Aequationes Math. 64 (2002) 297–303
0001-9054/02/030297-7

Aequationes Mathematicae

On iteration semigroups of mean-type mappings and invariant means

Janusz Matkowski

Summary. Iteration semigroups of weighted quasi-arithmetic mean-type mappings are considered. Existence and uniqueness of a mean which is invariant with respect to a continuous semigroup of mean-type mapping is proved.

Mathematics Subject Classification (2000). Primary 26A18, 26E60, 39B12.

Keywords. Mean, mean-type mapping, iteration semigroup, invariant mean.

1. Introduction

Let $I \subset \mathbb{R}$ be an interval. By a *mean* we mean a function $M : I^2 \to I$ such that

$$\min(x, y) \le M(x, y) \le \max(x, y), \qquad x, y \in I.$$

If for all $x, y \in I, x \neq y$, these inequalities are strict, we call M to be a *strict mean.* A mean M is *symmetric* if

$$M(x, y) = M(y, x), \qquad x, y \in I.$$

A mapping $\mathbf{M} : I^2 \to I^2$ is called a mean-type mapping if there are two means $M, N : I^2 \to I$ such that

$$\mathbf{M}(x, y) = (M(x, y), N(x, y)), \qquad (x, y) \in I^2,$$

shortly $\mathbf{M}=(M, N)$. The composition of two mean-type mappings is a mean-type mapping; moreover, if $M, N : I^2 \to I$ are strict continuous means, then the sequence of iterates of the mean-type mapping $\mathbf{M} : I^2 \to I^2$, $\mathbf{M}=(M, N)$, converges to a mean-type mapping $\mathbf{K} : I^2 \to I^2$, $\mathbf{K}=(K, K)$, where K is a unique continuous and $\mathbf{M}$-invariant mean, i.e. such that

$$K(M(x, y), N(x, y)) = K(x, y), \qquad x, y \in I.$$

(It is an immediate consequence of a more general result proved in [5].)

There are some important special classes of symmetric means, for instance, power means $\{M^{[p]} : p \in \mathbb{R}\}$, $M^{[p]} : (0,\infty)^2 \to (0,\infty)$, defined by

$$M^{[p]}(x,y) := \left(\frac{x^p + y^p}{2}\right)^{1/p}, \quad p \neq 0; \qquad M^{[0]}(x,y) := \sqrt{xy}.$$

Given $p, q \in \mathbb{R}$, denote by $\mathbf{M}^{[p,q]}$ the *power mean-type mapping* $(M^{[p]}, M^{[q]})$. It turns out that only in the following very special cases the composition of two power mean-type mappings is a power mean type mapping ([3]):

$$\mathbf{M}^{[p,q]} \circ \mathbf{M}^{[r,r]} = \mathbf{M}^{[r,r]}, \qquad \mathbf{M}^{[p,p]} \circ \mathbf{M}^{[p,0]} = \mathbf{M}^{[\frac{p}{2},\frac{p}{2}]},$$

$$\mathbf{M}^{[0,0]} \circ \mathbf{M}^{[-p,p]} = \mathbf{M}^{[0,0]}, \quad (p,q,r \in \mathbb{R}).$$

The first of these relations implies that the one-parameter family of mean-type mappings $\{\mathbf{M}^{[p,p]} : p \in \mathbb{R}\}$ with the composition operation is a semigroup but, obviously, it is not an iteration semigroup. Taking into account the remaining two relations it easy to observe that the family $\{\mathbf{M}^{[p,q]} : p,q \in \mathbb{R}\}$ does not contain an iteration semigroup. (Note that the third of these relations means that the mean-type mapping $\mathbf{M}^{[0,0]}$ is $\mathbf{M}^{[-p,p]}$-invariant and it is equivalent to the identity $G \circ (A,H) = G$, where A, H, G denote the arithmetic, harmonic and geometric mean, respectively.) Similar facts hold true for other known symmetric families of mean-type mappings like Gini's and Stolarsky's ones (for the definitions cf. [2]).

It turns out that the situation changes completely if the weighted means (non-symmetric) are admitted. In section 2 we introduce a *weighted quasi-arithmetic mean-type mapping* $\mathbf{M}^{[\varphi]}_{p,q}$ with *generator* φ and the *weights* $p,q \in (0,1)$, and we show that the family $\{\mathbf{M}^{[\varphi]}_{p,q} : p,q \in (0,1)\}$ contains iteration semigroups. In section 3 we show that for every continuous iteration semigroup of strict mean-type mappings there exists a unique continuous mean that is invariant with respect to each element of the semigroup. We end this paper with an open question.

