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ON (α, a)-CONVEX SET-VALUED FUNCTIONS

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Abstract

Let I be a real interval and $\alpha, a \in (0, 1)$ be fixed numbers. We prove that if α, a are not algebraically conjugate, then every (α, a)-convex set-valued function $F : I \to cl(Y)$, where $cl(Y)$ denotes all nonempty closed subsets of a locally convex linear topological space~~, is constant~~.

1. Introduction

Let $I \subset \mathbf{R}$ be an interval and $\alpha, a \in (0, 1)$ be arbitrarily fixed. A function $f : I \to [-\infty, +\infty)$ is said to be (α, a)-*convex* iff

$$f(\alpha s + (1 - \alpha)t) \leq af(s) + (1 - a)f(t)$$

for all $s, t \in I$.

This notion was considered by Kuhn [3]. J. Matkowski and M. Pycia [4] proved that if α and a are not conjugate (cf. Definition 2), then every (α, a)-convex function is constant.

In this paper, we give a set-valued version of this result.

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For a Hausdorff topological space Y denote by $cl(Y)$ the class of all nonempty closed subsets of Y.

We start with some definitions.

Definition 1. Let Y be a linear topological space, and let α, $a \in (0, 1)$ be fixed. A set-valued function $F : I \to cl(Y)$ is said to be (α, a)*-convex* iff

$$F(\alpha s + (1 - \alpha)t) \supset aF(s) + (1 - a)F(t)$$

for all $s, t \in I$.

Definition 2 (cf. for instance Kuczma [1, p. 106]). The elements $\alpha, a \in \mathbf{R}$ are said to be *conjugate* iff either both are transcendental or algebraically conjugate, i.e., both are algebraic and have the same minimal polynomial with rational coefficients.

Now we need the following.

Lemma. *Let A be a closed subset of linear topological space and $a \in (0, 1)$ be a fixed number. Then A is convex if and only if*

$$A \supset aA + (1 - a)A.$$

Proof. Take arbitrary $x, y \in A$. We shall show that the segment $[x; y] := \{tx + (1 - t)y : t \in [0, 1]\}$ is contained in A. For an indirect argument suppose that it is not true. Since $[x; y] \cap A$ is closed (even compact), the set $[x; y] \setminus A$ is open in $[x; y]$ (with the induced topology). Consequently, there exist $u, v \in [x; y] \cap A$, $u \neq v$, such that

$$(u; v) \cap A = \varnothing,$$

where $(u; v)$ is the open segment with endpoints u and v. In view of the assumption, $au + (1 - a)v \in aA + (1 - a)A \subset A$, and consequently $au + (1 - a)v \in (u; v) \cap A$, which is a desired contradiction. As the converse implication is obvious, the proof is completed.

Remark. It is well known that every closed subset A of a linear topological space satisfying the condition

$$A \supset \frac{1}{2}A + \frac{1}{2}A$$

is convex. Using this fact and the following identity (cf. Daroczy-Pàles [1])

$$\frac{s+t}{2} = \alpha\left[\alpha\frac{s+t}{2} + \beta t\right] + \beta\left[\alpha s + \beta\frac{s+t}{2}\right]$$

(we write here $\beta = 1 - \alpha$, $b = 1 - a$), one can give another proof of the above lemma.

The main result is the following.

Theorem. *Let Y be a locally convex linear topological space. Let $I \subset \mathbf{R}$ be an open interval. If $\alpha, a \in (0, 1)$ are not conjugate, then every (α, a)-convex set-valued function $F : I \to cl(Y)$ is constant.*

Proof. Take an arbitrary (α, a)-convex set-valued function $F : I \to cl(Y)$, where $\alpha, a \in (0, 1)$ are not conjugate. In view of our Lemma, F is convex-valued.

We divide the proof into two steps. The first shows that our theorem holds in the case when $Y = \mathbf{R}$. Next, applying a separation theorem, we prove that it remains true for an arbitrary locally convex linear topological space Y.

Step 1. Let $Y := \mathbf{R}$.

Since the values of F are convex, there exist two functions $f : I \to [-\infty, +\infty)$, $g : I \to (-\infty, +\infty]$, $f(s) \le g(s)$, $s \in I$, such that for every $s \in I$, the set $F(s)$ is an interval of the endpoints $f(s)$, $g(s)$. Thus we have

$$F(s) = \begin{cases} [f(s), g(s)] & \text{if } f(s), g(s) \in \mathbf{R} \\ [f(s), +\infty) & \text{if } g(s) = +\infty \\ (-\infty, g(s)] & \text{if } f(s) = -\infty \\ (-\infty, +\infty) & \text{if } f(s) = -\infty \text{ and } g(s) = +\infty. \end{cases}$$

Writing the (α, a)-convexity for this representation of F, it is easy to see that the functions f and $-g$ are (α, a)-convex. Applying the main result of [4] we infer that the functions f and g are constant. Hence the interval-valued function F must also be constant.

Step 2. Let Y be a locally convex linear topological space.

Take an arbitrary $y^* \in Y^*$, where Y^* denotes the space of all real continuous linear functional defined on Y. From the (α, a)-convexity of F, we obtain

$$(y^* \circ F)(\alpha s + (1-\alpha)t) \supset y^* \circ (aF(s) + (1-a)F(t))$$

$$= a(y^* \circ F)(s) + (1-a)(y^* \circ F)(t)$$

for all $s, t \in I$. Since the closure operation is monotonic and homogeneous, and the closure of the algebraic sum of two sets contains the sum of their closures, we have

$$\overline{(y^* \circ F)(\alpha s + (1-\alpha)t)} \supset \overline{a(y^* \circ F)(s) + (1-a)(y^* \circ F)(t)}$$

$$\supset a\overline{(y^* \circ F)(s)} + (1-a)\overline{(y^* \circ F)(t)}$$

for all $s, t \in I$. It means that the multifunction $G := \overline{y^* \circ F} : I \to cl(\mathbf{R})$ is (α, a)-convex. According to what we have shown in Step 1, $\overline{G}$ is constant. This implies that the multifunction F is also constant. For an indirect argument suppose that it is not a case. Then we could find such points $s, t \in I$ that $F(s) \setminus F(t) \neq \varnothing$. Hence there exists a point z such that $z \in F(s)$ and $z \notin F(t)$. Since the singleton $\{z\}$ is compact, and $F(t)$ is convex and closed, there exists a functional $y^* \in Y^*$ which separates these two sets (cf. Rolewicz [5], p. 98, Corollary 4.3.4). More precisely, there exist $k \in \mathbf{R}$ and $\varepsilon > 0$, such that

$$y^*(z) \geq k + \varepsilon \quad \text{and} \quad \sup_{r \in F(t)} y^*(r) \leq k.$$

Hence $y^*(z) \notin \overline{(y^* \circ F)(t)}$, which contradicts the fact that G is constant, and the proof is completed.

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