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Arithmetic mean as a linear combination of two quasi-arithmetic means

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Abstract. Under some regularity assumptions imposed on the generators f, g, we determine all the quasi-arithmetic means $M^{[f]}$, $M^{[g]}$ and all real numbers λ and μ such that $\lambda M^{[f]} + \mu M^{[g]} = A$, where A is the arithmetic mean.

1. Introduction

This paper is concerned with a general question when a given quasi-arithmetic mean is a linear combination of two other quasi-arithmetic means.

In Section 2, assuming some regularity conditions of generators of the means, we give only some necessary conditions.

The main result of the present paper is concerned with the case when the given mean is the arithmetic one, denoted by A. In Section 3 we determine the forms of the generators f, g of the quasi-arithmetic means $M^{[f]}$, $M^{[g]}$ and the real constants λ, μ such that

$$\lambda M^{[f]} + \mu M^{[g]} = A,$$

under the assumption that f and g are twice continuously differentiable. In the proof, one-parameter sub-family $\{E_{r+1,r} : r \in \mathbb{R}\}$ of Stolarsky means [11] occurs.

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In the case $\lambda = \mu = \frac{1}{2}$ SUTÔ [12] already in 1914 treated this equation assuming that f and g are analytic functions. It was rediscovered by MATKOWSKI [9] where the solutions of the class C^2 were found. This result has been considerably improved in the series of papers by DARÓCZY [2], DARÓCZY–MAKSA [4], DARÓCZY–MAKSA–PÁLES [5] and DARÓCZY–PÁLES [6].

The object of the present paper has appeared in connection with a problem by Z. DARÓCZY [3] to determine the class of "mixing-arithmetic means" which are quasi-arithmetic [7] where, unexpectedly, the mean $E_{-2,-3}$ appeared.

2. A general problem and some auxiliary results

Let $I \subset \mathbb{R}$ be an interval. A function $M : I \times I \to I$ is called a *mean in* I if

$$\min(x, y) \leq M(x, y) \leq \max(x, y), \qquad x, y \in I.$$

If M is a mean in I then M is *reflexive*, i.e.

$$M(x, x) = x, \qquad x \in I,$$

and, obviously, for every subinterval $J \subset I$, we have $M(J \times J) = J$.

A mean $M : I^2 \to I$ is said to be *homogeneous* if

$$x, y, tx, ty \in I \Rightarrow M(tx, ty) = tM(x, y).$$

Let $f : I \to \mathbb{R}$ be a continuous and strictly monotonic function. Recall that a function $M^{[f]} : I \times I \to \mathbb{R}$, defined by

$$M^{[f]}(x, y) := f^{-1}\left(\frac{f(x) + f(y)}{2}\right), \qquad x, y \in I,$$

is a mean, and it is said to be *quasi-arithmetic*. The function f is called a *generator* of the mean $M^{[f]}$.

Remark 1. If a quasi-arithmetic mean is a linear combination of two quasi-arithmetic means then this combination is actually an affine one. Indeed, if f, g, h are continuous and strictly monotonic functions defined on an interval $I \subset \mathbb{R}$ with a non-zero real and

$$M^{[h]} = \lambda M^{[f]} + \mu M^{[g]}$$

with some $\lambda, \mu \in \mathbb{R}$ then, by the reflexivity of the means, we have

$$x = M^{[h]}(x,x) = \lambda M^{[f]}(x,x) + \mu M^{[h]}(x,x) = (\lambda+\mu)x,$$

for every $x \in I$ and, consequently, $\lambda + \mu = 1$.

Let us note the following (cf. J. ACZÉL and J. DHOMBRES [1], p. 246):

Lemma 1. *Let $f, g : I \to \mathbb{R}$ be continuous and strictly monotonic functions. Then $M^{[f]} = M^{[g]}$ if and only if $g = \alpha f + \beta$ for some $\alpha \in \mathbb{R}\backslash\{0\}$ and $\beta \in \mathbb{R}$.*

In particular, in the definition of the quasi-arithmetic mean, one can always assume that its generator is strictly increasing.

