the following conditions are equivalent

1. there is a point $x_0 \in A$ such that f is differentiable at x_0;
2. f is differentiable at 0;
3. f is differentiable at every point of the set A.

The differentiability at 0 implies that there is a nontrivial maximal open interval $I \subset A$, $I = (a, b)$, $-\infty \leq a < b \leq \infty$, such that $0 \in I$. Now, by a similar reasoning as in the proof of Theorem 2, we can show that

$$f(x) = 1 + cx, \quad x \in I.$$

Assuming, for instance, that $b < \infty$ one can prove that $f(x) = 0$ for all $x \geq b$, $c = -\frac{1}{b}$ and $a = -\infty$. □

In connection with this remark we pose the following

Problem 1. *Determine all functions* $f : [0, \infty) \to [0, \infty)$ *satisfying the Gołab-Schinzel functional equation*

$$f(x + yf(x)) = f(x)f(y), \quad x, y \geq 0,$$

and such that $f(x_0) \neq 0$ *and* f *is differentiable at* x_0, *for some* $x_0 \geq 0$.

3. Applications to Nonlinear Processes of Meteorology and Fluid Mechanics

3.1. Evaporation of cloud droplets

The evaporation of a spherical cloud droplet in a dry environment may be described in the simplest way by a loss of liquid mass $m(t) \geq 0$, with time t, proportional to the droplet surface $A(t) \geq 0$:

$$\frac{dm}{dt} = -cA, \quad t \geq 0, \tag{7}$$

where $c > 0$ is a constant (incorporating the influence of factors like humidity gradient and diffusion coefficient). The idealized model can only describe single droplets, without interaction with their ("always dry") environment, e.g. the situation in dissipating fair weather clouds. (For more specific details cf. [23].)

The mass of a droplet is the product of volume $V(t) \geq 0$ and density $\rho > 0$ (assumed constant)

$$m = V\rho = \frac{4}{3}\pi r^3 \rho, \tag{8}$$

where $r(t) \geq 0$ is the radius of the droplet. Substituting (8) into (7) yields

$$4\pi\rho r^2 \frac{dr}{dt} = -4\pi c r^2,$$

or simply,

$$r^2\left(\rho\frac{dr}{dt} + c\right) = 0, \quad t \geq 0, \tag{9}$$

which is an (ordinary first-order) nonlinear differential equation. The initial condition is

$$r(0) = R, \tag{10}$$

where $R > 0$ is the initial droplet radius. The solution $r : [0, \infty) \to [0, \infty)$ reads

$$r(t) = \begin{cases} R - \frac{c}{\rho}t, & 0 \leq t \leq R\frac{\rho}{c} \\ 0, & t > R\frac{\rho}{c}, \end{cases} \tag{11}$$

or equivalently

$$r(t) = \max\left\{R - \frac{c}{\rho}t, 0\right\} = R\max\{1 - Ct, 0\}, \quad t \geq 0, \tag{11a}$$

where $C = \frac{c}{R\rho}$ is a positive constant. Evidently, the end of evaporation is reached (i.e. the droplet vanishes) after time

$$t_0 = R\frac{\rho}{c} = \frac{1}{C}$$

(the solution remaining valid also for $t > t_0$). An inspection of Eq. (9) reveals some symmetries: transforming the variables t (by scaling and translation) and r (by the same scaling) in the following way

$$[0, \infty) \ni t \to \sigma t + \tau, \qquad [0, \infty) \ni r \to \sigma r, \tag{12}$$

where $\sigma > 0, \tau \geq 0$ are parameters, we obtain in place of (9)

$$\sigma^2 r^2 \left(\rho \frac{\sigma\, dr}{\sigma\, dt} + c \right) = 0$$

which reduces to (9) again, meaning that (9) is invariant with respect to the transformation (12). [In other words: (12) denotes symmetries of (9).] To indicate the possibility that the parameters σ, τ may not be chosen independently of each other, we write $\sigma(\tau)$ instead of σ. Thus, the transformation (12) reads in detail

