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Results in Mathematics

An invariance of the geometric mean with respect to Stolarsky mean-type mappings

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ABSTRACT. We determine all pairs of Stolarsky means $(E_{r,s}, E_{k,m})$ such that $G \circ (E_{r,s}, E_{k,m}) = G$, where $G = E_{0,0}$ is the geometric mean. The convergence of the sequences of iterates of the mean-type mapping $(E_{r,s}, E_{k,m})$ to the mapping (G, G) is considered. An application to a functional equation is given.

1. Introduction

A well-known identity

$$G(x, y) = G(A(x, y), H(x, y)), \qquad x, y > 0,$$

written shortly as $G = G \circ (A, H)$, where G, A, H stand, respectively, for two variable geometric, arithmetic and harmonic means, is sometimes referred to as the invariance of the geometric mean with respect to the mean type mapping (A, H).

Recently this identity, together with the convergence of the sequence of iterates of the mapping (A, H) to (G, G), appeared to be helpful in solving a problem (cf. [4]) posed by H. Haruki and Th. M. Rassias [2].

The purpose of this paper is to determine all pairs $(E_{r,s}, E_{k,m})$ of Stolarsky means with respect to which G is invariant i.e. that

$$G \circ (E_{r,s}, E_{k,m}) = G. \tag{1}$$

One of the consequences of the invariance is the convergence of the sequence of iterates of the mapping $(E_{r,s}, E_{k,m})$ satisfying this equation to the mean-type mapping (G, G) (cf. [3]).

Section 2 is devoted to some basic definitions and auxiliary results.

In section 3 we prove the main result which says that G is $(E_{r,s}, E_{k,m})$-invariant if, and only if, one of the following conditions occurs

(i) $\quad k = m = r = s = 0$;

(ii) $\quad k = -r, m = s = 0$;

(iii) $\quad k = m \neq 0, \quad r = s \neq 0$ and $k = -r$;

(iv) $\quad rk \neq 0, r \neq s, k \neq m$ and either $r = -s$ and $k = -m$, or $k = -r$ and $m = -s$, or $k = -s$ and $m = -r$.

Moreover, G is a unique continuous mean which is invariant with respect to each pairs of these means.

For $r = 2, s = 1$ and $k = -1, r = -2$ we get the identity $G = G \circ (A, H)$.

In section 4 we apply these results to find a general continuous solution of the functional equation of the form

$$F(x, y) = F(M(x, y), N(x, y)),$$

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where M, N are some means. A special case of this functional equation appears in a problem posed by H. Haruki and Th. M. Rassias [2].

2. Some definitions and auxiliary results

Let $I \subset \mathbb{R}$ be an interval. A function $M : I^2 \to \mathbb{R}$ is said to be a *mean* on I^2 if

$$\min(x,y) \le M(x,y) \le \max(x,y), \qquad x,y \in I.$$

If moreover for all $x,y \in I$, $x \neq y$, these inequalities are sharp, the mean M is called *strict*, and M is called *symmetric*, if for all $x,y \in I$, $M(x,y) = M(y,x)$.

Note that if $M : I^2 \to \mathbb{R}$ is a mean, then M is *reflexive*, that is, $M(x,x) = x$ for all $x \in I$ and, consequently, for every interval $J \subset I$ we have $M(J^2) = J$; in particular, $M(I^2) = I$.

A mean $M : (0,\infty)^2 \to (0,\infty)$ is called *homogeneous* if

$$M(tx,ty) = tM(x,y), \qquad t,x,y > 0.$$

Let $M,N : I^2 \to I$ be means. A mean $K : I^2 \to I$ is called *invariant with respect to the mean-type mapping* $(M,N) : I^2 \to I^2$, shortly, *(M,N)-invariant*, if

$$K(M(x,y),N(x,y)) = K(x,y), \qquad x,y \in I.$$

Let us quote the following

THEOREM 1 (*cf.* [3]). *Let an interval $I \subset R$. If $(M,N) : I^2 \to I^2$ is a continuous mean-type mapping such that at most one of the coordinate means M and N is not strict, then:*

1^0 *there is a continuous mean $K : I^2 \to I$ such that the sequence of iterates $((M,N)^n)_{n=1}^{\infty}$ of the mapping (M,N) converges (pointwise) to a continuous mean-type mapping $(K,K) : I^2 \to I^2$;*

