

ON THE POLYNOMIAL-LIKE ITERATIVE FUNCTIONAL EQUATION

JANUSZ MATKOWSKI

*Institute of Mathematics, Pedagogical University
Pl.Slowiannski 9, PL-65069, Zielona Gora, Poland*

WEINIAN ZHANG

*Department of Mathematics, Sichuan University
Chengdu 610064, P.R. China*

Abstract. in this paper we give some properties of solutions of the iterative functional equation $f^{n+1}(x) = a_n f^n(x) + \dots + a_0 x$, considering its characteristic equation. A useful method to discuss the general case is detailed described for the case $n = 2$.

1 Introduction

The polynomial-like functional equation of the order $n + 1$

$$(1.1) \quad f^{n+1}(x) = a_n f^n(x) + a_{n-1} f^{n-1}(x) + \dots + a_0 x, \quad a_0 \neq 0,$$

where $x \in I$, an interval of \mathbf{R} , $f : I \rightarrow I$ is an unknown function, f^n denotes the n -th iterate of f , and a_0, a_1, \dots, a_n are real constants, is an important form of iterative functional equations (see [1]-[8]). There are many particular functional equations which can be reduced to Eq. (1.1) (see [8]-[10]).

If f satisfies equation (1.1) then all its iterates f^{k+n} , $k \in \mathbf{N}$, belong to the $(n+1)$ -dimensional linear space spanned by the functions $(id|_{\mathbf{R}}, f, \dots, f^n)$. It is especially interesting in the context that equation (1.1) is not linear one.

It is worth mentioning that Eq. (1.1), related to the $(n+1)$ -th order linear difference equation

$$x_{k+n+1} = a_n x_{k+n} + \dots + a_1 x_{k+1} + a_0 x_k,$$

is a nonlinear equation since the set of solutions does not span a linear space, and the Babbage functional equation

$$f^n(x) = x,$$

is a spacial case of Eq.(1.1).

An important role is played by the characteristic polynomial of Eq.(1.1). Theorem 0 says that if two equations of the type (1.1) are of the orders n and m , $n < m$, and the characteristic polynomial of Eq.(1.1) of order n divides the characteristic polynomial of Eq.(1.1) of order m , then every solution of the Eq.(1.1) of the smaller order satisfies the second equation.

This partially explain why we confine our considerations mainly to Eq.(1.1) of the smallest nontrivial order. We give a complete description of all continuous solutions of Eq.(1.1) for $n = 1$, i.e.,

$$(1.2) \quad f^2(x) = a_1 f(x) + a_0 x, \quad a_0 \neq 0, \quad x \in \mathbf{R},$$

through discussing its characteristic equation. A useful method to discuss the general case of Eq.(1.1) is described naturally in this procedure. Let us mention that Eq.(1.2) was considered by S.Nabeya [8] with using a little different methods (cf. Final Remarks in chapter III).

2 A Basic Result

Denote by \mathbf{N} and \mathbf{Z} the sets of positive integers and integers, respectively, and put $\mathbf{N}_0 := \mathbf{N} \cup \{0\}$.

For $n = 0$ the Eq.(1.1) reduces to $f(x) = a_0 x$. Thus, in this case, $f : \mathbf{R} \rightarrow \mathbf{R}$ satisfies (1.1) if, and only if, f is a linear function. Note that, in general case, a linear function

$$f(x) = rx, \quad x \in \mathbf{R},$$

where $r \in \mathbf{R}$ (or $r \in \mathbf{C}$) is fixed, satisfies Eq.(1.1) iff

$$r^{n+1} = a_n r^n + \dots + a_1 r + a_0.$$

This algebraic equation is called the *characteristic equation* of Eq.(1.1), and the polynomial

$$P(r) := r^{n+1} - a_n r^n - \dots - a_1 r - a_0,$$

the *characteristic polynomial* of Eq.(1.1). An r_0 is a *characteristic roots* iff $P(r_0) = 0$. The linear function $f(x) = r_0 x$ is termed the *characteristic solution* of the equation. Of course there is one-to-one correspondence between the characteristic roots and the characteristic solutions of Eq.(1.1), and there are at most $n + 1$ characteristic solutions. We are mainly interested in the real solutions.

The following basic result shows that the characteristic polynomials (equations) play an important role in theory of Eq.(1.1).