$$[0, \infty) \ni t \to \sigma(\tau)t + \tau, \qquad [0, \infty) \ni r \to \sigma(\tau)r, \tag{12a}$$

and the invariance of (9) with respect to (12a) may be expressed as

$$r(\sigma(\tau)t + \tau) = \sigma(\tau)r(t). \tag{13}$$

Changing the notation according to $(r, t, \sigma, \tau) = (F, y, s, x)$, the usual form of the (pexiderized) Gołab-Schinzel functional equation is obtained,

$$F(s(x)y + x) = s(x)F(y), \quad x, y \geq 0, \tag{14}$$

describing, in fact, the symmetries of the nonlinear differential equation (9). The solution of (13) [or (14)] coincides with (11a). It is worth noting that the extinction time t_0 is finite, and depends on the initial condition (typical properties of nonlinear processes). Explicitly, the scaling factor reads here

$$\sigma(\tau) = \max\{1 - C\tau, 0\}, \quad \tau \geq 0,$$

and the solution (11a) may be written as

$$r(t) = r(0)\sigma(t), \quad t \geq 0.$$

Remark 4. *It is easy to verify that a transformation* $T : [0, \infty)^2 \to [0, \infty)^2$ *of the form*

$$T(t, r) := (\varphi(t), \gamma(r)), \quad t, r \geq 0,$$

where $\varphi, \gamma : [0,\infty) \to [0,\infty)$ are some differentiable functions, is a symmetry of the differential equation (9) *iff there are $\sigma, \tau \geq 0$ such that*

$$\varphi(t) = \sigma t + \tau, \quad t \in [0,\infty); \qquad \gamma(r) = \sigma r, \quad r \in [0,\infty).$$

It is easy to see that the pair $(\mathcal{T}, \circ)$ where

$$\mathcal{T} = \{T_{\sigma,\tau} |\ \sigma > 0, \tau \geq 0\}$$

denotes the family of all transformations $T_{\sigma,\tau} : [0,\infty)^2 \to [0,\infty)^2$ defined by (12), *that is*

$$T_{\sigma,\tau}(t,r) := (\sigma t + \tau, \sigma r), \quad t, r \geq 0,$$

and "$\circ$" is the composition, forms a non-abelian semigroup (of symmetries of the differential equation (9)).

In this connection a problem to determine some abelian subsemigroups of $\mathcal{T}$ arises.

Applying Theorems 1 and 2 with $p = 1$ we obtain the following

Corollary 1. *Suppose that $\sigma : [0,\infty) \to [0,\infty)$ is continuous, or monotonic and not identically zero in $(0,\infty)$. Then a one-parameter subfamily of mappings*

$$\{T_{\sigma(\tau),\tau} |\ \tau \geq 0\}$$

(defined by (12a)) *of the semigroup $(\mathcal{T}, \circ)$ is an abelian subsemigroup iff, for some $c \geq 0$,*

$$\sigma(\tau) = \max\{1 + c\tau, 0\} = 1 + c\tau, \quad \tau \geq 0.$$

3.2. A generalized evaporation law

In the evaporation law (7), let us generalize the dependence on the droplet surface $A(t)$:

$$\frac{dm}{dt} = -cA^{1+q/2}, \quad t \geq 0, \tag{15}$$

where $c > 0$ and $q < 1$ are fixed. (We recover (7) by setting $q = 0$. The case $q = -1$ has been treated in [12].) Substituting Eq. (8) into (15) yields

$$r^2 \left(\rho \frac{dr}{dt} + c(4\pi)^{q/2} r^q \right) = 0, \quad t \geq 0, \tag{16}$$

which is an (ordinary first-order) nonlinear differential equation. The initial condition is again (10).

