ON INVARIANT GENERALIZED BECKENBACH-GINI MEANS

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Dedicated to Professor Peter Kahlig on the occasion of his sixtieth birthday

Abstract A functional equation that characterizes generalized Beckenbach-Gini means which are invariant with respect to Beckenbach-Gini mean-type mappings is considered. In the case when an invariant mean is either arithmetic or geometric or harmonic, without any regularity conditions, all solutions are found. In the general case, under some regularity assumptions, a necceasary condition is given. For positively homogeneous Beckenbach-Gini means a complete list of solutions is established. Translative Beckenbach-Gini means are also examined.

Keywords: mean-type mapping, iteration, invariant mean, Beckenbach-Gini mean, homogeneous mean, translative mean

Mathematics Subject Classification (2000): 39B22; 26A18

Let $I \subseteq \mathbb{R}$ be an interval. By a *mean* we mean a function $M : I^2 \to I$ such that

$$\min(x,y) \le M(x,y) \le \max(x,y) \qquad (x,y \in I).$$

If for all $x, y \in I, x \neq y$, these inequalities are sharp, we call M to be a *strict mean*. It is proved in [6] that if $M, N : I^2 \to I$ are strict continuous

Z. Daróczy and Z. Páles (eds.),
Functional Equations - Results and Advances, 219–230.

means then the sequence of iterates of the mean-type mapping $(M,N): I^2 \to I^2$ converges to a mean-type mapping $(K,K): I^2 \to I^2$ where K is a unique (M,N)-invariant mean, i.e.

$$K\left(M(x,y), N(x,y)\right) = K(x,y) \qquad (x,y \in I).$$

There are some important special classes of means, for instance, quasi-arithmetic means, Gini means, Stolarsky means (cf. [3]). In this connection the following general question arises. Given a class of means, determine all pairs (M,N) from this class such that their (unique) (M,N)-invariant mean K is also a member of this class. This problem, under some regularity assumption, has been solved for the class of quasi-arithmetic means in [7]. In the present paper we examine this problem for the class of generalized Beckenbach-Gini means $M_f: I^2 \to I$ which are of the form

$$M_f(x,y) := \frac{xf(x)+yf(y)}{f(x)+f(y)} \qquad (x,y \in I),$$

where $f: I \to (0,\infty)$ is a function, called a *generator* of the mean. This mean is a special quasi-arithmetic weighted mean. In the case when f is a power function, the mean M_f was considered by Gini [5] and Beckenbach [2]. Thus the above invariant mean relation reduces to the functional equation

$$M_h\left(M_f(x,y), M_g(x,y)\right) = M_h(x,y) \qquad (x,y \in I), \tag{1}$$

with three unknown functions $f,g,h: I \to (0,\infty)$.

As the arithmetic mean A is a Beckenbach-Gini mean, in section 2 we determine all pairs of functions (f,g) such that A is (M_f,M_g)-invariant. In this context the translative Beckenbach-Gini means are considered. Since the geometric mean G and harmonic mean H are also of Beckenbach-Gini type, in section 3 we establish all the pair of functions (f,g) such that G and H are (M_f,M_g)-invariant.

In section 4 we show that, under some regularity assumption, if M_h is (M_f,M_g)-invariant, i.e. eq. (1) is satisfied, then, necessarily, for some $c>0$,

$$h = c\sqrt{fg}.$$

Using this result, in section 5, we find all positively homogeneous means M_f, M_g and M_h such that M_h is (M_f,M_g)-invariant.

1. AUXILIARY RESULTS

Some properties of Beckenbach-Gini means can be found in [3]. We begin with recalling the following easy to verify

Remark 1. Let $I \subseteq \mathbb{R}$ be an interval and $f, g : I \to (0, \infty)$. Then $M_f = M_g$ if, and only if, $g = cf$ for some $c > 0$.