Now we prove the following

Lemma 2. *Let $\lambda \in \mathbb{R}$ be a fixed number, I a real non-trivial interval and let $f, g, h : I \to \mathbb{R}$ be twice differentiable strictly increasing functions having non-vanishing derivatives and satisfying the equation*

$$\lambda M^{[f]} + (1-\lambda)M^{[g]} = M^{[h]}. \tag{1}$$

Then

$$\lambda g'\left(M^{[g]}(x,y)\right)\left(f'(x)f'(y)^{\lambda}g'(y)^{1-\lambda} - f'(y)f'(x)^{\lambda}g'(x)^{1-\lambda}\right)$$

$$= (1-\lambda)f'\left(M^{[f]}(x,y)\right)\left(f'(x)^{\lambda}g'(x)^{1-\lambda}g'(y) - f'(y)^{\lambda}g'(y)^{1-\lambda}g'(x)\right)$$

for every $x, y \in I$.

PROOF. Differentiating both sides of equality (1) with respect to x we obtain

$$\lambda\frac{f'(x)}{f'\left(M^{[f]}(x,y)\right)} + (1-\lambda)\frac{g'(x)}{g'\left(M^{[g]}(x,y)\right)} = \frac{h'(x)}{h'\left(M^{[h]}(x,y)\right)} \tag{2}$$

for all $x, y \in I$. Changing the roles of x and y we get

$$\lambda\frac{f'(y)}{f'\left(M^{[f]}(x,y)\right)} + (1-\lambda)\frac{g'(y)}{g'\left(M^{[g]}(x,y)\right)} = \frac{h'(y)}{h'\left(M^{[h]}(x,y)\right)} \tag{3}$$

for all $x, y \in I$. It follows from (2) and (3) that

$$\lambda\frac{\frac{f'(x)-f'(y)}{x-y}}{f'\left(M^{[f]}(x,y)\right)} + (1-\lambda)\frac{\frac{g'(x)-g'(y)}{x-y}}{g'\left(M^{[g]}(x,y)\right)} = \frac{\frac{h'(x)-h'(y)}{x-y}}{h'\left(M^{[h]}(x,y)\right)}$$

for all $x, y \in I, x \neq y$. Letting here $y \to x$ we obtain

$$\lambda \frac{f''(x)}{f'(x)} + (1-\lambda)\frac{g''(x)}{g'(x)} = \frac{h''(x)}{h'(x)}, \qquad x \in I,$$

and, consequently, there is a positive constant k such that

$$f'(x)^{\lambda} g'(x)^{1-\lambda} = k h'(x), \qquad x \in I.$$

Hence, applying (2) and (3), for all $x, y \in I$ we get

$$\lambda \frac{f'(x)}{f'\left(M^{[f]}(x,y)\right)} + (1-\lambda)\frac{g'(x)}{g'\left(M^{[g]}(x,y)\right)} = \frac{f'(x)^{\lambda} g'(x)^{1-\lambda}}{f'(M^{[h]}(x,y))^{\lambda} g'(M^{[h]}(x,y))^{1-\lambda}},$$

and

$$\lambda \frac{f'(y)}{f'\left(M^{[f]}(x,y)\right)} + (1-\lambda)\frac{g'(y)}{g'\left(M^{[g]}(x,y)\right)} = \frac{f'(y)^{\lambda} g'(y)^{1-\lambda}}{f'(M^{[h]}(x,y))^{\lambda} g'(M^{[h]}(x,y))^{1-\lambda}}.$$

Dividing these equalities by sides and making some simple calculations we complete the proof. □

3. Results

Let I be a real non-trivial interval. In this section we deal with equation (1) with $h(x) = x$, $x \in I$, that is we assume that $M^{[h]} = A$ where

$$A(x,y) := \frac{x+y}{2}, \qquad x, y \in I,$$

is the arithmetic mean.

In what follows λ is a fixed real number and f, g are continuous strictly monotonic functions defined on a non-trivial real interval I. Consider the equation

$$\lambda M^{[f]} + (1-\lambda) M^{[g]} = A. \tag{4}$$

The main result reads as follows.