2. Iteration semigroups of weighted quasi-arithmetic mean-type mappings

Let a continuous and strictly monotonic function $\varphi : I \to \mathbb{R}$ and a number $a \in (0,1)$ be fixed. Then $M^{[\varphi]}_a : I^2 \to I$ defined by

$$M^{[\varphi]}_a(x,y) = \varphi^{-1}(a\varphi(x) + (1-a)\varphi(y)), \qquad x,y \in I,$$

is a mean and it is called *quasi-arithmetic weighted* mean with *generator* φ and *weight* a. For $a = \frac{1}{2}$ the mean $M^{[\varphi]}_a$ is called *quasi-arithmetic* and it is denoted by $M^{[\varphi]}$, i.e.

$$M^{[\varphi]}(x,y) = \varphi^{-1}\left(\frac{\varphi(x) + \varphi(x)}{2}\right), \qquad x,y \in I.$$

For every $a, b \in (0,1)$ the mapping $\mathbf{M}_{a,b}^{[\varphi]} : I^2 \to I^2$defined by

$$\mathbf{M}_{a,b}^{[\varphi]} := (M_a^{[\varphi]}, M_b^{[\varphi]})$$

is referred to as a *quasi-arithmetic weighted mean-type mapping* with generator φ and weights a and b.

Now we consider the question whether the family $\{\mathbf{M}_{a,b}^{[\varphi]} : a, b \in (0,1)\}$ contains a continuous iterations semigroup.

To answer the question we look for functions $a, b : (0, \infty) \to (0,1)$ such that the family of mappings $\{\mathbf{M}(t) : t > 0\}$ defined by

$$\mathbf{M}(t) := \mathbf{M}_{a(t),b(t)}^{[\varphi]}, \qquad t > 0,$$

satisfies the semigroup condition

$$\mathbf{M}(s) \circ \mathbf{M}(t) = \mathbf{M}(s+t), \qquad s, t > 0.$$

Since, for all $t > 0$, $x, y \in I$,

$$\begin{aligned}&\mathbf{M}(t)(x,y)\\ &= (\varphi^{-1}(a(t)\varphi(x) + (1-a(t))\varphi(y)), \varphi^{-1}(b(t)\varphi(x) + (1-b(t))\varphi(y)))\end{aligned}$$

comparing the coordinates of both sides in the semigroup condition we obtain

$$\begin{aligned}a(s)\left[a(t)\varphi(x) + (1-a(t))\varphi(y)\right] + (1-a(s))\left[b(t)\varphi(x) + (1-b(t))\varphi(y)\right]\\ = a\left[h^{-1}(h(s)+h(t)\right]\varphi(x) + (1 - a[h^{-1}(h(s)+h(t))]\varphi(y)\end{aligned}$$

and

$$\begin{aligned}b(s)\left[a(t)\varphi(x) + (1-a(t))\varphi(y)\right] + (1-b(s))\left[b(t)\varphi(x) + (1-b(t))\varphi(y)\right]\\ = b(s+t)\varphi(x) + (1 - b(s+t)\varphi(y))\end{aligned}$$

for all $s, t > 0$, $x, y \in I$.

Since x, y are arbitrary and φ is one-to-one, the first of these equations implies that

$$a(s)a(t) + (1-a(s))b(t) = a(s+t), \qquad s, t > 0. \tag{1}$$

Similarly the second equation implies that

$$b(s)a(t) + (1-b(s))b(t) = b(s+t), \qquad s, t > 0. \tag{2}$$

By the symmetry of the right-hand side of (1) we obtain

$$a(s)a(t) + (1-a(s))b(t) = a(t)a(s) + (1-a(t))b(s), \qquad s, t > 0,$$

which can be written in the form

$$\frac{1-a(t)}{b(t)} = \frac{1-a(s)}{b(s)}, \qquad s, t > 0.$$

Consequently, there exists a constant $c > 0$ such that

$$\frac{1-a(t)}{b(t)} = c, \qquad t > 0. \tag{3}$$

Similarly equation (2) implies that there is a constant $d > 0$ such that

$$\frac{1-b(t)}{a(t)} = d, \qquad t > 0. \tag{4}$$

From (3) and (4) we get

$$a(t)(dc-1) = c-1, \qquad t > 0. \tag{5}$$

If $dc - 1 \neq 0$ then the function a is constant. In this case equation (1) implies that b is also constant and $b = a$, and the family $\{\mathbf{M}(t) : t > 0\}$ reduces to a trivial one element set.