The solution $r : [0, \infty) \to [0, \infty)$ reads

$$r(t) = R(\max\{1 - Ct, 0\})^{1/(1-q)}, \quad t \geq 0, \tag{17}$$

with a positive constant

$$C = (4\pi)^{q/2} \frac{1-q}{R^{1-q}} \frac{c}{\rho}. \tag{18}$$

Evidently, the end of evaporation is reached (i.e. the drop vanishes) after time

$$t_0 = \frac{1}{C} \tag{19}$$

(the solution remaining valid also for $t > t_0$). An inspection of Eq. (16) reveals some symmetries: transforming the variables t and r in the following way

$$[0, \infty) \ni t \to \sigma^{1-q} t + \tau, \qquad [0, \infty) \ni r \to \sigma r, \tag{20}$$

where $\sigma > 0, \tau \geq 0$ are parameters, we obtain in place of (16)

$$\sigma^2 r^2 \left(\rho \frac{\sigma}{\sigma^{1-q}} \frac{dr}{dt} + c(4\pi)^{q/2} \sigma^q r^q \right) = 0$$

which reduces to (16) again, meaning that (16) is invariant with respect to the transformation (20). [In other words: (20) denotes symmetries of (16).] To indicate the possibility that the parameters σ, τ may not be chosen independently of each other, we write $\sigma(\tau)$ instead of σ. Thus, the transformation (20) reads in detail

$$[0, \infty) \ni t \to \sigma(\tau)^{1-q} t + \tau, \qquad [0, \infty) \ni r \to \sigma(\tau) r, \tag{20a}$$

and the invariance of (16) with respect to (20a) may be expressed as

$$r(\sigma(\tau)^{1-q} t + \tau) = \sigma(\tau) r(t). \tag{21}$$

Identifying $(r, t, \sigma, \tau) = (F, y, s, x)$, the usual form of the (pexiderized) modified Gołab-Schinzel functional equation is obtained,

$$F(x + s(x)^{1-q} y) = s(x) F(y), \quad x, y \geq 0, \tag{22}$$

describing the symmetries of the nonlinear differential equation (16). The solution of (21) [or (22)] coincides with (17). Again, it is worth noting that the extinction time (19) is finite, and depends on the initial condition (typical features of nonlinear processes).

Now we treat the case $q \to 1$. It turns out that this is the (limiting) linear case, with infinite extinction time (independent of the initial condition). For $q \to 1$, the differential equation (16) becomes

$$r^2\left(\rho\frac{dr}{dt} + 2\sqrt{\pi}cr\right) = 0, \quad t \geq 0, \tag{23}$$

yielding (apart from the trivial solution $r = 0$) the solution

$$r(t) = R\exp(-Ct), \quad t \geq 0, \tag{24}$$

for the initial condition $r(0) = R$, with a positive constant

$$C = 2\sqrt{\pi}\frac{c}{\rho} = \frac{1}{t_1},$$

this time scale t_1 being different from the time scale t_0 in (19); the quantity t_0 ceases to exist for $q \to 1$ (here $t_0 \to \infty$), and t_1 takes over the role of a physical parameter (*time constant*) in a natural way. The symmetries of (23) are, by inspection,

$$[0,\infty) \ni t \to t + \tau, \qquad [0,\infty) \ni r \to \sigma r, \tag{25}$$

[cf. Eq. (20) for $q \to 1$], and the invariance of (23) with respect to (25) may be expressed as

$$r(t+\tau) = \sigma(\tau)r(t), \quad t \geq 0. \tag{26}$$

Identifying $(r, t, \sigma, \tau) = (F, y, s, x)$, the usual form (6) of a (pexiderized) Cauchy functional equation is obtained, describing the symmetries of the (linear) differential equation (23). The solution of (26) [or (6)] coincides with (24).

Remark 5. *Note that a transformation* $T : [0,\infty)^2 \to [0,\infty)^2$,

$$T(t,r) := (\varphi(t), \gamma(r)), \quad t, r \geq 0,$$

where $\varphi, \gamma : [0,\infty) \to [0,\infty)$ *are some differentiable functions, is a symmetry of the differential equation* (16) *iff there are* $\sigma, \tau \geq 0$ *such that*

$$\varphi(t) = \sigma^{1-q}t + \tau, \quad t \in [0,\infty); \qquad \gamma(r) = \sigma r, \quad r \in [0,\infty).$$

It is easy to see that the pair $(\mathcal{T}, \circ)$ where

$$\mathcal{T} = \{\mathcal{T}_{\sigma,\tau} \mid \sigma > 0, \tau \geq 0\}$$

denotes the family of all transformations $T_{\sigma,\tau} : [0,\infty)^2 \to [0,\infty)^2$ *defined by* (20), *that is*

$$T_{\sigma,\tau}(t,r) := (\sigma^{1-q}t + \tau, \sigma r), \quad t, r \geq 0,$$

and "$\circ$*" is the composition, forms a non-abelian semigroup (of symmetries of the differential equation* (16)).