Theorem. *Assume that $f, g : I \to \mathbb{R}$ are twice continuously differentiable. Then equation (4) is satisfied if and only if one of the following cases occurs:*

1. *there are $a, c \in \mathbb{R}\backslash\{0\}$ and $b, d \in \mathbb{R}$ such that*

$$f(x) = ax + b, \quad g(x) = cx + d, \qquad x \in I;$$

2. *either $\lambda = 1$ and*

$$f(x) = ax + b, \qquad x \in I,$$

or $\lambda = 0$ and

$$g(x) = cx + d, \qquad x \in I,$$

for some $a, c \in \mathbb{R}\backslash\{0\}$ and $b, d \in \mathbb{R}$;

3. *$\lambda = \frac{1}{2}$ and there is a $t \in \mathbb{R}\backslash\{0\}$ such that*

$$f(x) = ae^{tx} + b, \quad g(x) = ce^{-tx} + d, \qquad x \in I,$$

for some $a, c \in \mathbb{R}\backslash\{0\}$ and $b, d \in \mathbb{R}$;

4. *either $\lambda = 2$ and there is an $x_0 \in \mathbb{R}\backslash I$ such that*

$$f(x) = a\sqrt{|x - x_0|} + b, \quad g(x) = c\log|x - x_0| + d, \qquad x \in I,$$

or $\lambda = -1$ and there is an $x_0 \in \mathbb{R}\backslash I$ such that

$$f(x) = a\log|x - x_0| + b, \quad g(x) = c\sqrt{|x - x_0|} + d, \qquad x \in I,$$

for some $a, c \in \mathbb{R}\backslash\{0\}$ and $b, d \in \mathbb{R}$.

In the proof we need some lemmas.

Lemma 3. *Let $\lambda \neq 0, 1$. Assume that twice continuously differentiable and strictly increasing functions $f, g : I \to \mathbb{R}$ with non-vanishing derivatives f', g' satisfy equation (4). If f'' does not vanish then*

$$M^{[F]}(x, y) = \frac{r}{1+r}\,\frac{x^{1+r} - y^{1+r}}{x^r - y^r}, \quad x, y \in f'(I),\ x \neq y,$$

where

$$F := f \circ (f')^{-1}, \quad r := \frac{\lambda}{1 - \lambda}.$$

Proof. Assume that $f''(x) \neq 0$, $x \in I$. Then f' is strictly monotonic. By (4) we have

$$\frac{1}{1-\lambda}A - \frac{\lambda}{1-\lambda}M^{[f]} = M^{[g]}.$$

Thus, applying Lemma 2 where λ, f, g, h are replaced by $\frac{1}{1-\lambda}, \mathrm{id}_I, f, g$, respectively, we get

$$f'\left(M^{[f]}(x,y)\right) = \frac{r}{1+r}\frac{f'(x)^{-r}f'(y) - f'(y)^{-r}f'(x)}{f'(x)^{-r} - f'(y)^{-r}}, \quad x, y \in I, \ x \neq y,$$

whence

$$f'\left(M^{[f]}\left((f')^{-1}(x), (f')^{-1}(y)\right)\right) = \frac{r}{1+r}\frac{x^{-r}y - y^{-r}x}{x^{-r} - y^{-r}},$$

for $x, y \in f'(I)$, $x \neq y$, which gives the assertion. □

Remark 2. Note that the equality of the above lemma can be written in the form

$$M^{[F]}(x,y) = E_{r,r+1}(x,y), \quad x, y \in f'(I), \ x \neq y,$$

where $E_{r,s}$ denotes the Stolarsky mean with $s, r \in \mathbb{R}$ (cf. [11]), which for $r \neq s$, $rs \neq 0$, has the form

$$E_{r,s}(x,y) := \left(\frac{r}{s}\frac{x^s - y^s}{x^r - y^r}\right)^{1/(s-r)}, \quad x, y > 0, \ x \neq y.$$

Recall that for every $p \in \mathbb{R}$ the function $M_{[p]} : (0,\infty)^2 \to (0,\infty)$ defined by

$$M_{[p]}(x,y) := \begin{cases} \left(\dfrac{x^p + y^p}{2}\right)^{1/p} & \text{for } p \neq 0 \\ \sqrt{xy} & \text{for } p = 0 \end{cases}$$

is a mean and it is called a *power mean.*

Lemma 4. *Let $p \in \mathbb{R}$ and $r \in \mathbb{R}\backslash\{-1, 0\}$. Then*

$$E_{r,r+1}(x,y) = M_{[p]}(x,y), \quad x, y \in (0,\infty), \ x \neq y, \tag{5}$$

if and only if either $(p,r) = (-1,-2)$ *or* $(p,r) = (1,1)$ *or* $(p,r) = (0,-\frac{1}{2})$.