If the function a is not constant, from (5) we get $dc - 1 = 0 = c - 1$ and, consequently $c = d = 1$. Now (3) yields

$$b(t) = 1 - a(t), \qquad t > 0. \tag{6}$$

Hence, making use of (1), we obtain the functional equation

$$a(s)a(t) + (1-a(s))(1-a(t)) = a(s+t), \qquad s,t \in (0,\infty),$$

for an unknown function a. Setting here

$$\gamma(t) := 2\alpha(t) - 1, \qquad t \in (0,\infty),$$

we get

$$\gamma(s+t) := \gamma(s)\gamma(t), \qquad s,t \in (0,\infty).$$

Since, by assumption, $a : (0,\infty) \to (0,1)$, the range of γ is contained in $(0,1)$. It follows that $\log \circ \gamma$ is additive and bounded above. The Berstein–Doetsch theorem (cf. [4], p. 147) implies that γ is continuous and, consequently, there is a constant $k \in \mathbb{R}$ (cf. for instance [1] or [4]) such that

$$\gamma(t) := \exp(kt), \qquad t \in (0,\infty).$$

Since γ is not constant, we infer that $k \neq 0$. By the definitions of γ and a we infer that

$$a(t) = \frac{1}{2} + \frac{1}{2}\exp(kt), \qquad t > 0.$$

Hence, in view of (6),

$$b(t) = \frac{1}{2} - \frac{1}{2}\exp(kt), \qquad t > 0.$$

As the ranges of the functions a and b are contained in $(0,1)$, we infer that

$$k < 0.$$

Since it is easy to verify that the family of mappings $\{\mathbf{M}^{[\varphi]}_{a(t),b(t)} : t > 0\}$ satisfies the semigroup condition, we have proved the following

Theorem 1. *Let $I \subset \mathbb{R}$ be an interval and $\varphi : I \to \mathbb{R}$ continuous and strictly monotonic. A function $(0,\infty) \ni t \to \mathbf{M}(t)$ defined by*

$$\mathbf{M}(t) := (M^{[\varphi]}_{a(t)}, M^{[\varphi]}_{b(t)}), \qquad t > 0,$$

where $a, b : (0, \infty) \to (0, 1)$, *satisfies the functional equation*

$$\mathbf{M}(s) \circ \mathbf{M}(t) = \mathbf{M}(s + t), \qquad s, t > 0,$$

if, and only if, either the functions a *and* b *are constant and* $a = b$ *or there is a constant* $k < 0$ *such that*

$$a(t) = \frac{1}{2} + \frac{1}{2}\exp(kt), \qquad b(t) = \frac{1}{2} - \frac{1}{2}\exp(kt), \qquad t > 0.$$

3. Iteration semigroups and invariant mean-type mappings

Let us note the following easy to verify

Remark 1. Under the assumptions of Theorem 1, for every $t > 0$, the quasi-arithmetic mean $M^{[\varphi]}$ is $\mathbf{M}^{[\varphi]}_{[a(t), 1-a(t)]}$-invariant.

In this section we show the following more general

Theorem 2. *Let* $I \subset \mathbb{R}$ *be an interval. Suppose that* $\{\mathbf{M}^t : t > 0\}$ *is a family of continuous and strict mean-type mappings* $\mathbf{M}^t : I^2 \to I^2$ *such that*

$$\mathbf{M}^s \circ \mathbf{M}^t = \mathbf{M}^{s+t}, \qquad s, t > 0, \tag{7}$$

and, for every $x, y \in I$, *the function*

$$(0, \infty) \ni t \to \mathbf{M}^t(x, y) \tag{8}$$

is continuous. Then there exists a unique continuous mean $K : I^2 \to I$ *such that for every* $t > 0$, K *is* M^t*-invariant, i.e.*

$$K \circ \mathbf{M}^t = K, \qquad t > 0;$$

moreover K *is strict.*

Proof. According to Theorem 1 in [5] for every $t > 0$ there exists a unique continuous $\mathbf{M}^t$-invariant mean K^t, i.e.