Applying Theorems 1 and 2 with $p := \frac{1}{1-q}$ we obtain the following

Corollary 2. *Suppose that* $\sigma : [0,\infty) \to [0,\infty)$ *is continuous, or monotonic and not identically zero in* $(0,\infty)$. *Then a one-parameter subfamily*

$$\{T_{\sigma(\tau),\tau} \mid \tau \geq 0\}$$

(defined by (20a)) *of the semigroup* $(\mathcal{T},\circ)$ *is an abelian subsemigroup iff, for some* $c \geq 0$ *and* $q < 1$,

$$\sigma(\tau) = (\max\{1 + c\tau, 0\})^{1/(1-q)} = (1 + c\tau)^{1/(1-q)}, \quad \tau \geq 0.$$

Remark 6. *In the case* $q \to 1$ *the transformation* $T : [0,\infty)^2 \to [0,\infty)^2$,

$$T(t,r) := (\varphi(t), \gamma(r)), \quad t, r \geq 0,$$

where $\varphi, \gamma : [0,\infty) \to [0,\infty)$ *are some differentiable functions, is a symmetry of the differential equation* (23) *iff there are* $\sigma, \tau \geq 0$ *such that*

$$\varphi(t) = t + \tau, \quad t \in [0,\infty); \qquad \gamma(r) = \sigma r, \quad r \in [0,\infty).$$

In this case the pair $(\mathcal{T},\circ)$ *where* $\mathcal{T} = \{T_{\sigma,\tau} \mid \sigma > 0, \tau \geq 0\}$ *where*

$$T_{\sigma,\tau}(t,r) := (t + \tau, \sigma r),$$

and "$\circ$*" is the composition, is a two-parameter abelian semigroup of symmetries of the differential equation* (23).

3.3. Water discharging from a reservoir

In a cylindrical tank, at time t, water is contained up to height h above an orifice. The outflow velocity v is, according to Torricelli's law,

$$v = \sqrt{2gh}, \quad h \geq 0, \tag{27}$$

where g denotes the acceleration of gravity (a positive constant). Strictly speaking, Eq. (27) is exact for *steady* processes, but may be used as an excellent approximation for (sufficiently slow) *unsteady* processes as well (cf. [18, 26, 28]). Due to the continuity of liquid mass there holds

$$-A\frac{dh}{dt} = A_0 v, \quad t \geq 0, \tag{28}$$

where A, A_0 are the cross sections of tank and orifice, respectively, and the minus sign in (28) indicates *falling* water level.

Substitution of (27) in (28) gives $dh/dt = -(A_0/A)\sqrt{2gh}$ which is, in general, modified by engineers (to account for losses e.g. due to flow contraction near the orifice) by a semi-empirical *discharge coefficient* of about 0.60 (cf. [20]):

$$\frac{dh}{dt} = -0.60\frac{A_0}{A}\sqrt{2gh}, \quad t \geq 0. \tag{29}$$

Taking as an abbreviation the positive constant

$$k = 0.60\frac{A_0}{A}\sqrt{2g},$$

there results the (ordinary first-order) nonlinear differential equation

$$\frac{dh}{dt} = -k\sqrt{h}, \quad t \geq 0. \tag{30}$$

The initial condition is

$$h(0) = H, \tag{31}$$

where $H > 0$ is the initial filling height.

