Reprint from
aequationes mathematicae
Vol. 8, fasc. 3, 1972 BIRKHÄUSER VERLAG BASEL pages 233–237

# The Uniqueness of Solutions of a System of Functional Equations in Some Classes of Functions

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In the present paper we consider the problem of the uniqueness of solutions of the system of functional equations

$$\varphi_i(x) = h_i(x, \varphi_1[f_{i1}(x)], \ldots, \varphi_n[f_{in}(x)]), \quad i = 1, \ldots, n, \tag{1}$$

where $f_{ik}$ and $h_i$ are given functions and $\varphi_i$ are unknown.

Theorem 1 below yields the uniqueness of solutions $\varphi_i$ $(i=1,\ldots,n)$ in certain classes $\mathcal{B}_i^r$. In the case $n=1$ this result is related to the investigations of B. Choczewski [2] and M. Kuczma [4] regarding the asymptotic behavior at the fixed points of the function $f$ of solutions (the so-called regular solutions) of the equation

$$\varphi(x) = h(x, \varphi[f(x)]) \tag{2}$$

or of its particular cases.

As a corollary to Theorem 1 we obtain a uniqueness theorem for the differentiable solutions of the system (1). The main results concerning the differentiable solutions of equation (2) are due to B. Choczewski [1] (cf. also [3], Chapter IV). Some recent contributions are contained in the author's papers [7] and [8].

In general the solution of equation (2) as well as that of system (1) depends on an arbitrary function (cf. [3], Theorems 3.1, 4.1, 12.5), therefore conditions ensuring the uniqueness of solutions are of considerable importance in the theory of functional equations in a single variable (cf. [5]).

**1.** Let $I$ be an interval. We assume that

(i) Each function $f_{ik}$ is defined in $I$ and there exists a $\xi \in I$ such that

$$0 < \frac{f_{ik}(x) - \xi}{x - \xi} < 1 \quad \text{for} \quad x \in I \setminus \{\xi\}, \quad f_{ik}(\xi) = \xi, \quad i, k = 1, \ldots, n.$$

*Remark* 1. This condition implies that for every interval $I_1 \subset I$ such that $\xi \in I_1$ and for every $i, k = 1, \ldots, n$ we have $f_{ik}(I_1) \subset I_1$. If, moreover, $f_{ik}$ is continuous in $I$, then for every $x \in I$ we have $\lim_{n\to\infty} f_{ik}^n(x) = \xi$, where $f_{ik}^n$ denotes the $n$-th iterate of the function $f_{ik}$.

Setting $x = \xi$ in (1) we obtain for the values $\eta_i = \varphi_i(\xi)$ $(i = 1, \ldots, n)$ the conditions

$$h_i(\xi, \eta_1, \ldots, \eta_n) = \eta_i, \; i = 1, \ldots, n. \tag{3}$$

Thus we assume

(ii) For every $i = 1, \ldots, n$ the function $h_i$ is defined in a domain $\Omega \subset R^{n+1}$; there exists a point $(\xi, \eta_1, \ldots, \eta_n) \in \Omega$ such that condition (3) is fulfilled.

*Received March 27, 1971 and in revised form June 23, 1971*

Let $\eta_i^{(1)}, \ldots, \eta_i^{(r-1)}$ $(i=1,\ldots,n)$ be a fixed system of $(r-1)\eta$ numbers. We introduce the following definition of the function classes $B_i^r$.

DEFINITION. $\varphi_i \in B_i^r$ $(i=1,\ldots,n)$ if and only if $\varphi_i$ is defined in $I$ and there exists a number $\mu_i > 0$ such that

$$\varphi_i(x) = P_i(x) + (x-\xi)^r \Phi_i(x), \quad |\Phi_i(x)| \leq \mu_i, \quad x \in I, \tag{4}$$

where

$$P_i(x) = \eta_i + \sum_{k=1}^{r-1} \frac{\eta_i^{(k)}}{k!}(x-\xi)^k. \tag{5}$$

We shall prove the following

THEOREM 1. *If conditions* (i), (ii) *are fulfilled and there exist positive numbers* $\alpha, \beta, s_{ik}, l_{ik}$ *such that*

$$|f_{ik}(x) - \xi| \leq s_{ik}|x-\xi|, \quad x \in I \cap \langle \xi - \alpha, \xi + \alpha \rangle, \quad i, k = 1, \ldots, n; \tag{6}$$

$$\left. \begin{array}{l} |h_i(x, y_1, \ldots, y_n) - h_i(x, \bar{y}_1, \ldots, \bar{y}_n)| \leq \sum_{k=1}^{n} l_{ik}|y_k - \bar{y}_k|, \\ i = 1, \ldots, n, (x, y_1, \ldots, y_n), (x, \bar{y}_1, \ldots, \bar{y}_n) \in \Omega_0 \end{array} \right\} \tag{7}$$