An important role is played by

Lemma 1. *Let $I \subseteq \mathbb{R}$ be an interval. If $f : I \to (0, \infty)$ is continuously differentiable then*

$$\lim_{y \to x} \frac{\frac{\partial M_f}{\partial x}(x,y) - \frac{\partial M_f}{\partial y}(x,y)}{x-y} = \frac{f'(x)}{f(x)} \qquad (x \in I). \tag{2}$$

Proof. From the definition of M_f we have

$$\frac{\partial M_f}{\partial x}(x,y) = \frac{f(x)^2 + f(x)f(y) + xf'(x)f(y) - yf'(x)f(y)}{\left(f(x)+f(y)\right)^2} \quad (x, y \in I),$$

and, by the symmetry of M_f,

$$\frac{\partial M_f}{\partial y}(x,y) = \frac{\partial M_f}{\partial x}(y,x) \qquad (x, y \in I).$$

Hence, by simple calculations, we get

$$\frac{\frac{\partial M_f}{\partial x}(x,y) - \frac{\partial M_f}{\partial y}(x,y)}{x-y} = \frac{\frac{f(x)-f(y)}{x-y}\left(f(x)+f(y)\right) + f'(x)f(y) + f(x)f'(y)}{\left(f(x)+f(y)\right)^2},$$

and, letting y tend to x, we obtain formula (2). □

In section 4 we need the following.

Lemma 2. *Let $f : (0, \infty) \to (0, \infty)$ be an arbitrary function. The Beckenbach-Gini mean $M_f : (0, \infty)^2 \to (0, \infty)$ is positively homogeneous, i.e.*

$$M_f(tx, ty) = tM_f(x,y) \qquad (x, y, t > 0),$$

if, and only if, the function $\frac{f}{f(1)}$ is multiplicative. If moreover f is measurable or the graph of f is not dense in $(0, \infty)^2$ then $f(x) = f(1)x^p, x > 0$, for some $p \in \mathbb{R}$, and

$$M_f(x,y) = \frac{x^{p+1} + y^{p+1}}{x^p + y^p} \qquad (x, y > 0).$$

Proof. For every $t > 0$ define $f_t : (0,\infty) \to (0,\infty)$ by $f_t(x) := f(tx)$, $x > 0$. Then, by Remark 1, M_f is homogeneous iff $M_f = M_{f_t}$ for all $t > 0$, that is, there exists a function $c : (0,\infty) \to (0,\infty)$ such that

$$f(tx) = c(t)f(x) \qquad (x, t > 0).$$

Setting here $x = 1$ we get $f(t) = f(1)c(t)$ for all $t > 0$, and, consequently,

$$f(1)f(tx) = f(t)f(x) \qquad (x, t > 0),$$

which means that the function $\frac{f}{f(1)}$ is multiplicative. The converse implication is obvious. The second part follows from the first one (cf. [1], Theorem 3, p. 14). □

2. WHEN THE ARITHMETIC MEAN IS INVARIANT; TRANSLATIVE BECKENBACH-GINI MEANS

Taking a constant generator in the definition of Beckenbach-Gini mean we get the arithmetic mean $A(x,y) = \frac{x+y}{2}$, $x, y \in \mathbb{R}$.

Theorem 1. *Let $I \subseteq \mathbb{R}$ be an interval and $f, g : I \to (0,\infty)$. The following conditions are equivalent:*

(1) the arithmetic mean A is (M_f, M_g)-invariant, i.e.

$$M_f(x,y) + M_g(x,y) = x + y \qquad (x, y \in I); \tag{3}$$

(2) there is a $c > 0$ such that

$$f(x)g(x) = c \qquad (x \in I);$$

(3) $M_g = M_{1/f}$, i.e.

$$M_g(x,y) = \frac{yf(x) + xf(y)}{f(x) + f(y)} \qquad (x, y \in I).$$

Moreover, for every continuous function $f : I \to (0,\infty)$, the sequence of iterates of the mapping $(M_f, M_{1/f})$ converges to the arithmetic mean.

Proof. Setting $h(x) = 1$, $x \in I$, in (1) gives the equation $A(M_f, M_g) = A$, i.e. (3), which is equivalent to

$$\frac{xf(x) + yf(y)}{f(x) + f(y)} + \frac{xg(x) + yg(y)}{g(x) + g(y)} = x + y \qquad (x, y \in I).$$

This functional equation reduces to the relation

$$f(x)g(x) = f(y)g(y) \qquad (x, y \in I),$$

and the proofs of equivalences of the conditions 1-3 are obvious. Since the means $M_f, M_{1/f}$ are strict, the remaining part of the theorem is a consequence of Theorem 1 in [6]. $\square$

Remark 2. A counterpart of equation (3) for quasi-arithmetic means (which is much more difficult and yet not completely solved) leads to an interesting one-parameter family of translative quasi-arithmetic means (cf. Z. Daróczy [4]).

