

*Separatum*

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Another characterization of the gamma function

## Another characterization of the gamma function

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**Abstract.** The function  $\frac{\log \Gamma(x)}{\log x}$  is characterized to be the only convex solution of the functional equation

$$f(x+1) = \frac{\log x}{\log(x+1)}(f(x)+1), \quad x \in (0, \infty).$$

Some relations to the function  $\log \Gamma(x+1)/x^a$ ,  $0 < a \leq 1$  are shown.

### 0. Introduction

In this paper we examine the behavior of the Euler gamma function  $\Gamma$  in the logarithmically scaled coordinate system. More exactly, we show that the function  $g : (0, \infty) \rightarrow \mathbb{R}$  defined by

$$g(x) := \begin{cases} \frac{\log \Gamma(x)}{\log x} & \text{for } x \neq 1, \\ -\gamma & \text{for } x = 1, \end{cases}$$

(where  $\gamma$  is the Euler gamma constant) is increasing, convex,  $g(0+) = -1$  and  $g(2) = 0$ . The main result of our paper states that the function  $g$  is the only convex solution of the functional equation

$$f(x+1) = \frac{\log x}{\log(x+1)}(f(x)+1), \quad x \in (0, \infty). \quad (1)$$

One can weaken the supposition of convexity of the solution in this way that only convexity is supposed in a neighborhood of infinity.

Note that no initial condition is required.

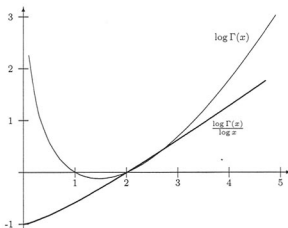
We would like to remark that this characterization of the gamma function is not a consequence of the famous Bohr–Møllerup characterization of the gamma function (see e.g. [2] or [3], p. 288) as the only log-convex solution of the functional equation

$$f(x+1) = x \cdot f(x), \quad x \in (0, \infty); \quad f(1) = 1. \quad (2)$$

And, the more, it cannot be derived from the recent generalization of the Bohr–Møllerup theorem [5] that says that the gamma function is the only solution of (2), which is geometrically convex on a neighborhood of infinity. In these characterizations the initial condition is indispensable.

As an interesting consequence we infer that the function  $G(x) = \frac{\log \Gamma(\exp x)}{\log x}$  is strictly increasing and strictly convex on  $\mathbb{R}$ .

We further consider the functions  $\frac{\log \Gamma(x+1)}{\log x^\alpha}$  and  $\frac{\log \Gamma(x)}{\log x^\alpha}$  for a fixed real  $\alpha$ ,  $0 < \alpha \leq 1$ , relating it with a recent paper of GRABNER *et al.* [4].



1. Some properties of  $\log \Gamma(x)/\log x$ 

The function  $g(x) := \frac{\log \Gamma(x)}{\log x}$  is analytic on  $(0, \infty)$  with a removable singularity at  $x = 1$ ,  $g(1) = -0.577215 \dots = -\gamma$  (where  $\gamma = \lim_{n \rightarrow \infty} (\sum_{k=1}^n \frac{1}{k} - \log n)$ , the Euler gamma constant). We further have  $g(0+) = \lim_{x \rightarrow 0+} g(x) = -1$ , and  $g'_+(0) = 0$ .

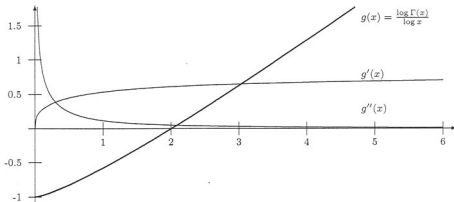
To show this the following representations are useful (see [3], p. 287f.):

$$\log \Gamma(x) = -\log x - \gamma x - \sum_{n=1}^{\infty} \left( \log \left( 1 + \frac{x}{n} \right) - \frac{x}{n} \right), \quad x > 0;$$

$$\Psi(x) = \frac{d}{dx} \log \Gamma(x) = \frac{\Gamma'(x)}{\Gamma(x)} = -\frac{1}{x} - \gamma + \sum_{n=1}^{\infty} \frac{x}{(x+n)n}, \quad x > 0.$$

**Proposition.** The function  $g(x) = \frac{\log \Gamma(x)}{\log x}$  is strictly monotone increasing and strictly convex on  $(0, \infty)$ .