*where*

$$\Omega_0 = \Omega \cap \langle \xi - \alpha, \xi + \alpha \rangle \times \langle \eta_1 - \beta, \eta_1 + \beta \rangle \times \ldots \times \langle \eta_n - \beta, \eta_n + \beta \rangle$$

*and*

$$c_k = \sum_{i=1}^{n} (s_{ik})^r l_{ik} < 1; \quad k = 1, \ldots, n. \tag{8}$$

*then:*

(a) *for every two solutions* $\varphi = (\varphi_1, \ldots, \varphi_n)$, $\psi = (\psi_1, \ldots, \psi_n)$, $\varphi_i, \psi_i \in B_i^r$, *of the system* (1) *there exists an interval* $I_1$ *such that* $\xi \in I_1$ *and* $\varphi_i(x) = \psi_i(x)$ *for* $x \in I_1$, $i = 1, \ldots, n$;

(b) *if, moreover,* $s_{ik} < 1$ *and* $I \subset \langle \xi - \alpha, \xi + \alpha \rangle$ *or* $f_{ik}$ *are continuous in* $I$ $(i, k = 1, \ldots, n)$, *then the system* (1) *has at most one solution* $\varphi = (\varphi_1, \ldots, \varphi_n)$ *such that* $\varphi_i \in B_i^r$ $(i = 1, \ldots, n)$.

*Proof.* (a) Without loss of generality we may assume that

$$\xi = \eta_1 = \cdots = \eta_n = 0. \tag{9}$$

Suppose that $\varphi = (\varphi_1, \ldots, \varphi_n)$, $\psi = (\psi_1, \ldots, \psi_n)$, $\varphi_i, \psi_i \in B_i^r$, are solutions of the system (1). Thus we have

$$\varphi_i(x) = P_i(x) + x^r \Phi_i(x), \quad |\Phi_i(x)| \leq \mu_i, \quad x \in I \quad i = 1, \ldots, n; \tag{10}$$

$$\psi_i(x) = P_i(x) + x^r \Psi_i(x), \quad |\Psi_i(x)| \leq \mu_i, \quad x \in I, \quad i = 1, \ldots, n. \tag{11}$$

Inserting $\varphi_i$ and $\psi_i$ into (1) we easily verify that the functions $\Phi=(\Phi_1,\dots,\Phi_n)$, $\Psi=(\Psi_1,\dots,\Psi_n)$ fulfil in $I\setminus\{0\}$ the following system of equations

$$\Phi_i(x)=H_i(x,\Phi_1[f_{i1}(x)],\dots,\Phi_n[f_{in}(x)]),\quad i=1,\dots,n, \tag{12}$$

where

$$H_i(x,w_1,\dots,w_n)=\frac{h_i(x,P_1[f_{i1}(x)]+[f_{i1}(x)]^r w_1,\dots,P_n[f_{in}(x)]+[f_{in}(x)]^r w_n)-P_i(x)}{x^r}. \tag{13}$$

Since $P_k$ and $f_{ik}$ are continuous at $x=0$ and $P_k(0)=f_{ik}(0)=0$, it follows from (8) and (9) that there exists an interval $I_1\subset I$ such that $0\in I_1$ and

$$(x,P_1[f_{i1}(x)]+[f_{i1}(x)]^r w_1,\dots,P_n[f_{in}(x)]+[f_{in}(x)]^r w_n)\in\Omega_0 \tag{14}$$

for $x\in I_1$, $|w_k|\le\mu_k$, $i,k=1,\dots,n$.

Now it follows from (14), (13), (6) and (7) that

$$|H_i(x,w_1,\dots,w_n)-H_i(x,\bar w_1,\dots,\bar w_n)|\le\sum_{k=1}^{n}L_{ik}|w_k-\bar w_k|,\quad i=1,\dots,n \tag{15}$$

for $x\in I_1\setminus\{0\}$, $|w_k|\le\mu_k$, $|\bar w_k|\le\mu_k$, $k=1,\dots,n$, where

$$L_{ik}=(s_{ik})^r l_{ik}. \tag{16}$$

According to (15) we obtain for $x\in I_1\setminus\{0\}$

$$|\Phi_i(x)-\Psi_i(x)|\le\sum_{k=1}^{n}L_{ik}|\Phi_k[f_{ik}(x)]-\Psi_k[f_{ik}(x)]|,\quad i=1,\dots,n.$$

Since $f_{ik}(I_1)\subset I_1$ (cf. Remark 1) we obtain

$$\sup_{I_1\setminus\{0\}}|\Phi_i(x)-\Psi_i(x)|\le\sum_{k=1}^{n}L_{ik}\sup_{I_1\setminus\{0\}}|\Phi_k(x)-\Psi_k(x)|,\quad i=1,\dots,n.$$