PROOF. At first assume (5) with $p = 0$. Then

$$\sqrt{x} = \frac{r}{1+r}\,\frac{x^{1+r}-1}{x^r-1}, \quad x \in (0,\infty)\backslash\{1\}.$$

Since

$$\lim_{x\to\infty} \frac{x^{1+r}-1}{x^{\frac{1}{2}+r}-x^{\frac{1}{2}}} = 0$$

if $r < -\frac{1}{2}$ and

$$\left|\lim_{x\to\infty} \frac{x^{1+r}-1}{x^{\frac{1}{2}+r}-x^{\frac{1}{2}}}\right| = \infty$$

in the case when $r > -\frac{1}{2}$, we have $r = -\frac{1}{2}$. Conversely, it is obvious that

$$E_{-\frac{1}{2},\frac{1}{2}}(x,y) = \sqrt{xy} = M_{[0]}(x,y), \quad x,y \in (0,\infty),\ x \neq y.$$

In the case when $p \neq 0$ the assertion follows from the identity

$$M_{[p]}(x,y) = E_{p,2p}(x,y), \quad x,y \in (0,\infty),\ x \neq y,$$

and a result of PÁLES [10, Corollary] (in fact the last can be deduced from [8, Theorem 4] by LEACH and SHOLANDER). The reader may also give a self-contained proof using some standard considerations a little bit more complicated than above. □

Lemma 5. *Assume that* $f : I \to \mathbb{R}$ *is continuously differentiable and* f' *is strictly monotonic. Let* $F := f \circ (f')^{-1}$.

(i) *If* $M^{[F]}$ *is the arithmetic mean then there is a* $t \in \mathbb{R}\backslash\{0\}$ *such that*

$$f(x) = ae^{tx} + b, \quad x \in I,$$

with some $a \in \mathbb{R}\backslash\{0\}$ *and* $b \in \mathbb{R}$.

(ii) *If* f'*does not vanish and* $M^{[F]}$ *is the geometric mean then there is an* $x_0 \in \mathbb{R}\backslash\{0\}$ *such that*

$$f(x) = a\log|x - x_0| + b, \quad x \in I,$$

with some $a \in \mathbb{R}\backslash\{0\}$ *and* $b \in \mathbb{R}$.

(iii) *If f'does not vanish and $M^{[F]}$ is the harmonic mean then there is an $x_0 \in \mathbb{R}\backslash I$ such that*

$$f(x) = a\sqrt{|x - x_0|} + b, \quad x \in I,$$

with some $a \in \mathbb{R}\backslash\{0\}$ and $b \in \mathbb{R}$.

PROOF. (i) Assume that $M^{[F]}$ is the arithmetic mean. In view of Lemma 1 we have $F(x) = \alpha x + \beta$, $x \in f'(I)$, for some $\alpha \in \mathbb{R}\backslash\{0\}$ and $\beta \in \mathbb{R}$, that is

$$f \circ (f')^{-1}(x) = \alpha x + \beta, \quad x \in f'(I),$$

or, equivalently,

$$f(x) = \alpha f'(x) + \beta, \quad x \in I.$$

Solving this differential equation we get the assertion.

(ii) Assume that $f'(x) \neq 0$, $x \in I$, and $M^{[F]}$ is the geometric mean. Lemma 1 now yields $F(x) = \alpha \log|x| + \beta$, $x \in f'(I)$, for some $\alpha \in \mathbb{R}\backslash\{0\}$ and $\beta \in \mathbb{R}$. In other words

$$f(x) = \alpha \log|f'(x)| + \beta, \quad x \in I.$$

Solving this differential equation we find a real value $x_0 \notin I$ such that

$$f(x) = -\alpha \log\left|\frac{x - x_0}{\alpha}\right| + \beta, \quad x \in I.$$

Putting $a := -\alpha$ and $b := \alpha \log|\alpha| + \beta$ we come to the desired form of f.

