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On a Problem of Alsina Again

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We give some new results concerning the functional equation $f(x+y)+f(f(x)+f(y))=f[f(x+f(y))+f(f(x)+y)]$, proposed by C. Alsina in 1986. We also investigate a related equation $zf(x+y)+f(f(x)+f(y))=f[f(x+zf(y))+f(zf(x)+y)]$ and we solve it in the class of decreasing involutions of $(0,\infty)$ onto itself.

1. INTRODUCTION

At the 24th International Symposium on Functional Equations, Alsina (cf. [1]) proposed to determine all continuous and decreasing involutions f: $(0,\infty)\to(0,\infty)$ satisfying the functional equation

$$f(x+y)+f(f(x)+f(y))=f[f(x+f(y))+f(f(x)+y)]. \quad (1)$$

It is easy to check that for every positive c the function $f(x)=cx^{-1}$, $x>0$, satisfies (1) and the remaining requirements. A partial answer to Alsina's question was given in [5]. However, the original problem remains open.

In [5] we have noticed that without loss of generality we may restrict ourselves to the case where 1 is the only fixed point of f. It turns out that

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then any solution of (1) satisfies also the equation

$$f(x+1)+f(f(x)+1)=1.$$

The above equation has already been dealt with. Benz and Elliger [2] proved that the inverse is the only endomorphism (or antiendomorphism) of the multiplicative group $\mathbf{K}^*$ of a field $\mathbf{K}$ which satisfies the above equation. The result has applications in the theory of groups of permutations and in geometry (cf. [2]). The same equation appears also in the definition of the so-called KT-nearfields.

Also the form of every decreasing and convex solution of (1) is an open question. In this context it seems to be interesting that, as we show in Section 3, all decreasing and geometrically convex (or geometrically concave) solutions of (1) are of the form $f(x) = \frac{c}{x}$, $x > 0$.

In Section 4 we prove that power functions are the only continuous (at least at one point) solutions of the system

$$\gamma(at) = \alpha\gamma(t),\ t > 0; \qquad \gamma(bt) = \beta\gamma(t),\ t \in (0,1),$$

where $a, b, \alpha, \beta \in (0,\infty)$ are fixed and such that $a < 1 < b$ and $(\ln b/\ln a) \notin Q$. This is a generalization of a similar result from [3] where γ was supposed to be positive while both equations were assumed on $(0,\infty)$.

The main part of the paper is contained in Section 5. We deal with the following functional equation

$$zf(x+y)+f(f(x)+f(y))=f[f(x+zf(y))+f(zf(x)+y)] \quad (2)$$

of which (1) is a particular case; namely putting $z = 1$ in (2) we see that any solution of (2) satisfies (1) as well. Applying the results of Section 4 we prove that every decreasing solution of Eq. (2) is of the form $f(x) = cx^{-1}$ for some $c > 0$.

2. SOME AUXILIARY RESULTS

In this section we recall some known facts about Alsina's Eq. (1) (cf. [5]). Let us begin with

Remark 2.1. If f: $(0,\infty) \to (0,\infty)$ is a continuous and decreasing solution of (1) then for every $c > 0$ the function cf enjoys the above properties as well. It follows that without loss of generality we may assume $f(1) = 1$.

LEMMA 2.2 (cf. [5, Lemma 1]). *Let f: $(0,\infty) \to (0,\infty)$ be a strictly decreasing solution of* (1) *with* $f(1) = 1$. *Then f is an involution of $(0,\infty)$, i.e.,* $f \circ f = \mathrm{id}_{(0,\infty)}$.

LEMMA 2.3 (cf. [5, Theorem 5]). *Let $f\colon (0,\infty) \to (0,\infty)$ be a continuous and strictly decreasing solution of* (1). *If the functions $g, h\colon (0,\infty) \to (0,\infty)$ defined by*

$$g(x) = \frac{f[2f(x)]}{x} \quad \text{and} \quad h(x) = \frac{f[3f(x)]}{x},$$

are monotonic then there exists a $c > 0$ such that

$$f(x) = cx^{-1}$$

for every $x > 0$.