The solution $h : [0, \infty) \to [0, \infty)$ reads

$$h = H(\max\{1 - Ct, 0\})^2, \quad t \geq 0, \tag{32}$$

with a positive constant

$$C = \frac{k}{2\sqrt{H}}. \tag{33}$$

Evidently, the end of discharge is reached (i.e. the tank is empty) after time

$$t_0 = \frac{1}{C} = \frac{2\sqrt{H}}{k} \tag{34}$$

(the solution remaining valid also for $t > t_0$). An inspection of Eq. (30) reveals some symmetries: transforming the variables t and h in the following way

$$t \to \sigma^{1/2}t + \tau, \qquad h \to \sigma h \tag{35}$$

where $\sigma, \tau > 0$ are parameters, we obtain in place of (30)

$$\frac{\sigma}{\sigma^{1/2}}\frac{dh}{dt} = -k\sigma^{1/2}h^{1/2}$$

which reduces to (30) again, meaning that (30) is invariant with respect to the transformation (35). [In other words: (35) denotes symmetries of (30).] To indicate the possibility that the parameters

σ, τ may not be chosen independently of each other, we write $\sigma(\tau)$ instead of σ. Thus, the transformation (35) reads in detail

$$t \to \sigma(\tau)^{1/2}t + \tau, \qquad h \to \sigma(\tau)h, \tag{35a}$$

and the invariance of (30) with respect to (35a) may be expressed as

$$h(\sigma(\tau)^{1/2}t + \tau) = \sigma(\tau)h(t). \tag{36}$$

Identifying $(h, t, \sigma, \tau) = (F, y, s, x)$, the usual form of the (pexiderized) modified Gołab-Schinzel functional equation is obtained [cf. Eq. (1)],

$$F(x + y\, s(x)^{1/2}) = s(x)F(y), \quad x, y \geq 0, \tag{37}$$

describing the symmetries of the nonlinear differential equation (30). The solution of (36) [or (37)] coincides with (32). The extinction time t_0 is finite, and depends on the initial condition (typical properties of nonlinear processes).

Remark 7. *A transformation* $T : [0,\infty)^2 \to [0,\infty)^2$,

$$T(t,h) := (\varphi(t), \gamma(h)), \quad t, h \geq 0,$$

where $\varphi, \gamma : [0,\infty) \to [0,\infty)$ *are some differentiable functions, is a symmetry of the differential equation* (30) *iff there are* $\sigma, \tau \geq 0$ *such that*

$$\varphi(t) = \sigma^{1/2}t + \tau, \quad t \in [0,\infty); \qquad \gamma(h) = \sigma h, \quad h \in [0,\infty).$$

It is easy to see that the pair $(\mathcal{T}, \circ)$ *where*

$$\mathcal{T} = \{T_{\sigma,\tau} \mid \sigma > 0, \tau \geq 0\}$$

denotes the family of all transformations $T_{\sigma,\tau} : [0,\infty)^2 \to [0,\infty)^2$ *defined by* (35), *that is*

$$T_{\sigma,\tau}(t,h) := (\sigma^{1/2}t + \tau, \sigma h), \quad t, h \geq 0,$$

and "$\circ$" is the composition, forms a non-abelian semigroup (of symmetries of the differential equation (30)).

Applying Theorems 1 and 2 with $p := 2$ we obtain the following

Corollary 3. *Suppose that* $\sigma : [0,\infty) \to [0,\infty)$ *is continuous, or monotonic and not identically zero in* $(0,\infty)$. *Then a one-parameter subfamily*

$$\{T_{\sigma(\tau),\tau} \mid \tau \geq 0\}$$

(defined by (35a)*) of the semigroup* $(\mathcal{T}, \circ)$ *is an abelian subsemigroup iff, for some* $c \geq 0$,

$$\sigma(\tau) = (\max\{1 + c\tau, 0\})^2 = (1 + c\tau)^2, \quad \tau \geq 0.$$

3.4. Water clock (clepsydra)

If we choose in (29) the area ratio as

$$\frac{A_0}{A} = \frac{b}{\sqrt{h}} \tag{38}$$

(where $b > 0$ plays the role of a design constant), there results the simple differential equation for $h \geq 0$

$$\frac{dh}{dt} = -B, \quad t \geq 0, \tag{39}$$

with a positive constant $B = 0.60b\sqrt{2g}$. Equation (39) describes a *uniformly* falling water level (which is expedient for the measurement of time). From (38) we obtain the shape of the water reservoir for this case, namely

$$h = b^2 \left(\frac{A}{A_0}\right)^2 = b^2 \left(\frac{r^2}{R^2}\right)^2$$

where r is the tank radius at height h, and R is the tank radius at the initial filling height H, or briefly

$$h = Kr^4, \quad r > 0, \tag{40}$$

with a positive constant $K = b^2/R^4$.

