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A solution of a problem of Z. Daróczy on mixing-arithmetic means

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Abstract. Under some regularity assumptions, a problem of Z. Daróczy on mixed quasi-arithmetic means is solved.

1. Introduction

In 1999, Z. Daróczy [3] posed a problem to determine the class of "mixing-arithmetic means" which are quasi-arithmetic. In this paper we solve this problem, under some regularity assumptions of the generators of these means.

In Section 2 we introduce necessary definitions to formulate the problem and recall Aczél's theorem on bisymetry functional equation which is applied in the proof of the main result. In Section 3, assuming that the generators of the occurring means are of the class C^2, we prove that the "mixing-arithmetic mean" is quasi-arithmetic if, and only if, it is the arithmetic mean. In the proof, unexpectedly, a Stolarsky mean $E_{-2,-3}$, related to Cauchy's mean-value theorem, appears.

2. Some definitions and Aczél's theorem

Let $I \subset \mathbb{R}$ be an open interval. A function $M: I \times I \to I$ is called a *mean* if

$$\min(x,y) \le M(x,y) \le \max(x,y), \qquad x,y \in I.$$

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A mean M is called *strict* if for all $x, y \in I$, $x \neq y$, these inequalities are sharp. A mean M is called *symmetric* if for all $x, y \in I$, $M(x,y) = M(y,x)$. A mean $M: (0,\infty)^2 \to (0,\infty)$ is called *homogeneous if*

$$M(tx, ty) = tM(x,y), \qquad t.x, y > 0.$$

By $CM(I)$ denote the set of all continuous and strictly monotonic functions $\varphi: I \to \mathbb{R}$. Recall that a mean $M: I \times I \to I$ is called *quasi-arithmetic* if there exists a function $\varphi \in CM(I)$ such that $M = M^{[\varphi]}$, where

$$M^{[\varphi]}(x,y) := \varphi^{-1}\left(\frac{\varphi(x)+\varphi(y)}{2}\right), \qquad x, y \in I.$$

Following Z. Daróczy, a mean $M: I \times I \to I$ is called *mixing-arithmetic* if there exists a function $\psi \in CM(I)$ such that

$$M_\psi(x,y) = \psi^{-1}\left(\frac{\psi(x)+\psi(y)+\psi(\frac{x+y}{2})}{3}\right), \qquad x, y \in I.$$

In the proof of the main result we need the following (cf. [1] or J. Aczél and J. Dhombres [2], Theorem 1, p. 287-288).

Theorem 1. (J. Aczél). *Let $I \subset \mathbb{R}$ be an interval. Suppose that $M: I \times I \to I$ is a symmetric and continuous mean which is strictly increasing with respect to each variable. Then M is quasi-arithmetic if, and only if, M satisfies the bisymmtry functional equation*

$$M\left(M(x,y), M(z,w)\right) = M\left(M(x,z), M(y,w)\right), \qquad x, y, z, w \in I.$$

3. Main result

In this paper we prove the following

Theorem 2. *Let $\varphi, \psi \in CM(I)$ be twice continuously differentiable. Then*

$$M_\psi = M^{[\varphi]}$$

if, and only if,

$$\varphi(x) = ax + b, \qquad \psi(x) = cx + d, \qquad x \in I,$$

for some $a, b, c, d \in \mathbb{R}$, $a \neq 0 \neq c$.

Proof. Suppose that $M^{[\varphi]} = M_\psi$, i.e. that

$$\varphi^{-1}\left(\frac{\varphi(x)+\varphi(y)}{2}\right) = \psi^{-1}\left(\frac{\psi(x)+\psi(y)+\psi(\frac{x+y}{2})}{3}\right), \qquad x,y \in I.$$

Setting

$$f := \varphi \circ \psi^{-1}, \qquad g := \psi^{-1}, \qquad J := \psi(I),$$

we hence get

$$3M^{[f]}(x,y) = M^{[g]}(x,y) + x + y, \qquad x,y \in J, \tag{1}$$

that is,

$$3f^{-1}\left(\frac{f(x)+f(y)}{2}\right) = g^{-1}\left(\frac{g(x)+g(y)}{2}\right) + x + y, \qquad x,y \in J. \tag{2}$$

Assume additionally that

$$\varphi'(x) \neq 0 \neq \psi'(x), \qquad x \in I. \tag{3}$$

Now the assumptions of φ and ψ imply that f and g are twice continuously differentiable in J. Differentiating both sides of this equation with respect to x we obtain

$$3\frac{f'(x)}{f'\left(M^{[f]}(x,y)\right)} = \frac{g'(x)}{g'\left(M^{[g]}(x,y)\right)} + 2, \qquad x,y \in J. \tag{4}$$

Similarly, differentiating both sides with respect to y, we obtain

$$3\frac{f'(y)}{f'\left(M^{[f]}(x,y)\right)} = \frac{g'(y)}{g'\left(M^{[g]}(x,y)\right)} + 2, \qquad x,y \in J.$$