PROOF. We show that  $g'$  and  $g''$  are positive on  $(0, \infty)$ . To do this we use asymptotic expansions for  $\log \Gamma$ ,  $\Psi$  and  $\Psi'$  which will show us, that  $g'(x)$  and  $g''(x)$  are positive for large  $x$ . For smaller  $x$  one can see this from the graph of these functions:



The following asymptotic formulas are valid for positive  $x$ :

$$\log \Gamma(x) = \left( x - \frac{1}{2} \right) \log x - x + \frac{1}{2} \log(2\pi) + \frac{1}{12x} - \frac{1}{360x^3} + \dots \quad (3)$$

$$\Psi(x) = \log x - \frac{1}{2x} - \frac{1}{12x^2} + \frac{1}{120x^4} - \frac{1}{252x^6} + \dots \quad (4)$$

$$\Psi'(x) = \frac{1}{x} + \frac{1}{2x^2} + \frac{1}{6x^3} - \frac{1}{30x^5} + \frac{1}{42x^7} - \dots \quad (5)$$

If we only take a partial sum of one of these series then the error will be less than the first term neglected and has the same sign ([1], p. 257f.).

1. Note that

$$g'(x) = \frac{\Psi(x)}{\log x} - \frac{\log \Gamma(x)}{x(\log x)^2}, \quad x > 0,$$

with a removable singularity at  $x = 1$ . By (3) and (4) we have

$$\log \Gamma(x) < (x - 1/2x) \log x \quad \text{and} \quad \Psi(x) > \log x - \frac{1}{2x} - \frac{1}{12x^2}.$$

Hence

$$\begin{aligned} g'(x)x(\log x)^2 &= x(\log x)\Psi(x) - \log \Gamma(x) \\ &> x(\log x) \left( \log x - \frac{1}{2x} - \frac{1}{12x^2} \right) - \left( x - \frac{1}{2} \right) \log x > 0 \end{aligned}$$

if  $x > 2.7484\dots$ .

2.  $g''(x) = \frac{\Psi'(x)}{\log x} - \frac{2\Psi(x)}{x(\log x)^2} + \left( \frac{1}{x^2(\log x)^2} + \frac{2}{x^2(\log x)^3} \right) \log \Gamma(x)$  for  $x > 0$ , again with a removable singularity at  $x = 1$ . We get

$$\begin{aligned} x^2(\log x)^3 g''(x) &= x^2(\log x)^2 \Psi'(x) + (\log x + 2) \log \Gamma(x) \\ &\quad - 2x \log x \Psi(x). \end{aligned} \quad (6)$$

Here we use the inequalities following from (5), (3) (where  $\frac{1}{2} \log(2\pi) = 0.9189\dots$ ) and (4):

$$\Psi'(x) > \frac{1}{x} + \frac{1}{2x^2}, \quad \log \Gamma(x) > \left( x - \frac{1}{2} \right) \log x - x + 0.9 \quad \text{and} \quad \Psi(x) < \log x.$$

Herewith we get a lower bound for (6) by

$$\left( x - \frac{1}{10} \right) \log x - \frac{10x - 9}{5},$$

which is positive for say  $x \geq 6$  (more exactly  $x > 5.491776524\dots$ ). Thus also  $g''(x) > 0$  for at least  $x > 5.491776524\dots$   $\square$

*Remark 1.* In the previous proof we found it more appropriate to use computer aided calculations to show the convexity of  $g(x)$  for small  $x$  than to tackle complicate inequalities. For those who are not convinced by these methods we have a second version of our main result (see Theorem 2 below).

*Remark 2.* The function  $G : \mathbb{R} \rightarrow \mathbb{R}$  defined by  $G = g \circ \exp$ , i.e.

$$G(x) = \frac{\log \Gamma(\exp x)}{x},$$

as a composition of two strictly increasing and strictly convex functions is again *strictly increasing and strictly convex* on  $\mathbb{R}$ .