2^0 *K is (M,N)-invariant;*

3^0 *a continuous (M,N)-invariant mean-type mapping is unique;*

4^0 *if M and N are strict means then so is K;*

5^0 *if $I = (0,\infty)$ and M,N are homogeneous, then K is homogeneous.*

Throughout this paper we assume that $\mathbb{R}_+ = (0,\infty)$. Let us take arbitrary $r,s \in \mathbb{R}$. Recall that a function $E_{r,s} : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ is said to be a *Stolarsky mean* on $\mathbb{R}_+^2$ if

$$E_{r,s}(x,y) := \begin{cases} \left(\frac{r}{s}\frac{x^s-y^s}{x^r-y^r}\right)^{\frac{1}{s-r}}, & r \neq s,\ r \neq 0 \\[2ex] \left(\frac{x^s-y^s}{s(\ln x-\ln y)}\right)^{\frac{1}{s}}, & s \neq 0,\ r = 0 \\[2ex] \left(\frac{x^r-y^r}{r(\ln x-\ln y)}\right)^{\frac{1}{r}}, & r \neq 0,\ s = 0 \\[2ex] e^{-\frac{1}{r}}\left(\frac{x^{x^r}}{y^{y^r}}\right)^{\frac{1}{x^r-y^r}}, & r = s \neq 0 \\[2ex] \sqrt{xy}, & r = s = 0, \end{cases}$$

for all $x, y \in \mathbb{R}_+$ and $x \neq y$; if $x = y$ then $E_{r,s}(x,y) := x$ for all $r, s \in \mathbb{R}$ and $x \in \mathbb{R}_+$.
In the sequel we denote $E_{0,0}$ by G.

REMARK 1. *For every $x, y \in R_+$, the function $R^2 \ni (r,s) \to E_{r,s}(x,y)$ is continuous* (Stolarsky [4]).

REMARK 2. *Note that $E_{r,0} = E_{0,s}$ if $r = s$.*

3. Main results

We begin this section with a result of a negative character.

PROPOSITION 1. *Let $r, k, \in R \setminus \{0\}$. A geometric mean G is not $(E_{r,s}, E_{k,m})$-invariant if one of the following conditions holds true*

1. $k \neq 0$, $m = 0$ *and* $r = s \neq 0$;
2. $k \neq 0$, $m = 0$, $rs \neq 0$ *and* $r \neq s$;
3. $k \neq 0$, $m \neq 0$ *and* $r = s \neq 0$.

PROOF. *Part* 1. Suppose, for an indirect proof, that there exist $r, k \in \mathbb{R}$, $rk \neq 0$, such that (1) holds true with $s = r$ and $m = 0$, that is that

$$e^{-\frac{1}{r}} \left(\frac{x^{x^r}}{y^{y^r}} \right)^{\frac{1}{x^r - y^r}} \left(\frac{x^k - y^k}{k(\ln x - \ln y)} \right)^{\frac{1}{k}} = xy, \qquad x, y > 0, \quad x \neq y.$$

Setting here $y = 1$ we get

$$e^{-\frac{1}{r}} x^{\frac{x^r}{x^r - 1}} \left(\frac{x^k - 1}{k \ln x} \right)^{\frac{1}{k}} = x, \qquad x > 0, \quad x \neq 1. \tag{2}$$

This equation can be written in the form

$$-\frac{1}{r} + \frac{1}{k} \ln\left(\frac{x^k - 1}{k \ln x} \right) = \frac{-1}{x^r - 1} \ln x, \qquad x > 0, \quad x \neq 1.$$

Differentiation of both sides gives

$$\frac{kx^k \ln x - x^k + 1}{k(x^k - 1)\ln x} = \frac{rx^r \ln x - x^r + 1}{(x^r - 1)^2}, \qquad x > 0, \quad x \neq 1.$$

Defining $f_k : \mathbb{R}_+ \to \mathbb{R}$, $g_k : \mathbb{R}_+ \to \mathbb{R}$ and $h_r : \mathbb{R}_+ \to \mathbb{R}_+$ by

$$f_k(x) := kx^k \ln x - x^k + 1, \qquad x > 0, \quad x \neq 1,$$

$$g_k(x) := k(x^k - 1)\ln x, \qquad x > 0, \quad x \neq 1,$$

$$h_r(x) := (x^r - 1)^2, \qquad x > 0, \quad x \neq 1,$$

we can write this equation in the form

$$\frac{f_k(x)}{g_k(x)} = \frac{f_r(x)}{h_r(x)}, \qquad x > 0, \quad x \neq 1.$$

Differentiation of both sides gives

$$\frac{f_k'(x)g_k(x) - f_k(x)g_k'(x)}{g_k^2(x)} = \frac{f_r'(x)h_r(x) - f_r(x)h_r'(x)}{h_r^2(x)},$$

for all $x > 0, x \neq 1$, where

$$f_k'(x) = k^2 x^{k-1} \ln x,$$

$$g_k'(x) = kx^{k-1}(k\ln x + 1) - \frac{k}{x},$$

$$h_r'(x) = 2rx^{r-1}(x^r - 1).$$

Letting $x \to 1$ in this equation, we obtain

$$\frac{k}{12} = -\frac{r}{6}. \tag{3}$$

Taking the fourth derivatives of both sides of equation (2), leads to the equation

$$\left(\frac{f_k(x)}{g_k(x)}\right)^{(3)} = \left(\frac{f_r(x)}{h_r(x)}\right)^{(3)}, \qquad x > 0, \quad x \neq 1.$$