Subtracting these two equations we get

$$3\frac{f'(x)-f'(y)}{f'\left(M^{[f]}(x,y)\right)} = \frac{g'(x)-g'(y)}{g'\left(M^{[g]}(x,y)\right)}, \qquad x,y \in J,$$

which implies that

$$3\frac{\frac{f'(x)-f'(y)}{x-y}}{f'\left(M^{[f]}(x,y)\right)} = \frac{\frac{g'(x)-g'(y)}{x-y}}{g'\left(M^{[g]}(x,y)\right)}, \qquad x,y \in J, x \neq y.$$

Letting here $y \to x$ we get

$$3\frac{f''(x)}{f'(x)} = \frac{g''(x)}{g'(x)}, \qquad x \in J,$$

and, consequently, there is a constant $k \in \mathbb{R}$, $k \neq 0$, such that

$$g' = k \cdot (f')^3.$$

Hence, applying (4), we obtain

$$3\frac{f'(x)}{f'\left(M^{[f]}(x,y)\right)} = \frac{[f'(x)]^3}{[f'\left(M^{[g]}(x,y)\right)]^3} + 2, \qquad x, y \in J,$$

which implies that

$$f'\left(M^{[g]}(x,y)\right) = \frac{f'(x)}{\left(3\frac{f'(x)}{f'\left(M^{[f]}(x,y)\right)} - 2\right)^{1/3}}, \qquad x, y \in J.$$

Suppose that there is an $x_0 \in J$ such that $f''(x_0) \neq 0$. Since f'' is continuous, there is a subinterval K of the interval J such that f' is strictly monotonic in K. As $M^{[g]}$ is a mean, we have $M^{[g]}(K,K) = K$ and, consequently,

$$M^{[g]}(x,y) = (f')^{-1}\left(\frac{f'(x)}{\left(3\frac{f'(x)}{f'\left(M^{[f]}(x,y)\right)} - 2\right)^{1/3}}\right), \qquad x, y \in K.$$

Hence, making use of the relations (1) or (2), we get

$$3M^{[f]}(x,y) = (f')^{-1}\left(\frac{f'(x)}{\left(3\frac{f'(x)}{f'\left(M^{[f]}(x,y)\right)} - 2\right)^{1/3}}\right) + x + y, \qquad x, y \in K,$$

which can written in the equivalent form

$$\left[f'\left(3M^{[f]}(x,y) - x - y\right)\right]^3 = \frac{[f'(x)]^3 f'\left(M^{[f]}(x,y)\right)}{3f'(x) - 2f'\left(M^{[f]}(x,y)\right)}, \qquad x, y \in K.$$

The symmetry of the left-hand side of this equation implies that

$$\left[f'\left(3M^{[f]}(x,y)-x-y\right)\right]^3=\frac{[f'(y)]^3 f'\left(M^{[f]}(x,y)\right)}{3f'(y)-2f'\left(M^{[f]}(x,y)\right)},\qquad x,y\in K.$$

From the last two equations we obtain

$$\frac{[f'(x)]^3 f'\left(M^{[f]}(x,y)\right)}{3f'(x)-2f'\left(M^{[f]}(x,y)\right)}=\frac{[f'(y)]^3 f'\left(M^{[f]}(x,y)\right)}{3f'(y)-2f'\left(M^{[f]}(x,y)\right)},\qquad x,y\in K,$$

or, equivalently,

$$\frac{[f'(x)]^3}{3f'(x)-2f'\left(M^{[f]}(x,y)\right)}=\frac{[f'(y)]^3}{3f'(y)-2f'\left(M^{[f]}(x,y)\right)},\qquad x,y\in K.$$

Hence, after simple calculations, we obtain

$$f'\left(M^{[f]}(x,y)\right)=\frac{3f'(x)f'(y)\left[f'(x)+f'(y)\right]}{2\left[f'(x)^2+f'(x)f'(y)+f'(y)^2\right]},\qquad x,y\in K.$$

Putting

$$h:=f'\circ f^{-1},$$

and making use of the definition of the mean $M^{[f]}$, we hence get

$$h\left(\frac{x+y}{2}\right)=\frac{3h(x)h(y)\left[h(x)+h(y)\right]}{2\left[h(x)^2+h(x)h(y)+h(y)^2\right]},\qquad x,y\in f(K).$$

Since h is continuous and strictly monotonic in $f(K)$, we can write this functional equation in the form

$$h\left(\frac{h^{-1}(x)+h^{-1}(y)}{2}\right)=\frac{3xy(x+y)}{2(x^2+xy+y)^2},\qquad x,y\in f'(K). \tag{5}$$