In this connection recall that a mean $M : \mathbb{R}^2 \to \mathbb{R}$ is called *translative* if

$$M(x+t, x+t) = M(x,y) + t \qquad (x, y, t \in \mathbb{R}).$$

Lemma 3. *Let $f : R \to (0, \infty)$. The Beckenbach-Gini mean M_f is translative if, and only if, the function $\frac{f}{f(0)}$ is exponential, i.e.*

$$f(0)f(x+y) = f(x)f(y) \qquad (x, y \in \mathbb{R}).$$

Proof. For every $t \in \mathbb{R}$ define $f_t : \mathbb{R} \to (0, \infty)$ by $f_t(x) := f(x+t)$, $x \in \mathbb{R}$. Then, by Remark 1, M_f is translative iff $M_f = M_{f_t}$ for all $t \in \mathbb{R}$, that is, when there is a function $c : \mathbb{R} \to (0, \infty)$ such that

$$f(x+t) = c(t)f(x) \qquad (x, t \in \mathbb{R}).$$

Setting here $x = 0$ gives $f(t) = f(0)g(t)$, $t \in \mathbb{R}$. Hence we get

$$f(0)f(x+t) = f(x)f(t) \qquad (x, y, t \in \mathbb{R}),$$

which means that the function $\frac{f}{f(0)}$ is exponential.

Since the converse implication is easy to verify, the proof is completed. $\square$

Hence we obtain (cf. [1], Theorem 3, p. 14)

Corollary 1. *Let $f : \mathbb{R} \to (0, \infty)$ be measurable or its graph be not dense in $(0, \infty) \times \mathbb{R}$. The Beckenbach-Gini mean M_f is translative if, and only if, there exists a constant $p > 0$ such that*

$$M_f(x,y) = \frac{xp^x + yp^y}{p^x + p^y} \qquad (x, y \in \mathbb{R}).$$

For arbitrary $p > 0$, we define $T_{[p]} : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ by

$$T_{[p]}(x,y) := \frac{xp^x + yp^y}{p^x + p^y} \qquad (x, y \in \mathbb{R}).$$

Thus $\{T_{[p]} : p > 0)\}$ is a one-parameter family of translative Beckenbach-Gini means. Applying Theorem 1 we obtain

Corollary 2. *For all $p, r > 0$ the arithmetic mean A is $(T_{[p]}, T_{[r]})$ invariant, if and only if, $pr = 1$.*

3. THE INVARIANCE OF THE GEOMETRIC AND HARMONIC MEANS

Taking the generator $x \to x^{-1/2}$ in the definition of Beckenbach-Gini mean we get the geometric mean $G(x,y) := \sqrt{xy}$, $x, y > 0$.

Theorem 2. *Let $I \subseteq (0,\infty)$ be an interval and $f, g : I \to (0,\infty)$. The geometric mean is (M_f, M_g)-invariant if, and only if, there is a $c > 0$ such that*

$$f(x)g(x) = \frac{c}{x} \qquad (x \in I).$$

Moreover, for all continuous functions $f : I \to (0,\infty)$, $c > 0$, and $g(x) := \frac{c}{xf(x)}$, $x \in I$, the sequence of iterates of the mapping (M_f, M_g) converges to the geometric mean.

Proof. Writing in the explicit form the equation $G(M_f, M_g) = G$ we get

$$\frac{xf(x) + yf(y)}{f(x) + f(y)} \frac{xg(x) + yg(y)}{g(x) + g(y)} = xy \qquad (x, y \in I),$$

which, after simple calculations reduces to the equivalent condition

$$f(x)g(x)x = f(y)g(y)y \qquad (x, y \in I),$$

and the result follows. The second statement is an immediate consequence of Theorem 1 in [6]. □

Taking the generator $x \to x^{-1}$ in the definition of Beckenbach-Gini mean we get the harmonic mean H.

Theorem 3. *Let $I \subseteq (0,\infty)$ be an interval and $f, g : I \to (0,\infty)$. The harmonic mean is (M_f, M_g)-invariant if, and only if, there is a $c > 0$ such that*

$$f(x)g(x) = \frac{c}{x^2} \qquad (x \in I).$$

Moreover, for all continuous functions $f : I \to (0,\infty)$, $c > 0$, and $g(x) := \frac{c}{f(x)}$, $x \in I$, the sequence of iterates of the mapping (M_f, M_g) converges to the geometric mean.