Since

$$\left(\frac{f_k}{g_k}\right)^{(3)} = \frac{f_k^{(3)}g_k - f_k g_k^{(3)} - 3(f_k'' g_k' + f_k' g_k'')}{(g_k)^2} + \frac{6f_k g_k g_k' g_k'' + 6(g_k')^2(f_k' g_k - f_k g_k')}{(g_k)^4},$$

interchanging here the roles of f_k and f_r as well as g_k and h_r, we obtain the formula for $\left(\frac{f_r}{h_r}\right)^{(3)}$. Applying these formulas, and letting $x \to 1$ in this equation, we get

$$-\frac{k(k^2 - 20)}{4} = r(r^2 - 10). \tag{4}$$

Now, from (3) and (4) we infer that $rk = 0$, which contradicts to the assumption that $r, k \in \mathbb{R}\backslash\{0\}$.

Part 2. Suppose, that there exist $r, s, k \in \mathbb{R}\backslash\{0\}$, $r \neq s$, such that (1) holds with $m = 0$ that is that

$$\left[\frac{r}{s}\frac{x^s - y^s}{x^r - y^r}\right]^{\frac{1}{s-r}}\left[\frac{x^k - y^k}{k(\ln x - \ln y)}\right]^{\frac{1}{k}} = xy, \qquad x, y > 0, \quad x \neq y.$$

Setting here $y = 1$ we get

$$\left(\frac{r}{s}\frac{x^s - 1}{x^r - 1}\right)^{\frac{1}{s-r}}\left(\frac{x^k - 1}{k\ln x}\right)^{\frac{1}{k}} = x, \qquad x > 0, \quad x \neq 1.$$

Defining $f_{r,s} : \mathbb{R}_+ \to \mathbb{R}$, $g_k : \mathbb{R}_+ \to \mathbb{R}$ by

$$f_{r,s}(x) := \left(\frac{r}{s}\frac{x^s - 1}{x^r - 1}\right)^{\frac{1}{s-r}}, \qquad x > 0, \quad x \neq 1,$$

$$g_k(x) := \left(\frac{x^k-1}{k\ln x}\right)^{-\frac{1}{k}}, \qquad x>0, \quad x \neq 1,$$

we can write this equation in the equivalent form

$$f_{r,s}(x) = x\, g_k(x), \qquad x>0, \quad x \neq 1. \tag{5}$$

Calculating the second derivatives of both sides we get

$$f_{r,s}''(x) = 2g_k'(x) + xg_k''(x), \qquad x>0, x \neq 1,$$

where

$$f_{r,s}'(x) = \frac{r}{s(s-r)}\left(\frac{r}{s}\frac{x^s-1}{x^r-1}\right)^{\frac{1+r-s}{s-r}} \frac{x^{r+s}(r+s)-sx^s-rx^r}{x(x^r-1)^2},$$

$$g_k'(x) = -\frac{x^k k\ln x-(x^k-1)}{x(k\ln x)^2}\left(\frac{x^k-1}{k\ln x}\right)^{-\frac{1+k}{k}}.$$

Letting here $x \to 1$ we obtain

$$k+r+s = 0. \tag{6}$$

Similarly, since

$$f_{r,s}^{(4)}(x) = 4g_k^{(3)}(x) + xg_k^{(4)}(x), \qquad x>0, \quad x \neq 1,$$

and

$$f_{r,s}^{(6)}(x) = 6g_k^{(5)}(x) + xg_k^{(6)}(x), \qquad x>0, \quad x \neq 1,$$

letting $x \to 1$ in these equations we obtain

$$-2(k^3+r^3+s^3)+5(-k^2+r^2+s^2)+70(k+r+s)-2rs(r+s-5) = 0, \tag{7}$$

and

$$16(k^5+r^5+s^5)-42(-k^4+r^4+s^4)-1687(k^3+r^3+s^3)$$

$$+4305(-k^2+r^2+s^2)+15519(k+r+s)+8610rs-1617rs(r+s)$$

$$+84rs(r^2+s^2+rs)+16rs(r+s)(r^2+s^2) = 0.$$

Solving the system of equations (6), (7) and (8) we infer that $ksr = 0$. This contradiction completes the proof in this case.

