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On subadditive functions and ψ-additive mappings

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Abstract: In [4], assuming among others subadditivity and submultiplicavity of a function $\psi : [0, \infty) \to [0, \infty)$, the authors proved a Hyers-Ulam type stability theorem for "ψ-additive" mappings of a normed space into a normed space. In this note we show that the assumed conditions of the function ψ imply that $\psi = 0$ and, consequently, every "ψ-additive" mapping must be additive.
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1 Introduction

This note is motivated by some recent papers [1], [3] and [4] concerning the Hyers-Ulam type stability theorems for *ψ-additive* mapping where the subadditive and submultiplicative functions were used. Recall the definition of *ψ-additive* mapping from [3]:

Let $\psi : [0, \infty) \to [0, \infty)$ be function, E_1 and E_2 normed spaces. A mapping $F : E_1 \to E_2$ is called *ψ-additive* if there is $\theta > 0$ such that

$$\|F(x + y) - F(x) - F(y)\| \leq \theta \left(\psi(\|x\|) + \psi(\|y\|)\right)$$

for all $x, y \in E_1$ (cf. [3]).

In [4] the following Hyers-Ulam type stability result is proved:

Theorem 1.1. Suppose that $\psi : [0, \infty) \to [0, \infty)$ satisfies the following conditions
(1) $\lim_{t\to\infty} \frac{\psi(t)}{t} = 0$

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(2) $\psi(st) \le \psi(s)\psi(t)$ for all $s, t \ge 0$
(3) $\psi(s+t) \le \psi(s) + \psi(t)$ for all $s, t \ge 0$
(4) ψ is monotone increasing
(5) $\psi(t) < t$ for all $t > 0$.

Then a mapping $F : E_1 \to E_2$ of normed space E_1into a normed space E_2 is ψ-additive if, and only if, there exist a constant $c > 0$ and an additive mapping $T : E_1 \to E_2$ such that

$$|F(x) - T(x)| \le c\psi(\|x\|), \qquad \text{for all } x \in E_1.$$

The ψ-additive functions were also considered in [3] and in a recent paper [1].

In this note we prove some properties of subadditive and submultiplicative functions. By applying them we infer that every function $\psi : [0, \infty) \to [0, \infty)$ satisfying the conditions 2 and 5 must be the zero function. Thus, every ψ-additive mapping with ψ satisfying only these two conditions is additive.

2 Some remarks on subadditive and multiplicative functions

We begin with the following

Proposition 2.1. A function $f : \mathbb{R} \to \mathbb{R}$ satisfies the following two conditions:
(i) f is subadditive, that is

$$f(s+t) \le f(s) + f(t), \qquad s, t \in \mathbb{R};$$

(ii) there is a $c \in \mathbb{R}$ such that for all $t \in \mathbb{R}$,

$$f(t) \le ct,$$

if, and only if, $f(t) = ct$ for all $t \in \mathbb{R}$.

Proof. Suppose that $f : \mathbb{R} \to \mathbb{R}$ satisfies condition (i) and (ii). Hence, for all $s, t \in \mathbb{R}$,

$$f(t) = f((t-s)+s) \le f(t-s) + f(s) \le c(t-s) + f(s),$$

whence

$$f(t) - ct \le f(s) - cs, \qquad s, t \in \mathbb{R}.$$

Changing the roles of s and t, we get the converse inequality, and consequently,

$$f(t) - ct = f(s) - cs, \qquad s, t \in \mathbb{R}.$$

Taking $s = 0$ gives

$$f(t) - ct = f(0), \qquad t \in \mathbb{R}.$$

Setting $s = t = 0$ in (i) we get $f(0) \ge 0$ and from (ii) we have $f(0) \le 0$. Thus $f(0) = 0$. This shows that

$$f(t) = ct, \qquad t \in \mathbb{R}.$$

Since the converse implication is obvious, the proof is complete.

Remark 2.2. To show that assumption (i) cannot be replaced by the subadditivity of f on $(0,\infty)$ or $[0,\infty)$, it is enough to observe that $f:[0,\infty)\to[0,\infty)$, given by $f(t):=\frac{t}{t+1}$, is subadditive and satisfies the inequality $f(t)\le t$ for all $t\ge 0$.

Corollary 2.3. A function $\psi:(0,\infty)\to(0,\infty)$ satisfies the following two conditions:
(i) f is submultiplicative, that is

$$\psi(st)\le\psi(s)\psi(t),\qquad s,t>0;$$

(ii) there is a $c\in\mathbb{R}$ such that for all $t>0$,

$$\psi(t)\le t^c,$$

if, and only if, $\psi(t)=t^c$ for all $t>0$.

Proof. Suppose that $\psi:(0,\infty)\to(0,\infty)$ satisfies conditions (i) and (ii). Then $f:\mathbb{R}\to\mathbb{R}$ defined by $f:=\log\circ\psi\circ\exp$ satisfies the conditions (i)–(ii) of Proposition 2.1. Thus $f(t)=ct$ for all $t\in\mathbb{R}$, and consequently, $\psi(t)=t^c$ for all $t>0$. The converse implication is obvious.

Assuming here $c:=1$ we obtain the following

Corollary 2.4. If a function $\psi:[0,\infty)\to[0,\infty)$ is submultiplicative, that is

$$\psi(st)\le\psi(s)\psi(t),\qquad s,t\ge 0,$$

and

$$\psi(t)<t,\qquad t>0,$$

then $\psi(t)=0$ for all $t\ge 0$.