3. GEOMETRICALLY CONVEX SOLUTIONS OF ALSINA'S EQUATION

Let $I \subset (0,\infty)$ be an interval. A function $f\colon I \to (0,\infty)$ is said to be *geometrically Jensen convex* iff

$$f\left(\sqrt{xy}\right) \le \sqrt{f(x)f(y)}$$

for all $x, y \in I$. If a function f satisfies the reverse inequality then we say that it is *geometrically Jensen concave*.

Remark 3.1. Let $f\colon (0,\infty) \to (0,\infty)$ be continuous. It is easy to verify that the following conditions are equivalent

(i) f is geometrically Jensen convex (concave);

(ii) $\log \circ f \circ \exp$ is convex (concave) on $\mathbb{R}$;

(iii) for every $t > 1$ the function

$$(0,\infty) \ni x \to \frac{f(tx)}{f(x)}$$

is nondecreasing (nonincreasing).

It was noticed in [3] (see also [4]) that the condition (iii) appears in a natural way in some problems connected with a characterization of the L^p norm and in iteration theory. The main result of the present section is the following.

THEOREM 3.2. *If $f\colon (0,\infty) \to (0,\infty)$ is a strictly decreasing and geometrically Jensen convex (concave) solution of Eq.* (1) *then there exists a $c > 0$*

such that

$$f(x) = \frac{c}{x}$$

for every $x > 0$.

Proof. Suppose that f is geometrically Jensen concave in $(0, \infty)$. Since f is strictly decreasing it has to be continuous everywhere. From Remark 3.1 we infer that the function

$$(0, \infty) \ni x \to \frac{f(tx)}{f(x)}$$

is nondecreasing for every $t > 1$. Hence for every $t > 1$ the function

$$(0, \infty) \ni x \to \frac{f(tf(x))}{x}$$

is nonincreasing, as a composition of a nondecreasing function and f which is both decreasing and involutory (cf. Lemma 2.2). It remains to apply Lemma 2.3 to conclude the proof. The argument is analogous in the case where convexity is replaced by concavity.

Remark 3.3. Note that for every $c > 0$ the function f: $(0, \infty) \to (0, \infty)$ given by $f(x) = cx^{-1}$ is a geometrically Jensen function, i.e., f satisfies

$$f\left(\sqrt{xy}\right) = \sqrt{f(x)f(y)}$$

for all positive x and y.

Remark 3.4. It is an open question whether every strictly decreasing and convex solution of (1) is of the form $f(x) = cx^{-1}$, $x > 0$.

4. A SYSTEM OF FUNCTIONAL EQUATIONS

To determine all continuous and decreasing solutions of the functional Eq. (2) we need the following.

PROPOSITION 4.1. *Let a, b, α, and β be positive reals and suppose that γ: $(0, \infty) \to \mathbb{R} \setminus \{0\}$ is continuous at a point $s \in (0, 1)$ and satisfies the system of functional equations*

$$\begin{cases} \gamma(at) = \alpha\gamma(t) & \text{if } t > 0, \\ \gamma(bt) = \beta\gamma(t) & \text{if } 0 < t < 1. \end{cases} \tag{3}$$

If $a < 1 < b$ *and* $\frac{\log b}{\log a}$ *is irrational then there exists a* $p \in \mathbb{R}$ *such that* $\gamma(t) = \gamma(1)t^p$ *for all* $t \in (0, \infty)$.