Part 3. Suppose, for an indirect argument, that there exist $k, m, r \in \mathbb{R}$, $kr \neq 0$, $k \neq m$ such that

$$e^{-\frac{1}{r}}\left(\frac{x^{x^r}}{y^{y^r}}\right)^{\frac{1}{x^r-y^r}}\left(\frac{k}{m}\frac{x^m-y^m}{x^k-y^k}\right)^{\frac{1}{m-k}} = xy, \qquad x,y>0, x \neq y.$$

Setting here $y = 1$ we get

$$e^{-\frac{1}{r}}(x)^{\frac{x^r}{x^r-1}}\left(\frac{k}{m}\frac{x^m-1}{x^k-1}\right)^{\frac{1}{m-k}} = x, \qquad x>0, \quad x\neq 1.$$

This equation can be written in the form

$$-\frac{1}{r}+\frac{1}{m-k}\ln\left(\frac{k}{m}\frac{x^m-1}{x^k-1}\right) = \frac{-1}{x^r-1}\ln x, \qquad x>0, \quad x\neq 1. \tag{9}$$

Calculating the second derivatives of both sides we get

$$\frac{mx^m(x^k-1)^2(x^m-1+m)-kx^k(x^m-1)^2(x^k-1+k)}{(k-m)[(x^m-1)(x^k-1)]^2} =$$

$$= \frac{(x^r-1)(x^r+2rx^r-1)-rx^r\ln x(x^r-1+r(x^r+1))}{(x^r-1)^3}.$$

Letting here $x \to 1$ we obtain

$$\frac{k+m-6}{2} = -\frac{r+3}{3}. \tag{10}$$

Taking the third derivatives of both sides of (9) we get

$$\frac{1}{(k-m)}(f_k(x)-f_m(x)) = g_r(x)$$

where

$$f_k(x) = \frac{kx^k\left[3k(x^k-1)+2(x^k-1)^2+k^2(x^k+1)\right]}{(x^k-1)^3},$$

$$g_r(x) = \frac{rx^r\ln x\left[2(x^r-1)^2+3r(x^{2r}-1)+r^2(x^{2r}+4x^r+1)\right]}{(x^r-1)^4}$$

$$-\frac{2(x^r-1)^2+3rx^r(r(x^r+1)+2(x^r-1))}{(x^r-1)^3}.$$

Letting here $x \to 1$ we obtain

$$-\frac{k+m-4}{6} = r+2. \tag{11}$$

Similarly, calculating the fourth derivatives of both sides of (9) (we omit writing too long explicit formulas) and letting $x \to 1$, we get

$$\frac{(110-km)(k+m)-(k^3+m^3)}{4} = r^3-55r-90. \tag{12}$$

Solving the system of equations (10), (11) and (12) we infer that either $r=0$ and $k=-m$, which contradicts to the assumption that $k,m,r \in \mathbb{R}\backslash\{0\}$, or $r=-m$ and $k=m$, which contradicts to the assumption that $k \neq m$. Thus the proof is completed.

Now we prove the main result:

THEOREM 2. *Let $k, m, r, s \in R$. A geometric mean G is $(E_{r,s}, E_{k,m})$-invariant if, and only if, one of the following conditions holds*

(i) $$k = m = r = s = 0;$$

(ii) $$k = -r \neq 0 \quad \text{and} \quad m = s = 0;$$

(iii) $$k = m \neq 0, \quad r = s \neq 0 \quad \text{and} \quad k = -r;$$

(iv) $$rk \neq 0, \quad r \neq s, \quad k \neq m, \quad \text{and}$$

either

$$r = -s \quad \text{and} \quad k = -m,$$

or

$$k = -r \quad \text{and} \quad m = -s,$$

or

$$k = -s \quad \text{and} \quad m = -r.$$

PROOF. Taking into account Remark 2, we can divide the proof into four possible cases depending on the form of the mean $E_{k,m}$ in equation (1).

Case 1. $k = m = 0$.
In this case we have $E_{k,m} = G$, and, consequently, equation (1) can be written in the form

$$G(E_{r,s}(x,y), G(x,y)) = G(x,y), \qquad x, y > 0.$$

Hence, by the reflexivity of G, we have

$$G(E_{r,s}(x,y), G(x,y)) = G(G(x,y), G(x,y)), \qquad x, y > 0.$$

The increasing monotonicity of G, implies that

$$E_{r,s}(x,y) = G(x,y), \qquad x, y > 0,$$

and, consequently, $r = s = 0$. Obviously for $r = s = k = m = 0$ equation (1) is satisfied.