Proof. Writing the equation $H(M_f, M_g) = H$ in the explicit form we get

$$\frac{\frac{xf(x)+yf(y)}{f(x)+f(y)} \cdot \frac{xg(x)+yg(y)}{g(x)+g(y)}}{\frac{xf(x)+yf(y)}{f(x)+f(y)} + \frac{xg(x)+yg(y)}{g(x)+g(y)}} = \frac{xy}{x+y} \qquad (x, y \in I).$$

Simple calculation shows that this functional equation is equivalent to the relation

$$f(x)g(x)x^2 = f(y)g(y)y^2 \qquad (x, y \in I),$$

which completes the proof. □

Remark 3. Taking $h(x) = x, x \in I$, gives a Beckenbach-Gini mean $M_h(x,y) = \frac{x^2+y^2}{x+y}$ (which is the contra-harmonic one). It is not difficult to show that this mean is (M_f, M_g)-invariant for some functions $f, g : I \to (0,\infty)$, i.e.

$$\frac{\left(\frac{xf(x)+yf(y)}{f(x)+f(y)}\right)^2 + \left(\frac{xg(x)+yg(y)}{g(x)+g(y)}\right)^2}{\frac{xf(x)+yf(y)}{f(x)+f(y)} + \frac{xg(x)+yg(y)}{g(x)+g(y)}} = \frac{x^2+y^2}{x+y} \qquad (x, y \in I),$$

if, and only if, $f = ah$ and $g = bh$, for some positive $a, b \in \mathbb{R}$, i.e., if, and only if, $M_f = M_h = M_g$ (cf. Remark 1).

This remark shows that the classical means A, G and H play a special role in the theory of invariant Beckenbach-Gini means.

4. A NECESSARY CONDITION

In this section we prove the following

Theorem 4. *Let $I \subseteq \mathbb{R}$ be an interval. Suppose that $f, g : I \to (0,\infty)$ are differentiable and $h : I \to (0,\infty)$ is twice differentiable. If the mean M_h is (M_f, M_g)-invariant then there is a constant $c > 0$ such that*

$$h = c\sqrt{fg}.$$

Proof. Suppose that a mean M_h is (M_f, M_g)-invariant. Then the functions f, g and h satisfy equation (1) which, by the definition of Beckenbach-Gini mean, can be written in the form

$$\begin{aligned}&[M_f(x,y)h\left(M_f(x,y)\right) + M_g(x,y)h\left(M_g(x,y)\right)]\left(h(x)+h(y)\right)\\ &\quad = \left(xh(x)+yh(y)\right)\left[h\left(M_f(x,y)\right) + h\left(M_g(x,y)\right)\right] \qquad (x, y \in I).\end{aligned}$$

Denote, for convenience, the expressions of the left and right hand sides of this equation by $L(x,y)$ and $R(x,y)$, respectively. Since $L = R$ we have

$$\frac{\partial L}{\partial x} = \frac{\partial R}{\partial x}, \qquad \frac{\partial L}{\partial y} = \frac{\partial R}{\partial y},$$

and, consequently,

$$\frac{\frac{\partial L}{\partial x}(x,y) - \frac{\partial L}{\partial y}(x,y)}{x-y} = \frac{\frac{\partial R}{\partial x}(x,y) - \frac{\partial R}{\partial y}(x,y)}{x-y} \qquad (x, y \in I).$$

By simple calculations we have

$$\frac{\frac{\partial L}{\partial x}(x,y)-\frac{\partial L}{\partial y}(x,y)}{x-y}=\left[h\left(M_f\right)+M_f h'\left(M_f\right)\right]\frac{\frac{\partial M_f}{\partial x}-\frac{\partial M_f}{\partial y}}{x-y}[h(x)+h(y)]$$
$$+\left[h(M_g)+M_g h'(M_g)\right]\frac{\frac{\partial M_g}{\partial x}-\frac{\partial M_g}{\partial y}}{x-y}[h(x)+h(y)]$$
$$+\left[M_f h\left(M_f\right)+M_g h(M_g)\right]\frac{h'(x)-h'(y)}{x-y},$$

and

$$\frac{\frac{\partial R}{\partial x}(x,y)-\frac{\partial R}{\partial y}(x,y)}{x-y}=\left[h\left(M_f\right)+h\left(M_g\right)\right]\frac{h(x)+xh'(x)-h(y)-yh'(y)}{x-y}$$
$$+\left[xh(x)+yh(y)\right]\left[h'(M_f)\frac{\frac{\partial M_f}{\partial x}-\frac{\partial M_f}{\partial y}}{x-y}+h'(M_g)\frac{\frac{\partial M_g}{\partial x}-\frac{\partial M_g}{\partial y}}{x-y}\right],$$