Proof. An easy induction shows that (3) implies

$$\gamma(a^n b^m t) = \alpha^n \beta^m \gamma(t) \tag{4}$$

for every $(n, m, t) \in \mathbb{N} \times \mathbb{N} \times (0, \infty)$ such that $b^{m-1}a^n t \in (0, 1)$. Observe that by the well known Kronecker theorem the set

$$A := \{a^n b^m : n, m \in \mathbb{N}\}$$

is dense in $(0, \infty)$. Now, fix arbitrarily a $t \in (0, \infty)$ and choose two sequences $(n_k)_{k \in \mathbb{N}}$ and $(m_k)_{k \in \mathbb{N}}$ of positive integers such that

$$\lim_{k \to \infty} n_k = \lim_{k \to \infty} m_k = \infty \tag{5}$$

and

$$\lim_{k \to \infty} a^{n_k} b^{m_k} = \frac{s}{t}. \tag{6}$$

It follows from (6) that for k large enough, say $k \geq k_0$, we have

$$a^{n_k} b^{m_k} t \in (0, 1),$$

because we assumed $s \in (0, 1)$. Since $b > 1$, we can use (4) to get

$$\gamma(a^{n_k} b^{m_k} t) = \alpha^{n_k} \beta^{m_k} \gamma(t)$$

for every $k \geq k_0$. Hence, letting $k \to \infty$ and using continuity of γ at s we obtain

$$\gamma(s) = \gamma(t) \lim_{k \to \infty} \alpha^{n_k} \beta^{m_k}. \tag{7}$$

Note that in particular $\gamma(s)$ and $\gamma(t)$ are of the same sign; without loss of generality we may assume that both are positive. If $\alpha = \beta = 1$ then (7) implies the assertion with $p = 0$. Observe that cases $\alpha = 1 \neq \beta$ and $\alpha \neq 1 = \beta$ are impossible. Indeed, otherwise (7) would imply in view of (5) that $\gamma(s) \in \{0, \infty\}$, which is impossible in view of our assumptions. It remains therefore to consider the case $\alpha \neq 1 \neq \beta$. We have assumed that s, t, $\gamma(s)$, and $\gamma(t)$ are positive. Hence we can take logarithms of both sides in (6) and (7). If we divide them by m_k, let $k \to \infty$ and use (5). Then we see that

$$\lim_{k \to \infty} \frac{n_k}{m_k} = -\frac{\log b}{\log a} = -\frac{\log \beta}{\log \alpha}.$$

Thus in particular there exists a $p \in \mathbb{R}$ such that

$$\frac{\log \beta}{\log b} = \frac{\log \alpha}{\log a} =: p$$

or

$$\beta = b^p \qquad \text{and} \qquad \alpha = a^p.$$

Inserting the above into (7) and using (6) we obtain the equality

$$\gamma(s) = \left(\frac{s}{t}\right)^p \gamma(t) \tag{8}$$

which holds for every $t \in (0, \infty)$. In particular, inserting $t = 1$ into (8) we get

$$\gamma(1) = \frac{\gamma(s)}{s^p}$$

which concludes the proof in view of (8).

5. DECREASING SOLUTIONS OF EQ. (2)

Now we are in position to prove the main result concerning Eq. (2).

THEOREM 5.1. *A function $f\colon (0, \infty) \to (0, \infty)$ is a non-increasing solution of (2) if and only if there exists a positive constant c such that*

$$f(x) = cx^{-1}$$

for every $x \in (0, \infty)$.

Proof. The "if" part is trivial to check. Let f be a non-increasing solution of Eq. (2). Suppose that there are $a, b > 0$, $a < b$, such that $f(a) = f(b)$. Then, by the monotonicity of f,

$$f(x) = f(a), \qquad x \in [a, b].$$

Take arbitrary $x, y \in (a, b)$. For all positive and small enough numbers z we have

$$x + zf(y), \qquad zf(x) + y \in [a, b],$$

and, by Eq. (2), we get

$$zf(x + y) + f(2f(a)) = f(2f(a)),$$

i.e., $zf(x+y)=0$. This contradiction shows that f is in fact strictly decreasing. On the other hand, the monotonicity of f implies that

$$c := \lim_{x\to\infty} f(x)$$

exists, is finite, and non-negative. If c were positive then letting x and y tend to ∞ in (2) we would get

$$zc + f(2c) = f(2c),$$

whence $zc = 0$ for all $z > 0$. This contradiction shows that $c = 0$. Hence, taking into account that z in (2) is arbitrary, we infer that

$$f((0,\infty)) = (0,\infty).$$

Thus we have shown that f is a decreasing bijection of $(0,\infty)$ onto itself and, consequently, f is continuous.