Case 2. $k \neq 0, m = 0$.
In view of parts 1 – 2 of Proposition 1, it is enough to consider the case when $r \neq 0$ and $s = 0$. So, according to the definition of Stolarsky means, assume that

$$\left(\frac{x^r - y^r}{r(\ln x - \ln y)}\right)^{\frac{1}{r}} \left(\frac{x^k - y^k}{k(\ln x - \ln y)}\right)^{\frac{1}{k}} = xy, \qquad x, y > 0, \quad x \neq y.$$

Setting here $y = 1$ we get

$$\left(\frac{x^r - 1}{r \ln x}\right)^{\frac{1}{r}} \left(\frac{x^k - 1}{k \ln x}\right)^{\frac{1}{k}} = x, \qquad x > 0, \quad x \neq 1.$$

Defining $f_r : \mathbb{R}_+ \to \mathbb{R}_+, g_k : \mathbb{R}_+ \to \mathbb{R}_+$ by

$$f_r(x) := \left(\frac{x^r - 1}{r \ln x}\right)^{\frac{1}{r}}, \qquad x \neq 1,$$

$$g_k(x) := \left(\frac{x^k - 1}{k \ln x}\right)^{-\frac{1}{k}}, \qquad x \neq 1,$$

we can write this equation in the following form

$$f_r(x) = x\, g_k(x), \qquad x > 0, \quad x \neq 1.$$

Hence

$$f_r'(x) = g_k(x) + x g_k'(x), \qquad x > 0, \quad x \neq 1,$$

where

$$f_r'(x) = \frac{x^r(r\ln x - 1) + 1}{x(r\ln x)^2}\left(\frac{x^r - 1}{r\ln x}\right)^{\frac{1-r}{r}},$$

$$g_k'(x) = -\frac{x^k(k\ln x - 1) + 1}{x(k\ln x)^2}\left(\frac{x^k - 1}{k\ln x}\right)^{-\frac{1+k}{k}}.$$

Differentiation of both sides of this equation yields

$$f_r''(x) = 2g_k'(x) + x, \qquad x > 0, \quad x \neq 1,$$

where

$$f_r''(x) = -\frac{(r-1)(r\ln x)^2 x^r + r(x^{2r} - 1)\ln x - (r+1)(x^r - 1)^2}{(rx(x^r - 1)\ln x)^2}\left(\frac{x^r - 1}{r\ln x}\right)^{\frac{1-2r}{r}}$$

Letting here $x \to 1$ we get $k = -r$.

Now assume that $k = -r$. Then, for $x \neq y$,

$$G(E_{r,0}(x,y), E_{-r,0}(x,y)) = \left[\left(\frac{x^r - y^r}{r(\ln x - \ln y)}\right)^{\frac{1}{r}}\left(\frac{x^{-r} - y^{-r}}{-r(\ln x - \ln y)}\right)^{-\frac{1}{r}}\right]^{\frac{1}{2}}$$

$$= \left(\frac{1}{r}\frac{x^r - y^r}{\ln x - \ln y}\frac{(-r)(\ln x - \ln y)(x^r y^r)}{y^r - x^r}\right)^{\frac{1}{2r}} = \sqrt{xy} = G(x,y),$$

which completes the proof in this case.

Case 3. $k = m \neq 0$.

In view of parts 1 and 3 of Proposition 1, it is enough to consider the case when $r \neq 0$. So, according to the definition of Stolarsky means, (1) takes the form

$$e^{-\frac{1}{r}}\left(\frac{x^{x^r}}{y^{y^r}}\right)^{\frac{1}{x^r - y^r}} e^{-\frac{1}{k}}\left(\frac{x^{x^k}}{y^{y^k}}\right)^{\frac{1}{x^k - y^k}} = xy, \qquad x, y > 0, \quad x \neq y.$$

Setting here $y = 1$ we obtain

$$e^{-\frac{1}{r}}\left(x^{x^r}\right)^{\frac{1}{x^r-1}} e^{-\frac{1}{k}}\left(x^{x^k}\right)^{\frac{1}{x^k-1}} = x, \qquad x > 0, \quad x \neq 1,$$

or, equivalently,

$$\left(\frac{x^r}{x^r-1} + \frac{x^k}{x^k-1} - 1\right)\ln x = \frac{1}{r} + \frac{1}{k}, \qquad x > 0, \quad x \neq 1.$$

Differentiating both sides we obtain

$$\frac{x^r}{x^r-1}\left(1 - \frac{r\ln x}{x^r-1}\right) + \frac{x^k}{x^k-1}\left(1 - \frac{k\ln x}{x^k-1}\right) = 1, \qquad x > 0, \quad x \neq 1.$$

Defining $f_r : \mathbb{R}_+ \to \mathbb{R}$ by

$$f_r(x) := \frac{x^r}{x^r-1}\left(1 - \frac{r\ln x}{x^r-1}\right), \qquad x > 0, \quad x \neq 1,$$

we can write this equation in the form

$$f_r(x) + f_k(x) = 1, \qquad x > 0, \quad x \neq 1.$$

Hence

$$f_r'(x) + f_k'(x) = 0, \qquad x > 0, \quad x \neq 1,$$

where

$$f_r'(x) = \frac{rx^{r-1}(r(x^r+1)\ln x - 2(x^r-1))}{(x^r-1)^3}.$$

Letting $x \to 1$ in this equation we obtain

$$\frac{r+k}{6} = 0,$$

and, consequently, $k = -r$.