where, for short, $M_f=M_f(x,y)$ and $M_g=M_g(x,y)$. Letting here $y\to x$ and applying Lemma 1, the continuity of the means M_f, M_g, and the relation $M_f(x,x)=x=M_g(x,x)$, $x\in I$, we get

$$h(x)+xh'(x)\left[\frac{f'(x)}{f(x)}+\frac{g'(x)}{g(x)}\right]h(x)+xh(x)h''(x)$$
$$=h(x)[2h'(x)+xh''(x)]+xh(x)h'(x)\left[\frac{f'(x)}{f(x)}+\frac{g'(x)}{g(x)}\right],$$

which, after reduction, can be written in the form

$$2\frac{h'(x)}{h(x)}=\frac{f'(x)}{f(x)}+\frac{g'(x)}{g(x)}\qquad(x\in I).$$

It follows that there is a $c>0$ such that

$$h(x)^2=cf(x)g(x)\qquad(x\in I),$$

which completes the proof. □

Remark 4. The condition given in this theorem is not sufficient. To show that the converse implication is not true take $f,g,h:(0,\infty)\to(0,\infty)$ defined by

$$f(x):=\frac{x}{x+1},\qquad g(x):=x(x+1),\qquad h(x):=x\qquad(x>0).$$

Then $h = \sqrt{fg}$ and, it is easy to verify the mean M_h is not (M_f, M_g)-invariant (cf. also Remark 3.

Thus the problem to determine some neccesary and sufficient conditions is open.

5. HOMOGENEOUS BECKENBACH-GINI MEANS

In this section we present the following

Theorem 5. *Let each of the functions $f, g, h : (0, \infty) \to (0, \infty)$ be measurable or its graph be not dense in $(0, \infty)^2$. Suppose that the Beckenbach-Gini means M_f, M_g, M_h are positively homogeneous. Then M_h is (M_f, M_g)-invariant if, and only if, one of the following cases occurs:*

(1) there is a $p \in R$ such that

$$\frac{f(x)}{f(1)} = \frac{g(x)}{g(1)} = \frac{h(x)}{h(1)} = x^p \qquad (x > 0),$$

and, consequently,

$$M_f(x, y) = M_g(x, y) = M_h(x, y) = \frac{x^{p+1} + y^{p+1}}{x^p + y^p} \qquad (x, y > 0);$$

(2) there exists a $p \in \mathbb{R}$ such that

$$f(x) = f(1)x^p, \qquad g(x) = g(1)x^{-p}, \qquad h(x) = h(1) \qquad (x > 0),$$

and, consequently,

$$M_f(x, y) = \frac{x^{p+1} + y^{p+1}}{x^p + y^p}, \qquad M_g(x, y) = \frac{x^{1-p} + y^{1-p}}{x^{-p} + y^{-p}}, \qquad M_h = A$$

(3) there is a $p \in \mathbb{R}$ such that

$$f(x) = f(1)x^p, \qquad g(x) = g(1)x^{-p-1}, \qquad h(x) = h(1)x^{-1/2} \quad (x > 0),$$

and, consequently,

$$M_f(x, y) = \frac{x^{p+1} + y^{p+1}}{x^p + y^p}, \qquad M_g(x, y) = \frac{x^{-p} + y^{-p}}{x^{-p-1} + y^{-p-1}}, \qquad M_h = G;$$

(4) there is a $p \in \mathbb{R}$ such that

$$f(x) = f(1)x^p, \qquad g(x) = g(1)x^{-p-2}, \qquad h(x) = h(1)x^{-1} \qquad (x > 0),$$

and, consequently,

$$M_f(x,y) = \frac{x^{p+1}+y^{p+1}}{x^p+y^p}, \qquad M_g(x,y) = \frac{x^{-p-1}+y^{-p-1}}{x^{-p-2}+y^{-p-2}}, \qquad M_h = H.$$