$$K^t = K^t \circ \mathbf{M}^t.$$

moreover this mean is strict. Hence, for a fixed $t > 0$ and arbitrary positive integer n, by induction, we have

$$K^t = K^t \circ \mathbf{M}^{nt}.$$

Since $\mathbf{M}^{nt}$ is also a strict continuous mean-type mapping, we have

$$K^{nt} = K^{nt} \circ \mathbf{M}^{nt}.$$

Thus K^t and K^{nt} are $\mathbf{M}^{nt}$-invariant. Now the uniqueness of the invariant means implies that

$$K^{nt} = K^t.$$

Replacing here t by $\frac{t}{n}$ we get

$$K^{\frac{t}{n}} = K^t.$$

Hence, making use of the semigroup condition (7), we have, for arbitrary positive integers n and m,

$$K^t = K^t \circ (\mathbf{M}^{\frac{t}{m}})^n = K^t \circ \mathbf{M}^{\frac{n}{m}t},$$

and, consequently, for every positive rational number r,

$$K^t = K^t \circ \mathbf{M}^{rt}.$$

Taking a sequence of positive rationals numbers (r_k) such that $\lim_{k\to\infty} r_k = \frac{1}{t}$, and making use of the continuity of the function (8), we obtain

$$K^t = K^t \circ \mathbf{M}^{r_k t} = \lim_{k\to\infty} K^t \circ \mathbf{M}^{r_k t} = K^t \circ \mathbf{M}^1,$$

i.e. K^t is $\mathbf{M}^1$-invariant. Now the uniqueness of the $\mathbf{M}^1$-invariant means implies that $K^t = K^1$. This completes the proof. □

Corollary 1. *Let $I \subset \mathbb{R}$ be an interval. Suppose that $\{\mathbf{M}^t : t > 0\}$ is a family of continuous and strict mean-type mappings $\mathbf{M}^t : I^2 \to I^2$ such that for every $x, y \in I$, the function*

$$(0, \infty) \ni t \to \mathbf{M}^t(x, y)$$

is continuous. Then, a necessary condition for the family of mean-type mappings $\{\mathbf{M}^t : t > 0\}$ to be a continuous iteration semigroup is the existence of a unique strict continuous mean $K : I^2 \to I$ such that K is M^t-invariant for every $t > 0$.

4. Final remark and a problem

As we have mentioned in the introduction, the arithmetic, harmonic, and geometric means, denoted respectively by A, H, and G, satisfy the relation $G = G \circ (A, H)$ which means that G is invariant with respect to the mean-type mapping (A, H). According to Theorem 1 in [5], the sequence of iterates of the mapping (A, H) converges to G in $(0, \infty)^2$.

We end this note with the following open

Problem 1. *Does there exist a continuous iteration semigroup of symmetric mean-type mappings defined on $(0, \infty)^2$ such that the mean-type mapping (A, H) is a member of this semigroup? If the answer is affirmative, find its form.*

Acknowledgement. I wish to thank to anonymous referees for their comments and suggestions that led to improvements of the first version of the paper.

References

[1] J. Aczél, *Lectures on functional equations and their applications*, Academic Press, New York and London, 1966.
[2] P. S. Bullen, D. S. Mitrinović and P. M. Vasić, *Means and their inequalities*, Mathematics and its Applications, D. Reidel Publishing Company, Dodrecht–Boston–Lancaster–Tokyo, 1988.
[3] P. Kahlig and J. Matkowski, *On the composition of homogeneous quasi-arithmetic means*, J. Math. Anal. Appl. *216* (1997), 69–85.
[4] M. Kuczma, *An introduction to the theory of functional equations and inequalities. Cauchy's equation and Jensen's inequality*, Uniw. Sl.–P.W.N., Warszawa–Kraków–Katowice, 1985.
[5] J. Matkowski, *Iterations of mean-type mappings and invariant means*, Ann. Math. Sil. *13* (1999), 211–226.

J. Matkowski
Institute of Mathematics
University of Zielona Góra
Podgórna 50
PL–85-246 Zielona Góra
Poland
and
Institute of Mathematics
Silesian University
Bankowa 14
PL–40-007 Katowice
Poland
e-mail: J.Matkowski@im.uz.zgora.pl

Manuscript received: February 22, 2001 and, in final form, August 21, 2001.