On the other hand, if $k = -r$, then, for all $x \neq y$,

$$G(E_{r,r}(x,y), E_{-r,-r}(x,y)) = \left(e^{-\frac{1}{r}}\left(\frac{x^{x^r}}{y^{y^r}}\right)^{\frac{1}{x^r-y^r}} e^{\frac{1}{r}}\left(\frac{x^{x^{-r}}}{y^{y^{-r}}}\right)^{\frac{1}{x^{-r}-y^{-r}}}\right)^{\frac{1}{2}}$$

$$= \left(x^{\frac{x^r}{x^r-y^r} + \frac{y^r x^r x^{-r}}{y^r-x^r}} y^{\frac{-y^r}{x^r-y^r} + \frac{x^r y^r(-y^{-r})}{y^r-x^r}}\right)^{\frac{1}{2}} = \sqrt{xy} = G(x,y),$$

which completes the proof in this case.

Case 4. $k \neq m$.

In view of parts 2 and 3 of Proposition 1, it is enough to consider the case when $kr \neq 0$, $k \neq m$, $r \neq s$.

According to the definition of Stolarsky means, equation (1) can be written in the following

$$\left(\frac{r}{s}\frac{x^s-y^s}{x^r-y^r}\right)^{\frac{1}{s-r}}\left(\frac{k}{m}\frac{x^m-y^m}{x^k-y^k}\right)^{\frac{1}{m-k}} = xy, \qquad x,y > 0, \quad x \neq y.$$

Setting here $y = 1$ we obtain

$$\left(\frac{r}{s}\frac{x^s-1}{x^r-1}\right)^{\frac{1}{s-r}}\left(\frac{k}{m}\frac{x^m-1}{x^k-1}\right)^{\frac{1}{m-k}}=x, \qquad x>0, \quad x\neq 1.$$

Defining $f_{r,s}:\mathbb{R}_+\to\mathbb{R}$ and $g_{k,m}:\mathbb{R}_+\to\mathbb{R}$ by

$$f_{r,s}(x):=\left(\frac{r}{s}\frac{x^s-1}{x^r-1}\right)^{\frac{1}{s-r}}, \qquad x>0, \quad x\neq 1,$$

$$g_{k,m}(x):=\left(\frac{k}{m}\frac{x^m-1}{x^k-1}\right)^{\frac{1}{k-m}}, \qquad x>0, \quad x\neq 1,$$

this equation can be written in the form

$$f_{r,s}(x)=xg_{k,m}(x), \qquad x>0, \quad x\neq 1. \tag{13}$$

Hence

$$f'_{r,s}(x)=g_{k,m}(x)+xg'_{k,m}(x), \qquad x>0, \quad x\neq 1,$$

where

$$f'_{r,s}(x)=\frac{r}{s(s-r)}\left(\frac{r}{s}\frac{x^s-1}{x^r-1}\right)^{\frac{1+r-s}{s-r}}\frac{x^{r+s}(r+s)-sx^s-rx^r}{x(x^r-1)^2},$$

$$g'_{k,m}(x)=\frac{k}{m(k-m)}\left(\frac{k}{m}\frac{x^m-1}{x^k-1}\right)^{\frac{1+m-k}{k-m}}\frac{x^{k+m}(k+m)-mx^m-kx^k}{x(x^k-1)^2}.$$

From (13) we have

$$f''_{r,s}(x)=2g'_{k,m}(x)+xg''_{k,m}(x), \qquad x>0, \quad x\neq 1.$$

Letting here $x\to 1$ we get

$$k+m+r+s=0. \tag{14}$$

Letting $x\to 1$ in the relation

$$f^{(4)}_{r,s}(x)=4g^{(3)}_{k,m}(x)+xg^{(4)}_{k,m}(x), \qquad x>0, \quad x\neq 1,$$

we get

$$\begin{gathered}2(r^3+s^3+k^3+m^3)-5(r^2+s^2-k^2-m^2)-70(r+s+k+m)\\ +2km(k+m+5)+2rs(r+s-5)=0.\end{gathered} \tag{15}$$

Similarly, letting $x\to 1$ in the relation

$$f^{(6)}_{r,s}(x)=6g^{(5)}_{k,m}(x)+xg^{(6)}_{k,m}(x), \qquad x>0, \quad x\neq 1,$$