(iii) Assume that f' does not vanish and $M^{[F]}$ is the harmonic mean. By Lemma 1 there are $\alpha \in \mathbb{R}\backslash\{0\}$ and $\beta \in \mathbb{R}$ such that $F(x) = \frac{\alpha}{x} + \beta$, $x \in f'(I)$, i.e.

$$f(x) = \frac{\alpha}{f'(x)} + \beta, \quad x \in I.$$

Then

$$|f(x) - \beta| = \sqrt{2\alpha(x - x_0)}, \quad x \in I,$$

with some $x_0 \notin I$ and it is enough to take $b := \beta$, and either $a := \sqrt{2|\alpha|}$ or $a := -\sqrt{2|\alpha|}$ depending on whether f is greater or less than β. □

PROOF of the Theorem. Assume that equation (4) is satisfied. By virtue of Lemma 1 it is enough to consider the case when both f and g

are strictly increasing. Observe that if $\lambda = 0$ or $\lambda = 1$ then, by Lemma 1, the forms f and g given in Case 2 are obvious. So in what follows we assume that $\lambda \neq 0, 1$. In the case when $f''(x) = 0$, $x \in I$, we infer that f is affine and non-constant. Then, by (4), $M^{[g]}$ is the arithmetic mean and, as follows from Lemma 1, also g is affine and non-constant. Consequently, Case 1 occurs.

Now assume that f'' is non-zero at a point and let $J \subset I$ be a non-trivial interval such that $f''(x) \neq 0$ for $x \in J$. If $g''(x) = 0$, $x \in J$, then $g|_J$ would be affine and non-constant and, by (4) and Lemma 1, so would be $f|_J$ which is impossible. Thus g'' is non-zero at a point of J and, consequently, on a nontrivial subinterval of J. Then each of f' and g', being strictly monotonic on this subinterval, vanishes at most at one point of it. Therefore replacing, if necessary, J by its non-trivial subinterval, we may assume that f'', f', g'' and g' do not vanish on J and, in addition, J is a maximal interval with this property contained in I.

Put $F := f \circ (f')^{-1}$ and $r := \frac{\lambda}{1-\lambda}$. Of course $r \neq 0, -1$. Moreover, since f is increasing and f' is non-zero on I, we have $f'(J) \subset (0, \infty)$. According to Lemma 3,

$$M^{[F]}(x,y) = \frac{r}{1+r}\,\frac{x^{1+r} - y^{1+r}}{x^r - y^r}, \quad x, y \in f'(J),\ x \neq y. \tag{6}$$

As the right-hand side of the above equality is a homogeneous mean, so is $M^{[F]}$ on $f'(J) \times f'(J)$. It follows (cf. [1; p. 249]) that

$$M^{[F]}(x,y) = M_{[p]}(x,y), \quad x, y \in f'(J),$$

for some $p \in \mathbb{R}$ and, consequently, (5) holds. In view of Lemma 4

$$\text{either } (p,r) = (-1,-2) \text{ or } (p,r) = (1,1) \text{ or } (p,r) = (0, -\frac{1}{2}).$$

Consider first the case $p = -1$ and $r = -2$. Then $\lambda = 2$ and, by (6), $M^{[F]}$ is the harmonic mean. On account of Lemma 5(iii) there is an $x_0 \in \mathbb{R}\backslash J$ such that

$$f(x) = a\sqrt{|x - x_0|} + b, \quad x \in J,$$

for some $a \in \mathbb{R}\backslash\{0\}$ and $b \in \mathbb{R}$. Putting $G := g \circ (g')^{-1}$ and making use of Lemma 3, where λ, f, g are replaced by $1 - \lambda$, $g|_J$, $f|_J$, we infer that $M^{[G]}$

is the geometric mean. According to Lemma 5(ii) there is an $x_1 \in \mathbb{R}\backslash J$ such that

$$g(x) = c \log|x - x_1| + d, \quad x \in J,$$

for some $c \in \mathbb{R}\backslash\{0\}$ and $d \in \mathbb{R}$. Inserting the above formulas for f and g into (4) we conclude that $x_1 = x_0$.

In the second case we have $p = 1$ and $r = 1$. Then $\lambda = \frac{1}{2}$ and, by (6), $M^{[F]}$ is the arithmetic mean. Using Lemma 3 to $g|_J$ and $f|_J$ instead of f and g, respectively, we infer that also $M^{[G]}$ is the arithmetic mean. By Lemma 5(i) there are $t, s \in \mathbb{R}\backslash\{0\}$ such that

$$f(x) = ae^{tx} + b, \qquad g(x) = ce^{sx} + d, \qquad x \in J,$$

with some $a, c \in \mathbb{R}\backslash\{0\}$ and $b, d \in \mathbb{R}$. Making use of (4) one can verify that $s = -t$.