## 2. The functional equation

The function  $g$  satisfies the functional equation

$$f(x+1) = \frac{\log x}{\log(x+1)}(f(x)+1), \quad x \in (0, \infty). \quad (7)$$

If  $f : (0, \infty) \rightarrow \mathbb{R}$  is an arbitrary solution of (7), then (7) with  $x = 1$  yields

$$f(2) = 0. \quad (8)$$

Thus, the initial condition (8) is forced by the functional equation (7) itself. Furthermore we have

$$f(n) = \frac{\log \Gamma(n)}{\log n}, \quad n \in \mathbb{N}, \quad n \geq 2.$$

and also

$$f(x+n) = \frac{\log x}{\log(x+n)}f(x) + \frac{\log[(x+n-1)\dots(x)]}{\log(x+n)}, \quad x \in (0, \infty), \quad n \in \mathbb{N}.$$

Hence also for  $x \in (0, \infty)$ ,  $n \in \mathbb{N}$ ,

$$f(x+n+1) = \frac{\log x}{\log(x+n+1)}f(x) + \frac{\log[(x+n)\dots(x)]}{\log(x+n+1)}, \quad (9)$$

**Theorem 1.** *The only solution of (7), convex on  $(0, \infty)$  is*

$$g(x) = \frac{\log \Gamma(x)}{\log x}.$$

PROOF. Let  $f : (0, \infty) \rightarrow \mathbb{R}$  a solution of (7), convex on  $(0, \infty)$ . Then we have automatically (8) and

$$f(n) = g(n) = \frac{\log(n-1)!}{\log n}, \quad n \in \mathbb{N}, \quad n \geq 2. \quad (10)$$

For  $x \in (0, 1]$ ,  $n \in \mathbb{N}$ ,  $n \geq 2$  the convexity condition yields:

$$f(n+1) - f(n) \leq \frac{f(x+n+1) - f(n+1)}{x} \leq f(n+2) - f(n+1), \quad (11)$$

whence

$$\begin{aligned} 0 &\leq \frac{1}{x} [f(x+n+1) - f(n+1) - x(f(n+1) - f(n))] \\ &\leq f(n+2) + f(n) - 2f(n+1). \end{aligned}$$

Hence, applying in turn: relation (9) and (10), multiplication by  $\log(x+n+1) > 0$ , the monotonicity of  $\log$  and  $\frac{\log \Gamma}{\log}$ , and, finally the Stirling formula which implies that  $\log n! < (n + \frac{1}{2}) \log n - n + 1$  (see [1], p. 257), we obtain

$$\begin{aligned} 0 &\leq \frac{1}{x} \left[ \log x \cdot f(x) + \log[(x+n) \dots x] \right. \\ &\quad \left. - x \log(x+n+1) \left( \frac{\log n!}{\log(n+1)} - \frac{\log(n-1)!}{\log n} \right) \right] \\ &\leq \log(x+n+1) \left[ \frac{\log(n+1)!}{\log(n+2)} + \frac{\log(n-1)!}{\log n} - 2 \frac{\log n!}{\log(n+1)} \right] \\ &\leq \log(n+2) \left[ \frac{\log(n+1)!}{\log(n+2)} + \frac{\log(n-1)!}{\log n} - 2 \frac{\log n!}{\log(n+1)} \right] \\ &= \log(n+2) \left[ \frac{\log(n+1)!}{\log(n+2)} - \frac{\log n!}{\log(n+1)} \right] \\ &\quad + \log(n+2) \left[ \frac{\log(n-1)!}{\log n} - \frac{\log n!}{\log(n+1)} \right] \end{aligned}$$

$$\begin{aligned}
&< \log(n+2) \left[ \frac{\log(n+1)!}{\log(n+2)} - \frac{\log n!}{\log(n+1)} \right] \\
&\quad + \log(n+1) \left[ \frac{\log(n-1)!}{\log n} - \frac{\log n!}{\log(n+1)} \right] \\
&= \log(n+1)! - \log n! - \frac{\log(n+2)}{\log(n+1)} \log n! + \frac{\log(n+1)}{\log n} \log(n-1)! \\
&= \log(n+1) - \frac{\log(n+1)}{\log n} (\log n! - \log(n-1)!) \\
&\quad + \left( \frac{\log(n+1)}{\log n} - \frac{\log(n+2)}{\log(n+1)} \right) \log n! \\
&= \left[ \frac{\log(n+1)}{\log n} - \frac{\log(n+2)}{\log(n+1)} \right] \log n! \\
&< \left[ \frac{\log(n+1)}{\log n} - \frac{\log(n+2)}{\log(n+1)} \right] \left[ \left( n + \frac{1}{2} \right) \log n - n + 1 \right] =: \theta(n).
\end{aligned}$$