gives

$$-16(k^5+m^5+r^5+s^5)-42(k^4+m^4-r^4-s^4)+1687(k^3+m^3+r^3+s^3)$$

$$+4305(k^2+m^2-r^2-s^2)-15519(k+m+r+s)-1617km(k+m)$$

$$-1617rs(r+s)-16[km(k^3+m^3)+k^2m^2(k+m)+rs(r^3+s^3)+r^2s^2(r+s)]$$

$$-1617rs(r+s)-84[km(k^2+km+m^2)-rs(r^2+rs+s^2)]-+8610(km-rs)=0. \tag{16}$$

Finally, taking the eighth derivatives of both sides of equation (13) we have

$$f_{r,s}^{(8)}(x)=8g_{k,m}^{(7)}(x)+xg_{k,m}^{(8)}(x)=0, \qquad x>0, x\neq 1.$$

Letting here $x\to 1$ we obtain

$$144(k^7+m^7+r^7+s^7)+404(k^6+m^6-r^6-s^6)-31260(k^5+m^5+r^5+s^5)$$

$$-82985(k^4+m^4-r^4-s^4)+782712(k^3+m^3+r^3+s^3)$$

$$+2130030(k^2+m^2-r^2-s^2)-3909420(k+m+r+s)-165270(k^2m^2-r^2s^2)$$

$$+644112[km(k+m)+rs(r+s)]-165620[km(k^2+m^2)-rs(r^2+s^2)]$$

$$-30420[km(k^3+m^3)+rs(r^3+s^3)]+4260060(km-rs) \tag{17}$$

$$+808[km(k^4+m^4)-rs(r^4+s^4)]+976(k^3m^3-r^3s^3)$$

$$+144[km(k^5+m^5)+k^2m^2(k^3+m^3)+k^3m^3(k^2+m^2)]$$

$$+144[rs(r^5+s^5)+r^2s^2(r^3+s^3)+r^3s^3(r^2+s^2)]$$

$$-30000[k^2m^2(k+m)+r^2s^2(r+s)]+892[k^2m^2(k^2+m^2)-r^2s^2(r^2+s^2)]=0.$$

Solving the system of equations (14), (15), (16) and (17), we infer that, either

$$r=-s \text{ and } k=-m,$$

or

$$k=-r \text{ and } m=-s,$$

or

$$k = -s \quad \text{and} \quad m = -r.$$

On the other hand:

if $r = -s$ then, for all $x,y > 0, x \neq y$,

$$G(E_{-s,s}(x,y), E_{-m,m}(x,y)) = \sqrt{\left[-\frac{s}{s}\frac{x^s - y^s}{x^{-s} - y^{-s}}\right]^{\frac{1}{2s}}\left[-\frac{m}{m}\frac{x^m - y^m}{x^{-m} - y^{-m}}\right]^{\frac{1}{2m}}}$$

$$= \sqrt{(x^s y^s)^{\frac{1}{2s}}(x^m y^m)^{\frac{1}{2m}}} = \sqrt{xy} = G(x,y),$$

if $k = -r$ and $m = -s$ then, for all $x,y > 0,\ x \neq y$,

$$G(E_{r,s}(x,y), E_{-r,-s}(x,y)) = \sqrt{\left(\frac{r}{s}\frac{x^s - y^s}{x^r - y^r}\right)^{\frac{1}{s-r}}\left(\frac{-r}{-s}\frac{x^{-s} - y^{-s}}{x^{-r} - y^{-r}}\right)^{\frac{1}{r-s}}}$$

$$= \sqrt{\left(\frac{r}{s}\frac{x^s - y^s}{x^r - y^r}\right)^{\frac{1}{s-r}}\left(\frac{s}{r}\frac{x^s y^s}{x^r - y^r}\frac{x^s - y^s}{x^r y^r}\right)^{\frac{1}{s-r}}} = \sqrt{((xy)^{s-r})^{\frac{1}{s-r}}} = G(x,y);$$

and similarly if $k = -s$ and $m = -r$. This completes the proof.

REMARK 3. Calculating the derivatives of the order sixth and eighth of the functions in (13) as well as the suitable limits to get equations (16), (17), we applied the software package Mathematica 4.0.

From Theorem 1 and Theorem 2 we obtain the following

COROLLARY 1. *Suppose that $k, m, r, s \in R$ satisfy one of the conditions $(i)-(iv)$ of Theorem 2. Then the geometric mean G is the only continuous and $(E_{r,s}, E_{k,m})$-invariant mean. Moreover the sequence of iterates $(E_{r,s}, E_{k,m})^n$ of the mapping $(E_{r,s}, E_{k,m})$ converges to the mapping (G, G) on R_+^2.*

EXAMPLE 1. Consider the logarithmic mean $L : (0,\infty)^2 \to (0,\infty)$,

$$L(x,y) = \begin{cases} \frac{x-y}{\ln x - \ln y}, & x \neq y \\ x, & x = y \end{cases},$$

and its "conjugate mean" $L^* : (0,\infty)^2 \to (0,\infty)$,

$$L^*(x,y) = \frac{1}{L(\frac{1}{x}, \frac{1}{y})}, \qquad x,y > 0.$$

Since $L = E_{1,0}$, $L^* = E_{-1,0}$, in view of Corollary 1, we have

$$\lim_{n\to\infty} (L, L^*)^n = (G,G) \quad \text{pointwise in } (0,\infty)^2.$$