Now, applying Remark 2.1 to Eq. (2) with $z = 1$ we may assume that $f(1) = 1$. Thus we only have to show that $f(x) = x^{-1}$ for all $x > 0$. We will use Proposition 4.1 and therefore our first step is to show that f satisfies a suitable system of functional equations. Setting $x = y = z = 1$ in (2) we get

$$2f(2) = f(2f(2)).$$

Since f is decreasing, it has only one fixed point at 1, and thus the above equality implies

$$f(2) = \frac{1}{2} \quad \text{and} \quad f\left(\frac{1}{2}\right) = 2, \tag{9}$$

because f is an involution (cf. Remark 2.1 and Lemma 2.2). Now, if we set $x = y = 1$ in (2) and use (9) we get

$$\frac{1}{2}(z+1) = f(2f(z+1))$$

for all $z > 0$. Substituting t for $z+1$ in the above equality and using again the relation $f^2 = \mathrm{id}$ we obtain

$$f\left(\frac{t}{2}\right) = 2f(t) \tag{10}$$

for every $t > 1$. From (9) and (10) we infer that $f(4) = \frac{1}{4}$. Hence substituting $x = y = \frac{1}{2}$ in (2) results in

$$z + \frac{1}{4} = f\left(2f\left(\frac{1}{2} + 2z\right)\right),$$

or, after applying f to both sides of the above equality,

$$f\left(z + \frac{1}{4}\right) = 2f\left(\frac{1}{2} + 2z\right)$$

for every $z > 0$. Put $t = z + \frac{1}{4}$ to see that the above is equivalent to

$$f(2t) = \frac{1}{2}f(t) \tag{11}$$

for every $t > 1$. Taking into account that f is strictly decreasing and $f(1) = 1$, we see that $f(x) > 1$ for every $x \in (0, 1)$. Thus we can put $f(x)$ instead of t in (10) and make sure that (11) holds for $t \in (0, 1)$ as well. Since $f(1) = 1$ we have proved that (11) holds for all $t > 0$. Replacing t by $f(t)$ in (11) and applying f to both sides we get

$$f\left(\frac{1}{2}t\right) = 2f(t) \tag{12}$$

for all $t > 0$.

Now, put $x = 1$ and $y = \frac{1}{2}$ in (2). We get

$$zf\left(\frac{3}{2}\right) + f\left(1 + f\left(\frac{1}{2}\right)\right) = f\left[f\left(1 + zf\left(\frac{1}{2}\right)\right) + f\left(z + \frac{1}{2}\right)\right]$$

for every $z > 0$. Using (9) and (12) for $t \in \{3, 2z + 1\}$ we can rewrite the above equality in the form

$$f(3)(2z + 1) = f(3f(2z + 1))$$

for every $z > 0$, or equivalently, since f is an involution as

$$f(f(3)(2z + 1)) = 3f(2z + 1)$$

for all $z > 0$. If we substitute $t = 2z + 1$ this implies

$$f(f(3)t) = 3f(t)$$

for every $t > 1$. Again, if $x \in (0,1)$ then $f(x) > 1$ and thus inserting $f(x)$ instead of t in the above equality we obtain after applying f to both sides

$$f(3x) = f(3)f(x) \tag{13}$$

for every $x \in (0,1)$. Put $a := \frac{1}{2}$, $b := 3$, $\alpha := 2$, and $\beta := f(3)$. Then $\frac{\log b}{\log a}$ is irrational and from (12) and (9) it follows now that f satisfies the assumptions of Proposition 4.1. Thus $f(x) = x^p$ for some $p \in \mathbb{R}$ and all $x > 0$. But f is a decreasing involution whence $p = -1$. This concludes the proof.

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