Proof. Without any loss of generality we may assume that at least two of the means M_f, M_g, M_h are not the same. The homogeneity of M_f, M_g, M_h, in view of Lemma 2, implies that there exist $p, q, r \in \mathbb{R}$ such that

$$f(x) = f(1)x^p, \qquad g(x) = g(1)x^q, \qquad h(x) = h(1)x^r \qquad (x > 0).$$

Since f, g, h satisfy the assumptions of Theorem 5, we infer that

$$h(x) = c\sqrt{f(x)g(x)} \qquad (x > 0),$$

for a positive $c > 0$, and consequently,

$$r = \frac{p+q}{2}.$$

Now the (M_f, M_g)-invariance of the mean M_h, i.e. equation (1), can be written in the form

$$\left(\frac{x^{p+1}+y^{p+1}}{x^p+y^P}\right)^{\frac{p+q+2}{2}} + \left(\frac{x^{q+1}+y^{q+1}}{x^q+y^q}\right)^{\frac{p+q+2}{2}}$$
$$= \frac{x^{\frac{p+q+2}{2}}+y^{\frac{p+q+2}{2}}}{x^{\frac{p+q}{2}}+y^{\frac{p+q}{2}}}\left[\left(\frac{x^{p+1}+y^{p+1}}{x^p+y^p}\right)^{\frac{p+q}{2}} + \left(\frac{x^{q+1}+y^{q+1}}{x^q+y^q}\right)^{\frac{p+q}{2}}\right],$$

for all $x, y > 0$.

Suppose that some real numbers p, q satisfy this equation for all $x, y > 0$. Setting here $y = 1$ gives

$$\left(\frac{x^{p+1}+1}{x^p+1}\right)^{\frac{p+q+2}{2}} + \left(\frac{x^{q+1}+1}{x^q+1}\right)^{\frac{p+q+2}{2}}$$
$$- \frac{x^{\frac{p+q+2}{2}}+1}{x^{\frac{p+q}{2}}+1}\left[\left(\frac{x^{p+1}+1}{x^p+1}\right)^{\frac{p+q}{2}} + \left(\frac{x^{q+1}+1}{x^q+1}\right)^{\frac{p+q}{2}}\right] = 0$$

for all $x > 0$. Denote by $F_{p.q}(x)$ the left hand side of this identity. Then, of course,

$$\frac{d^k}{dx^k}F_{p,q}(1) = 0$$

for all nonnegative integers k. Careful calculations show that

$$\frac{d^k}{dx^k}F_{p,q}(1) = 0 \qquad (k = 0, 1, ..., 5),$$

for all $p, q \in \mathbb{R}$. Only the condition

$$\frac{d^6}{dx^6} F_{p,q}(1) = 0$$

reduces to the condition

$$\frac{15}{32}(p-q)^2(p+q)(p+q+1)(p+q+2) = 0$$

(we omit long and tedious calculations). Consequently, either $q = p$ or $q = -p$ or $q = -p-1$ or $q = -p-2$. Since the converse implication is easy to verify, the proof is completed. $\square$

Recall that for every $p \in \mathbb{R}$, a power mean $M^{[p]} : (0,\infty)^2 \to (0,\infty)$ is defined by

$$M^{[p]}(x,y) := \left(\frac{x^p + y^p}{2}\right)^{1/p} \qquad (p \neq 0); \qquad M^{[0]}(x,y) = \sqrt{xy}.$$

Theorem 6. *Let $p, q \in R$. Then*

$$M^{[p]}(x,y) = \frac{x^{q+1} + y^{q+1}}{x^q + y^q} \qquad (x, y > 0), \tag{4}$$

if, and only if, either $q = 0, p = 1$ or $q = -1 = p$, or $q = -\frac{1}{2}, p = 0$.

Proof. Setting $y = 1$ in (4) gives

$$\left(\frac{x^p + 1}{2}\right)^{1/p} = \frac{x^{q+1} + 1}{x^q + 1} \qquad (x > 0).$$

Calculating the second and the forth derivatives of both sides and then substituting $x = 1$ gives the system of equations

$$p = 2q + 1, \qquad 8q(4 - q^2) = (1-p)(2p^2 - p - 15).$$

Hence

$$q(q+1)(2q+1) = 0,$$

and consequently, either $q = 0$ and $p = 1$, or $q = -1$ and $p = -1$, or $q = -\frac{1}{2}$ and $p = 0$. The converse implication is easy to verify. $\square$

As an immediate consequence we get the following

Corollary 3. *The Beckenbach-Gini and power means coincide if, and only if, they are equal either to A, or to G or to H.*

Applying Theorem 6 we obtain

Remark 5. A Beckenbach-Gini mean is invariant with respect to a homogeneous Beckenbach-Gini mean-type mapping if, and only if, it is a power mean i.e., if, and only if, it is either A, or G or H.

Acknowledgement. The author is indebted to the referees for their valuable remarks.

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