The left-hand side is a quasi-arithmetic mean with a generator h^{-1}. The right-hand side

$$M(x,y):=\frac{3xy(x+y)}{2(x^2+xy+y)^2},\qquad x,y\in f'(K),$$

is also a mean. However it is not quasi-arithmetic. In fact, if M were quasi-arithmetic, then, according to theorem of Aczél, we would have

$$M\left(M(x,y),M(z,w)\right)=M\left(M(x,z),M(y,w)\right)$$

for all $x, y, z, w \in f'(K)$ and, as M is real analytic in $(0,\infty)^2$, this relation would hold true for all positive x, y, z, w. Since

$$M\left(M(1,2), M(3,4)\right) \neq M\left(M(1,3), M(2,4)\right),$$

the mean M is not quasi-arithmetic. Thus relation (5) is false and, consequently,

$$f''(x) = 0, \qquad x \in J.$$

Hence

$$f(x) = \alpha x + \beta, \qquad x \in J,$$

for some $\alpha, \beta \in \mathbb{R}$, $\alpha \neq 0$. Setting this function into (2) we infer that

$$g^{-1}\left(\frac{g(x)+g(y)}{2}\right) = \frac{x+y}{2}, \qquad x, y \in J,$$

and, consequently,

$$g(x) = \alpha' x + \beta', \qquad x \in J.$$

for some $\alpha', \beta' \in \mathbb{R}$, $\alpha' \neq 0$. Now the definitions of the functions f and g imply that

$$\varphi(x) = ax + b, \qquad \psi(x) = cx + d, \qquad x \in I,$$

for some $a, b, c, d \in \mathbb{R}$, $a \neq 0 \neq c$.

Till now we have additionally assumed condition (3). According to the assumptions of the theorem the set

$$Z := \{x \in I : \varphi'(x) = 0 \text{ or } \psi'(x) = 0\}$$

is closed and its interior is empty. Thus

$$I \setminus Z = \bigcup_{s \in S} I_s$$

where $\{I_s : s \in S\}$ is a family of open disjoints intervals for some at most countable set S. According to what has already been proved, the functions φ and ψ are affine on each of the interval I_s, i.e.

$$\varphi(x) = a_s x + b_s, \qquad \psi(x) = c_s x + d_s, \qquad x \in I_s,$$

for some $a_s, b_s, c_s, d_s \in \mathbb{R}$, $a_s \neq 0 \neq c_s$. Now the differentiability of the functions φ and ψ implies that there are $a, b, c, d \in \mathbb{R}$, $a \neq 0 \neq c$, such that

$$a_s = a, \quad b_s = b, \quad c_s = c, \quad d_s = c,$$

and, consequently, the set Z must be empty. This completes the proof of the "only if" part of our theorem. Since the converse implication is obvious, the proof is completed.

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Corollary 1. *Let $\varphi, \psi \in CM(I)$ be twice continuously differentiable. Then $M_\psi = M^{[\varphi]}$ if, and only if, $M_\psi = A = M^{[\varphi]}$, where A denotes the arithmetic mean in I^2.*

In connection with relation (5) let us note

Remark 1. The function $M\colon (0,\infty)^2 \to (0,\infty)$,

(6)
$$M(x,y) := \frac{3xy(x+y)}{2(x^2+xy+y)^2}, \qquad x,y>0,$$

which has appeared in proof of the theorem, is a homogeneous mean. It is not difficult to show that the power mean

$$\left(\frac{x^p+y^p}{2}\right)^{1/p}, \qquad x,y>0,$$

with

$$p := \frac{\log 2}{\log 2 - \log 3}$$

is the closest to M of all the homogeneous quasi-arithmetic means.

Remark 2. Mean (6) is related to Cauchy mean-value theorem. In fact, applying the Cauchy's mean-value theorem for functions

$$x \to x^{-2}, \qquad x \to x^{-3}, \qquad (x>0),$$

for every fixed $x,y>0$, $x \neq y$, there exists an $M(x,y)$, $\min\{x,y\} < M(x,y) < \max\{x,y\}$, such that

$$\frac{x^{-2}-y^{-2}}{x^{-3}-y^{-3}} = \frac{-2(M(x,y))^{-3}}{-3(M(x,y))^{-4}} = \frac{2}{3}M(x,y),$$

and, consequently,

$$M(x,y) = \frac{3}{2}\frac{x^{-2}-y^{-2}}{x^{-3}-y^{-3}} = \frac{3xy(x+y)}{2(x^2+xy+y)^2}, \qquad x \neq y.$$

Note also that

$$M = E_{-2,-3},$$

where $\{E_{p,q}:\ p,q \in \mathbb{R}\}$ is a known family of Stolarski's means.

References

[1] J. Aczél, On mean values, *Bull. Amer. Math. Soc.*, **54** (1948), 392–400.

[2] J. Aczél and J. Dhombres, *Functional equations in several variables*, Encyclopedia Math. Appl., Cambridge University Press, 1989.

[3] Z. Daróczy, A private communication.

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