Thus

$$\sum_{i=1}^{n}\sup_{I_1\setminus\{0\}}|\Phi_i(x)-\Psi_i(x)|\le\sum_{i=1}^{n}\sum_{k=1}^{n}L_{ik}\sup_{I_1\setminus\{0\}}|\Phi_k(x)-\Psi_k(x)|$$
$$=\sum_{k=1}^{n}\Big(\sum_{i=1}^{n}L_{ik}\Big)\sup_{I_1\setminus\{0\}}|\Phi_k(x)-\Psi_k(x)|.$$

Writing $d_i=\sup_{I_1\setminus\{0\}}|\Phi_i(x)-\Psi_i(x)|$ we obtain from (16) and (8) the inequality

$$d_1+\dots+d_n\le c_1d_1+\dots+c_nd_n.$$

Since $c_i<1$, $d_i\ge0$ $(i=1,\dots,n)$, we must have $d_i=0$ for $i=1,\dots,n$. This means that $\Phi_i(x)=\Psi_i(x)$ for $x\in I_1\setminus\{0\}$. From (10) and (11) we obtain $\varphi_i(x)=\psi_i(x)$ for $x\in I_1$. Thus (a) is proved.

*Remark* 2. The interval $I_1$ depends on $\mu$, and hence on the particular functions $\varphi$ and $\psi$. A common interval of equality for all solutions need not exist (cf. [6]).

(b) Without loss of generality we may assume relation (9) and that $\xi=0$ is the left endpoint of $I$.

Suppose that $s_{ik}<1$ $(i, k=1,\dots, n)$, $I\subset\langle 0, a\rangle$, and let $(\varphi_1,\dots,\varphi_n)$, $(\psi_1,\dots,\psi_n)$ be two solutions of the system (1) such that $\varphi_i, \psi_i \in R_r^I$. Let us denote by $x_0$ the supremum of all numbers $d$ such that $\varphi_i(x)=\psi_i(x)$ for $x\in\langle 0, d\rangle$, $i=1,\dots, n$. Evidently it is sufficient to show that $I\subset\langle 0, x_0\rangle$. For the indirect proof suppose that $x_0$ is an inner point of $I$. From the inequalities $s_{ik}<1$ $(i, k=1,\dots, n)$ and from (i) we obtain $s \stackrel{\mathrm{df}}{=} \max_{i,k} s_{ik}<1$ and $f_{ik}(x_0)\le sx_0<x_0$. If $x\in\langle 0, x_0 s^{-1}\rangle\cap I$ then $f_{ik}(x)\le sx<x_0$, $i, k=1, \dots, n$. Thus for $x\in\langle 0, x_0 s^{-1}\rangle\cap I$ we have

$$\left.\begin{aligned}\varphi_i(x) &= h_i(x, \varphi_1[f_{i1}(x)], \dots, \varphi_n[f_{in}(x)]) \\ &= h_i(x, \psi_1[f_{i1}(x)], \dots, \psi_n[f_{in}(x)]) = \psi_i(x),\end{aligned}\right\} \tag{17}$$

which contradicts the definition of $x_0$, and proves that $I\subset\langle 0, x_0\rangle$.

If $f_{ik}$ $(i, k=1,\dots, n)$ are continuous in $I$, then the argument is similar. Namely, in the same manner as above we define the number $x_0$ and we suppose that $x_0$ is an inner point of $I$. It follows from the continuity of $f_{ik}$ and from (i) that there exists an $x_1\in I$, $x_1>x_0$ such that

$$f_{ik}(\langle 0, x_1\rangle)\subset\langle 0, x_0), \quad i, k=1, \dots, n.$$

Hence we obtain (17) for $x\in\langle 0, x_1\rangle$ which contradicts the definition of $x_0$. Thus the theorem is proved.

Immediately from Theorem 1 we obtain the following

THEOREM 2. *If conditions* (i), (ii) *are fulfilled, the functions $f_{ik}$ and $h_i$ are of class $C^r$ in $I$ and $\Omega$, respectively, and*

$$\sum_{i=1}^{n}\left|[f_{ik}'(\xi)]^r\frac{\partial h_i}{\partial y_k}(\xi, \eta_1, \dots, \eta_n)\right|<1, \quad k=1, \dots, n,$$

*then for every system of numbers $\eta_i^{(k)}$, $i=1,\dots, n$, $k=1,\dots, r$, there exists at most one solution $\varphi=(\varphi_1,\dots,\varphi_n)$, $\varphi_i\in C^r[I]$ $(i=1,\dots, n)$ such that $\varphi_i(\xi)=\eta_i$, $\varphi_i^{(k)}(\xi)=\eta_i^{(k)}$ $(i=1,\dots, n, k=1,\dots, r)$.*

*Remark* 3. If $I=\langle\xi-a, \xi+a\rangle$, $a>0$, then we can assume less than (i), namely that

$$|f_{ik}(x)-\xi|<|x-\xi| \quad \text{for} \quad x\in I\setminus\{\xi\}, \quad f_{ik}(\xi)=\xi.$$

This condition implies that for every interval $I_1=\langle\xi-b, \xi+b\rangle$ $0<b\le a$, we have $f_{ik}(I_1)\subset I_1$.

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