Thus, to construct a water clock, the ideal reservoir should have the cup-like shape (40) (cf. [17, 18]). Still existing Egyptian water clocks come close to this requirement (cf. [9, 20]). In ancient Greece, a water clock of similar construction was called clepsydra ("water stealer"), and was later adopted by the Romans (cf. [9]). Water clocks, sun dials and sand glasses remained the only time-measuring devices for centuries.

The differential equation (39), with initial condition $h(0) = H > 0$ (initial filling height), has the solution $h : [0, \infty) \to [0, \infty)$

$$h = \max\{H - Bt, 0\} = H \max\{1 - Ct, 0\}, \quad t \geq 0, \tag{41}$$

with a positive constant $C = B/H$. The reservoir is empty after time

$$t_0 = \frac{1}{C} = \frac{H}{B}. \tag{42}$$

An inspection of Eq. (39) reveals some symmetries: transforming the variables t and h in the following way

$$t \to \sigma t + \tau, \qquad h \to \sigma h, \tag{43}$$

where σ, τ are parameters, we obtain in place of (39)

$$\frac{\sigma}{\sigma}\frac{dh}{dt} = -B$$

which reduces to (39) again, meaning that (39) is invariant with respect to the transformation (43). [In other words: (43) denotes symmetries of (39).] To indicate the possibility that the parameters σ and τ may not be chosen independently of each other, we write $\sigma(\tau)$ instead of σ. Thus, the transformation (43) reads in detail

$$t \to \sigma(\tau)t + \tau, \qquad h \to \sigma(\tau)h, \tag{43a}$$

and the invariance of (39) with respect to (43a) may be expressed as

$$h(\sigma(\tau)t + \tau) = \sigma(\tau)h(t). \tag{44}$$

Identifying $(h, t, \sigma, \tau) = (F, y, s, x)$, the usual form of the (pexiderized) Gołab-Schinzel functional equation is obtained

$$F(x + ys(x)) = s(x)F(y), \quad x, y \geq 0, \tag{45}$$

describing symmetries of the differential equation (39). The solution of (44) [or (45)] coincides with (41). The extinction time t_0 is finite, and depends on the initial condition (typical features of nonlinear processes).

Remark 8. *A transformation* $T : [0, \infty)^2 \to [0, \infty)^2$ *of the form*

$$T(t, h) := (\varphi(t), \gamma(h)), \quad t, h \geq 0,$$

where $\varphi, \gamma : [0, \infty) \to [0, \infty)$ *are some differentiable functions, is a symmetry of the differential equation* (39) *iff there are* $\sigma, \tau \geq 0$ *such that*

$$\varphi(t) = \sigma t + \tau, \quad t \in [0, \infty); \qquad \gamma(h) = \sigma h, \quad h \in [0, \infty).$$

The pair $(\mathcal{T}, \circ)$ *where*

$$\mathcal{T} = \{T_{\sigma,\tau} |\, \sigma > 0, \tau \geq 0\}$$

denotes the family of all transformations $T_{\sigma,\tau} : [0, \infty)^2 \to [0, \infty)^2$ *defined by* (43), *that is*

$$T_{\sigma,\tau}(t, h) := (\sigma t + \tau, \sigma h), \quad t, h \geq 0,$$

and "$\circ$" is the composition, forms a non-abelian semigroup (of symmetries of the differential equation (39)).