In the third case we have $p = 0$ and $r = -\frac{1}{2}$. Consequently $\lambda = -1$ and, by (5),

$$M^{[F]}(x, y) = \sqrt{xy}, \quad x, y \in (0, \infty).$$

By Lemma 5(ii) there is an $x_0 \in \mathbb{R}\backslash J$ such that

$$f(x) = a \log|x - x_0| + b, \quad x \in J,$$

with some $a \in \mathbb{R}\backslash\{0\}$ and $b \in \mathbb{R}$. Applying Lemma 3 with $1 - \lambda, g|_J$ and $f|_J$ and Lemma 5(iii) we find an $x_1 \in \mathbb{R}\backslash J$ such that

$$g(x) = c\sqrt{|x - x_1|} + d, \quad x \in J,$$

with some $c \in \mathbb{R}\backslash\{0\}$ and $d \in \mathbb{R}$. Inserting the above formulas for f and g into (4) we conclude that $x_1 = x_0$.

If $I\backslash J \neq \emptyset$ then the maximality of J would imply the existence of a point $x \in I \cap \mathrm{cl}(J)$ such that at least one of the numbers $f''(x)$, $f'(x)$, $g''(x)$, $g'(x)$ equals 0. This, however, is impossible as f and g of the forms determined above do not have this property. Thus $J = I$ and one of the Cases 1–4 holds true. We omit a simple proof of the opposite implication. □

If $t \in \mathbb{R}$ and $t < x$ $[t > x]$ for every $x \in I$ we simply write $t < I$ $[t > I]$.

Let us introduce the following families of means. For every $t \in \mathbb{R}\setminus\{0\}$ put

$$S_t(x,y) := \frac{1}{t}\log\left(\frac{e^{tx}+e^{ty}}{2}\right), \quad x,y \in I.$$

Given a $t \in \mathbb{R}$ we let

$$R_t^+(x,y) := t + \left(\frac{\sqrt{|x-t|}+\sqrt{|y-t|}}{2}\right)^2$$

and

$$G_t^+(x,y) := t + \sqrt{(x-t)(y-t)}$$

for $x,y \in I$ if $t < I$, and

$$R_t^-(x,y) := t - \left(\frac{\sqrt{|x-t|}+\sqrt{|y-t|}}{2}\right)^2$$

and

$$G_t^-(x,y) := t - \sqrt{(x-t)(y-t)}$$

for $x,y \in I$ if $t > I$.

As a simple consequence of Theorem 1 we obtain the following

Corollary 1. *Assume that $f,g : I \to \mathbb{R}$ are twice continuously differentiable. Then equation (4) is satisfied if and only if one of the following cases occurs*

1. $M^{[f]} = M^{[g]} = A$;

2a. $\lambda = 1$ *and* $M^{[f]} = A$;

2b. $\lambda = 0$ *and* $M^{[g]} = A$;

3. $\lambda = \frac{1}{2}$ *and there is a* $t \in \mathbb{R}\setminus\{0\}$ *such that* $M^{[f]} = S_t$ *and* $M^{[g]} = S_{-t}$;

4a. $\lambda = 2$ *and either* $M^{[f]} = R_t^+$ *and* $M^{[g]} = G_t^+$ *with some* $t < I$, *or* $M^{[f]} = R_t^-$ *and* $M^{[g]} = G_t^-$ *with some* $t > I$;

4b. $\lambda = -1$ *and either* $M^{[f]} = G_t^+$ *and* $M^{[g]} = R_t^+$ *with some* $t < I$, *or* $M^{[f]} = G_t^-$ *and* $M^{[g]} = R_t^-$ *with some* $t > I$.

Remark 3. For $\lambda = \frac{1}{2}$ equation (4) reduces to the functional equation

$$A\left(M^{[f]}, M^{[g]}\right) = A,$$

which means that A is invariant with respect to the mapping $\left(M^{[f]}, M^{[g]}\right)$ (cf. [9]).

Let us also note the following

Corollary 2. *Assume that $f, g : I \to \mathbb{R}$ are twice continuously differentiable and strictly monotonic in an interval I. Then the arithmetic mean A is a convex combination of quasi-arithmetic means $M^{[f]}$ and $M^{[g]}$, i.e.* (4) *holds true for some $\lambda \in (0,1)$ if and only if one of the following cases occurs*

1. $M^{[f]} = M^{[g]} = A$;
2. $\lambda = \frac{1}{2}$ *and there is a* $t \in \mathbb{R}\backslash\{0\}$ *such that* $M^{[f]} = S_t$ *and* $M^{[g]} = S_{-t}$.

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