Proof. Suppose that $\psi(s)=0$ for some $s>0$. Then from the submultiplicativity of ψ, we have

$$\psi(t)=\psi\left(s\frac{t}{s}\right)\le\psi(s)\,\psi\left(\frac{t}{s}\right)=0$$

for all $t\ge 0$, which shows that $\psi=0$ in $[0,\infty)$.

Suppose that $\psi(s)>0$ for all $s>0$. Then with $c=1$, the function $\psi\,|_{(0,\infty)}$ satisfies all the assumptions of Corollary 2.3. Consequently, in the other case, we would obtain $\psi(t)=t$ for all $t>0$, which contradicts the assumption $\psi(t)<t$ for all $t>0$.

We shall prove the following

Proposition 2.5. If $f:(0,\infty)\to[0,\infty)$ is subadditive, that is

$$f(s+t)\le f(s)+f(t),\qquad s,t>0;$$

and

$$\lim_{t\to 0} f(t) = 0$$

then the limit

$$\lim_{t\to 0} \frac{f(t)}{t}$$

exists and

$$\lim_{t\to 0} \frac{f(t)}{t} = \sup\left\{\frac{f(t)}{t} : t > 0\right\}.$$

Proof. Set

$$\beta := \sup\left\{\frac{f(t)}{t} : t > 0\right\},$$

and assume first that $\beta < +\infty$. Take $\varepsilon > 0$ and $a > 0$ such that

$$\frac{f(a)}{a} > \beta - \frac{\varepsilon}{2}.$$

For every $t \in (0, a)$ there is a unique positive integer $n \in \mathbb{N}$, $n = n(t)$, such that $\frac{a}{n+1} \leq t < \frac{a}{n}$, i.e.

$$\frac{an}{n+1} \leq nt < a, \qquad n \in \mathbb{N}.$$

Now, by the definition of β and the subadditivity of f, we have

$$\beta \geq \frac{f(t)}{t} = \frac{nf(t)}{nt} \geq \frac{f(nt)}{nt} \geq \frac{f(a) - f(a-nt)}{nt} \geq \frac{f(a)}{a} - \frac{f(a-nt)}{\frac{an}{n+1}}.$$

Since $n = n(t) \to \infty$ iff $t \to 0$, and

$$0 < a - nt \leq \frac{a}{n+1},$$

we infer that

$$\lim_{t\to 0}(a - nt) = 0.$$

The assumption that $\lim_{t\to 0} f(t) = 0$ implies there is a $\delta > 0$ such that

$$0 < t < \delta \Rightarrow \frac{f(a-nt)}{\frac{n}{n+1}a} < \frac{\varepsilon}{2}.$$

Now for all $t \in (0, \delta)$ we have

$$\beta \geq \frac{f(t)}{t} \geq \frac{f(a)}{a} - \frac{f(a-nt)}{\frac{an}{n+1}} \geq \left(\beta - \frac{\varepsilon}{2}\right) - \frac{\varepsilon}{2} = \beta - \varepsilon,$$

which proves that $\lim_{t\to 0} \frac{f(t)}{t}$ exists and

$$\lim_{t\to 0} \frac{f(t)}{t} = \beta.$$

In the case when $\beta = \infty$ we can argue in a similar way replacing β by an arbitrary number $M > 0$. This completes the proof.

Remark 2.6. Under the measurability assumptions the above result is proved in [2].

Corollary 2.7. Suppose that $f : (0, \infty) \to [0, \infty)$ is subadditive, that is

$$f(s+t) \leq f(s) + f(t), \qquad s, t > 0;$$

and

$$\lim_{t \to 0} f(t) = 0.$$

Then

$$\liminf_{t \to 0} \frac{f(t)}{t} \leq \limsup_{t \to \infty} \frac{f(t)}{t} \tag{1}$$

if and only if, there is a $c \in \mathbb{R}$ such that $f(t) = ct$ for all $t > 0$.

Proof. The assumptions imply that f is bounded on every finite subinterval of $[0, \infty)$. By the subadditivity of f (cf. Hille, Phillips [2, p.244, Theorem7.6.1] there exists a finite limit

$$\alpha := \lim_{t \to \infty} \frac{f(t)}{t}$$

and

$$\alpha = \inf_{t>0} \left\{ \frac{f(t)}{t} : t > 0 \right\}.$$

Putting

$$\beta := \sup \left\{ \frac{f(t)}{t} : t > 0 \right\},$$

in view of Proposition 2.5, we have

$$\beta = \lim_{t \to 0} \frac{f(t)}{t}.$$

Of course we have $\beta \geq \alpha$. Condition (1) implies that $\alpha \geq \beta$ and, consequently, $\alpha = \beta$. Setting $c := \alpha$ we obtain $f(t) = ct$ for all $t > 0$.

3 Remarks on ψ-additive maps

From Corollary 2.4 we obtain the following

Remark 3.1. Suppose that $\psi : [0, \infty) \to [0, \infty)$ satisfies the conditions 2 and 5 of Theorem 1.1. Then every mapping of normed space E_1 into a normed space E_2 is ψ-additive if, and only if, it is additive.

Applying Proposition 2.5 we get the following

Remark 3.2. Let $\psi : [0, \infty) \to [0, \infty)$ satisfy the conditions 2 and 3 of Theorem 1.1. Suppose that $\lim_{t \to 0} \psi(t) = 0$ and 0 is a cluster point of the set

$$\{t > 0 : \psi(t) < ct\}$$

for some $c < 1$. Applying Proposition 2.5 we infer that $\psi(t) < t$ for all $t > 0$, that is, condition 5 is satisfied.

These remarks show that condition 5 of Theorem 1.1 should be either removed or replaced by a weaker one.

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