EXAMPLE 2. Consider the identric mean $E : (0,\infty)^2 \to (0,\infty)$,

$$E(x,y) = \begin{cases} \frac{1}{e}\left(\frac{x^x}{y^y}\right)^{\frac{1}{x-y}}, & x \neq y \\ x, & x = y \end{cases},$$

and its "conjugate mean" $E^* : (0,\infty)^2 \to (0,\infty)$, $E_{1,1} = E$ and $E_{-1,-1} = E^*$ where

$$E^*(x,y) = \frac{1}{E(\frac{1}{x},\frac{1}{y})}, \qquad x,y > 0.$$

Since $L = E_{1,0}$, $L^* = E_{-1,0}$, applying Corollary 1, we have

$$\lim_{n\to\infty} (E,E^*)^n = (G,G) \quad \text{pointwise in } (0,\infty)^2.$$

EXAMPLE 3. Taking $r = 2$, $s = 3$ and $k = -2$, $m = -3$ we have

$$E_{2,3}(x,y) = \frac{2}{3}\frac{x^2+xy+y^2}{x+y}, \qquad E_{-2,-3}(x,y) = \frac{3}{2}\frac{xy(x+y)}{x^2+xy+y^2}.$$

By Corollary 1

$$\lim_{n\to\infty} (E_{2,3},E_{-2,-3})^n = (G,G) \quad \text{pointwise in } (0,\infty)^2.$$

4. An application

H. Haruki and Th. M. Rassias [2] posed the following

PROBLEM. *Is it true that a continuous function $F : (0,\infty)^2 \to R$ satisfies the functional equation*

$$F(A(x,y),H(x,y)) = F(x,y), \qquad x,y > 0,$$

if, and only if, there is a continuous function $f : (0,\infty) \to R$ such that

$$F(x,y) = f(xy), \qquad x,y > 0?$$

An affirmative answer was given in [4]. Applying Corollary 1 we prove the following more general

THEOREM 3. *Let $k,m,r,s \in R$ satisfy one of conditions (i) – (iv) of Theorem 2. Suppose that a function $F : (0,\infty)^2 \to R$ is continuous on the diagonal $\Delta := \{(x,x) : x > 0\}$. Then F satisfies the functional equation*

$$F(E_{r,s}(x,y),E_{k,m}(x,y)) = F(x,y), \qquad x,y > 0, \tag{18}$$

if, and only if, there is a continuous function $f : (0,\infty) \to R$ such that

$$F(x,y) = f(xy), \qquad x,y > 0.$$

PROOF. Suppose that a function $F : (0,\infty)^2 \to \mathbb{R}$ satisfies equation (18). By induction we hence get

$$F(x,y) = F((E_{r,s},E_{k,m})^n(x,y)), \qquad x,y > 0, \quad n \in \mathbb{N}.$$

Letting here $n \to \infty$, and making use of Corollary 1 and the continuity of F on the diagonal Δ, we obtain

$$F(x,y) = F(G(x,y), G(x,y)), \qquad x,y > 0.$$

Putting

$$f(u) := F(\sqrt{u}, \sqrt{u}), \qquad u > 0,$$

we hence get $F(x,y) = f(xy)$ for all $x, y > 0$.

Since the converse implication is obvious, the proof is completed.

For $r = 1$, $s = 0$ and $k = -1$, $m = 0$ taking into account the notations introduced in Example 1, we obtain the following

COROLLARY 2. *Suppose that a function* $F : (0,\infty)^2 \to R$ *is continuous on the diagonal* $\{(x,x) : x > 0\}$. *Then F satisfies the functional equation*

$$F(x,y) = F(L(x,y), L^*(x,y)), \qquad x,y > 0,$$

if, and only if, there is a continuous function $f : (0,\infty) \to R$ *such that* $F(x,y) = f(xy)$, $x,y > 0$.

REMARK 4. *Theorem* 3 *remains true if* $(0,\infty)$ *is replaced by an arbitrary interval* $I \subset (0,\infty)$.

Applying Theorem 1 one gets the following generalization of Theorem 3.

PROPOSITION 2. *Let* $I \subset R$ *be an interval, and suppose that* $M : I^2 \to I$ *is an arbitrary strict and continuous mean. Then*

1^0 $M^* : I^2 \to I$ *defined by* $M^*(x,y) := \frac{xy}{M(x,y)}$ *is a mean;*

2^0 $\lim_{n\to\infty} (M, M^*)^n = (G,G)$ *pointwise in* I^2;

3^0 *a function* $F : I^2 \to R$ *which is continuous on the diagonal* $\{(x,x) : x \in I\}$, *satisfies the functional equation*

$$F(x,y) = F(M(x,y), M^*(x,y)), \qquad x,y \in I,$$

if, and only if, there is a continuous function $f : I \to R$ *such that* $F(x,y) = f(xy)$ *for all* $x,y \in I$.

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