Applying Theorems 1 and 2 with $p = 1$ we obtain the following

Corollary 4. *Suppose that $\sigma : [0, \infty) \to [0, \infty)$ is continuous, or monotonic and not identically zero in $(0, \infty)$. Then a one-parameter subfamily*

$$\{T_{\sigma(\tau),\tau} | \ \tau \geq 0\}$$

(defined by 43a) *of the semigroup $(\mathcal{T}, \circ)$ is an abelian subsemigroup iff, for some $c \geq 0$,*

$$\sigma(\tau) = \max\{1 + c\tau, 0\} = 1 + c\tau, \quad \tau \geq 0.$$

4. Final Remarks

In Section 3, we have shown how the symmetries of certain differential equations may be expressed by the (modified) Gołab-Schinzel functional equation. Now we show that these differential equations can be derived from the Gołab-Schinzel functional equation.

Suppose that $f : [0, \infty) \to [0, \infty)$ is a continuous or monotonic solution of Eq. (4). [If we want to start with (1), we can go to (4) via (2) and (3).] Applying, respectively, the result of [3] or Theorem 2 we infer that we have either $f(x) = \max\{1 + cx, 0\}$ for all $x \geq 0$ or $f = 0$ in $[0, \infty)$. It follows that f is differentiable everywhere except for at most one point $x_0 := \sup\{x > 0 : f(x) > 0\}$ and, if x_0 is finite, the left derivative $f'_-(x_0)$ and the right derivative $f'_+(x_0)$ exist. For convenience assume in the sequel that $f'(x_0) := f'_-(x_0)$. Let us fix an arbitrary $x \geq 0$. Differentiating both sides of Eq. (4) with respect to $y \geq 0$ we obtain

$$f'(x + yf(x))f(x) = f(x)f'(y).$$

Setting here $y = 0$ gives the differential equation

$$f(x)[f'(x) - f'(0)] = 0, \quad x \geq 0.$$

Suppose now that $f : [0, \infty) \to [0, \infty)$ satisfies this differential equation's initial condition $f(0) = 1$. Putting $c := f'(0)$ we infer that f must be of the form

$$f(x) = \max\{1 + cx, 0\}, \quad x \geq 0,$$

which coincides with the nontrivial solution of (4).

In a similar way Eq. (2a) for $s : [0,\infty) \to [0,\infty)$ and $p > 0$ leads to a differential equation

$$s'(x) = s'(0)[s(x)]^{1-1/p}, \quad x \geq 0,$$

integrating to

$$s(x) = (\max\{1 + cx, 0\})^p, \quad x \geq 0,$$

with some real constant $c := \frac{s'(0)}{p}$, which coincides with the nontrivial continuous solution of (2a). The limiting case $p \to \infty$ yields a differential equation for $s : [0,\infty) \to [0,\infty)$,

$$s'(x) = s'(0)s(x), \quad x \geq 0,$$

integrating to $s(x) = \exp(cx)$, $x \geq 0$ (with some real constant $c := s'(0)$), which coincides with the continuous (or measurable) solution of the Eq. (6) for $F = s$,

$$s(x + y) = s(x)s(y), \quad x, y \geq 0.$$

Traditionally, *differential* equations prevail in modelling time-dependent processes. On the other hand, *functional* equations can serve the same purpose. Looking at the differential equations of the previous section, denote one of them by (A), the respective functional equation by (B), and the solution by (C); we have a *correspondence*: the traditional approach $(A) \to (C)$ can be supplemented by considering *symmetry* properties of (A) and (C), expressed by (B).

Acknowledgement

We want to thank Professor János Aczél and Professor Ludwig Reich for constructive discussions.

References

[1] Aczél, J., Dhombres, J. (1989) Functional Equations in Several Variables. Cambridge University Press, Cambridge-New York

[2] Aczél, J., Gołab, St. (1970) Remarks on one-parameter subsemigroups of the affine group and their homo- and isomorphisms. – Aequationes Math. **4**: 1–10

[3] Aczél, J., Schwaiger, J. (1999) Continuous solutions of the Gołab-Schinzel equation on the nonnegative reals and on related domains. – Sitzungsber. Österr. Akad. Wiss., Math.-naturwiss. Kl., Abt.II **208**: 171–177

[4] Asimov's Biographical Encyclopedia of Science and Technology. Pan Reference Books, London & Sydney, 1978