With the aid of the expansions

$$\log(n+1) = \log n - \log \left( 1 - \frac{1}{n+1} \right) = \log n + \sum_{i=1}^{\infty} \frac{1}{i \cdot (n+1)^i},$$

$$\begin{aligned}
\log(n+2) &= \log(n+1) + \log \left( 1 + \frac{1}{n+1} \right) \\
&= \log(n+1) + \sum_{i=1}^{\infty} \frac{(-1)^{i+1}}{i \cdot (n+1)^i}
\end{aligned}$$

we get

$$\begin{aligned}
&\frac{\log(n+1)}{\log n} - \frac{\log(n+2)}{\log(n+1)} \\
&= \frac{1}{\log n} \cdot \sum_{i=1}^{\infty} \frac{1}{i \cdot (n+1)^i} - \frac{1}{\log(n+1)} \cdot \sum_{i=1}^{\infty} \frac{(-1)^{i+1}}{i \cdot (n+1)^i} \\
&= \left( \frac{1}{\log n} - \frac{1}{\log(n+1)} \right) \frac{1}{n+1} \\
&\quad + \left( \frac{1}{\log n} + \frac{1}{\log(n+1)} \right) \frac{1}{2(n+1)^2} + \cdots = o \left( \frac{1}{n \log n} \right).
\end{aligned}$$



From this it is easy to see that  $\theta(n)$  tends to 0 for  $n \rightarrow \infty$ . Therefore for  $x \in (0, 1]$  we have

$$f(x) = \lim_{n \rightarrow \infty} \left[ x \log(x+n+1) \left( \frac{\log n!}{\log(n+1)} - \frac{\log(n-1)!}{\log n} \right) - \log[(x+n) \dots x] \right] \frac{1}{\log x}$$

that is,  $f$  is uniquely determined on  $(0, 1]$ . According to well known uniqueness theorems on difference equations, we have that  $f$ , as a convex solution of (7), is uniquely determined on all of  $(0, \infty)$ . Since  $g$  is also a convex solution of (7), we have  $f = g$ .  $\square$

**Theorem 2.** *The only solution of (7), convex on a neighborhood of infinity is  $g(x) = \frac{\log \Gamma(x)}{\log x}$ .*

PROOF. Let  $f : (0, \infty) \rightarrow \mathbb{R}$  a solution of (7), convex say for  $x > a$  for some positive real  $a$ . We can proceed as in the proof of Theorem 1, except that we have suppose in (11) that  $n > a$  holds.  $\square$

### 3. Final remarks

In a recent paper [4] P. GRABNER *et al.* showed that the function  $x \mapsto \frac{\log \Gamma(x+1)}{x}$  is concave on  $(-1, \infty)$  and it is characterized as the only concave solution of its functional equation. In contrary to this the function  $\frac{\log \Gamma(x+1)}{\log x} = \frac{\log \Gamma(x)}{\log x} + 1$  is convex on  $(0, \infty)$ .

In this connection the following question arises. What is the behavior of the function  $x \mapsto \frac{\log \Gamma(x+1)}{x^\alpha}$  or  $x \mapsto \frac{\log \Gamma(x)}{x^\alpha}$  for a fixed real  $\alpha$ ? Here the case  $0 < \alpha < 1$  is of interest.

For  $\alpha \in (0, 1)$  the plotted graph of these functions looks like convex, at least for small  $x$ . Nevertheless it is easy to show that there is a constant  $c$ , depending on  $\alpha$ , such that these functions are concave for  $x > c$ .

The function  $x \mapsto \frac{\log \Gamma(x)}{x^\alpha}$  e.g. fulfills the functional equation

$$f(x+1) = f(x) \cdot \frac{x^\alpha}{(x+1)^\alpha} + \frac{\log(x)}{(x+1)^\alpha}, \quad x \in (0, \infty). \quad (12)$$

It is routine to show that  $x \mapsto \frac{\log \Gamma(x)}{x^\alpha}$  is the only solution of (12), concave in a neighborhood of infinity, together with the side condition  $f(1) = 0$ . A similar statement can be given for the function  $x \mapsto \frac{\log \Gamma(x+1)}{x^\alpha}$ .

By this we receive different forms of characterizations of the gamma function.

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