[5] Baron, K. (1989) On the continuous solutions of the Gołab-Schinzel equation. – Aequationes Math. **38**: 155–162

[6] Brzdek, J. (1995) On some conditional functional equations of Gołab-Schinzel type. – Ann. Math. Siles. **9**: 65–80

[7] Brzdek, J. (2001) On continuous solutions of a conditional Gołab-Schinzel equation. – Anzeiger Österr. Akad. Wiss., Math.-naturwiss. Kl. Abt. II **138**: 3–6
[8] Cehak, K. (1978) Allgemeine Meteorologie. Prugg, Wien
[9] Dilke, O. A. W. (1987) Mathematics and Measurement. British Museum Publications, London
[10] Fahrenheit, D. G. (1724) Experimenta et observationes de congelatione aquae in vacuo factae. – Phil. Trans. **33**: 78–89
[11] Frisinger, H. H. (1983) The History of Meteorology: to 1800. American Meteorological Society, Boston
[12] Fuchs, N. A. (1959) Evaporation and Droplet Growth in Gaseous Media. Pergamon Press, London
[13] Gołab, St., Schinzel, A. (1960) Sur l'équation fonctionnelle $f[x + yf(x)] = f(x)f(y)$. – Publ. Math. Debrecen **6**: 113–125
[14] Guyot, G. (1997) Climatologie de l'Environnement. Masson, Paris
[15] Kahlig, P., Matkowski, J. (1998) On relating interval scales. Remark 14. – Aequationes Math. **55**: 303–304
[16] Kahlig, P., Schwaiger, J. (1994) Transforming the functional equation of Gołab-Schinzel into one of Cauchy. – Ann. Math. Siles. **8**: 33–38
[17] Kotschin, N. J., Kibel, I. A., Rose, N. W. (1954) Theoretische Hydromechanik I. Akademie-Verlag, Berlin
[18] Middleman, S. (1998) An Introduction to Fluid Dynamics. Wiley, New York
[19] Mureńko, A. (2001) On solutions of the Gołab-Schinzel equation. – Internat. J. Math. Math. Sci. **27**(9): 541–546
[20] Nakayama, Y., Boucher, R. F. (1999) Introduction to Fluid Mechanics. Arnold, London
[21] Reich, L. (1999) Über die stetigen Lösungen der Gołab-Schinzel-Gleichung auf $\mathbb{R}_{\geq 0}$. – Sitzungsber. Österr. Akad. Wiss., Math.-naturwiss. Kl., Abt.II **208**: 165–170
[22] Reich, L. (2001) Über die stetigen Lösungen der Gołab-Schinzel-Gleichung auf $\mathbb{R}$ und auf $\mathbb{R}_{\geq 0}$. – Anzeiger Österr. Akad. Wiss., Math.-naturwiss. Kl., Abt.II **138**: 7–12
[23] Rogers, R. R., Yau, M. K. (1989) A Short Course in Cloud Physics. Pergamon Press, Oxford
[24] Sablik, M. (2000) A conditional Gołab-Schinzel equation. – Anzeiger Österr. Akad. Wiss., Math.-naturwiss. Kl., Abt.II **137**: 11–15
[25] Schirmer H. (ed.) (1987) Meyers Kleines Lexikon Meteorologie. Meyers Lexikonverlag, Mannheim/Wien/Zürich
[26] White, F. M. (1999) Fluid Mechanics. McGraw-Hill, New York
[27] Zey, R. (ed.) (1997) Lexikon der Forscher und Erfinder. Rowohlt, Hamburg
[28] Ziegler, F. (1991) Mechanics of Solids and Fluids. Springer, New York

Authors' addresses: P. Kahlig, c/o Institute of Meteorology and Geophysics, University of Vienna, UZA2, A-1090 Vienna, Austria, e-mail: Peter.Kahlig@univie.ac.at; J. Matkowski, Institute of Mathematics, University of Zielona Góra, Podgórna 50, PL-65246 Zielona Góra, Poland, e-mail: J.Matkowski@im.uz.zgora.pl