Theorem 1 (cf. Matkowski [6]). Let $m, n \in \mathbf{N}$ be such that $m \geq n \geq 1$. Suppose

$$P(x) = x^m - a_{m-1}x^{m-1} - \dots - a_0; \quad Q(x) = x^n - b_{n-1}x^{n-1} - \dots - b_0$$

are polynomials such that $Q|P$ (i.e. Q divides P). If $f: \mathbf{R} \rightarrow \mathbf{R}$ satisfies the functional equation

$$f^n(x) = b_{n-1}f^{n-1}(x) + \dots + b_1f(x) + b_0x, \quad x \in \mathbf{R},$$

then f satisfies the functional equation

$$f^m(x) = a_{m-1}f^{m-1}(x) + \dots + a_1f(x) + a_0x, \quad x \in \mathbf{R}.$$

As the proof of Theorem 0 was not published yet, we include it to the present paper. we begin with some notation.

Denote by $(r_k)_{k=1}^\infty$ an arbitrary sequence of complex or real numbers. For each fixed $n \in \mathbf{N}$, we can write the polynomial

$$p_n(x) := (x - r_1)(x - r_2) \cdots (x - r_n)$$

in the form

$$p_n(x) = \sum_{k=0}^n \tau_k^n(r_1, \dots, r_n) x^{n-k},$$

where

$$\begin{aligned} \tau_0^n(r_1, \dots, r_n) &:= 1, \\ \tau_1^n(r_1, \dots, r_n) &:= \sum_{j=1}^n r_j, \\ \tau_2^n(r_1, \dots, r_n) &:= \sum_{j_1, j_2=1; j_1 < j_2}^n r_{j_1} r_{j_2}, \\ &\dots\dots\dots \\ \tau_n^n(r_1, \dots, r_n) &:= \sum_{j_1, \dots, j_n=1; j_1 < \dots < j_n}^n r_{j_1} \cdots r_{j_n} = r_1 \cdots r_n, \end{aligned}$$

are the fundamental symmetric polynomials.

Lemma 1 . The fundamental symmetric polynomials satisfy the relations

$$\tau_k^{n+1}(r_1, \dots, r_{n+1}) = \tau_k^n(r_1, \dots, r_n) + r_{n+1} \tau_{k-1}^n(r_1, \dots, r_n),$$

for all $n \in \mathbf{N}, k = 1, \dots, n$.

Proof. We have

$$(x - r_1)(x - r_2) \cdots (x - r_n) = \sum_{k=0}^n (-1)^k \tau_k^n(r_1, \dots, r_n) x^{n-k},$$

and

$$(x - r_1)(x - r_2) \cdots (x - r_{n+1}) = \sum_{k=0}^{n+1} (-1)^k \tau_k^{n+1}(r_1, \dots, r_{n+1}) x^{n+1-k},$$

and, consequently,

$$\sum_{k=0}^{n+1} (-1)^k \tau_k^{n+1}(r_1, \dots, r_{n+1}) x^{n+1-k} = (x - r_{n+1}) \sum_{k=0}^n (-1)^k \tau_k^n(r_1, \dots, r_n) x^{n-k}.$$

Let us fix $k \in \{1, \dots, n\}$. Comparing the coefficients at x^{n+1-k} of both sides we obtain the relation

$$\begin{aligned} (-1)^k \tau_k^{n+1}(r_1, \dots, r_{n+1}) &= (-1)^k \tau_k^n(r_1, \dots, r_n) - r_{n+1} (-1)^{k-1} \tau_{k-1}^n(r_1, \dots, r_n) \\ &= (-1)^k (\tau_k^n(r_1, \dots, r_n) + r_{n+1} \tau_{k-1}^n(r_1, \dots, r_n)) \end{aligned}$$

which completes the proof of the claim. \square

In the sequel, for a function $f: \mathbf{R} \rightarrow \mathbf{R}$ we put

$$f^0 := id|_{\mathbf{R}}, \quad f^n := f \circ f^{n-1}, \quad n \in \mathbf{N}.$$

Now can can prove the following

Lemma 2 . *Suppose that*

$$P(x) = x^{n+1} + a_n x^n + \dots + a_1 x + a_0; \quad Q(x) = x^n + b_{n-1} x^{n-1} + \dots + b_1 x + b_0$$

are polynomials such that $Q|P$. If a function $f: \mathbf{R} \rightarrow \mathbf{R}$ satisfies the functional equation

$$(2.3) \quad f^n(x) + b_{n-1} f^{n-1}(x) + \dots + b_1 f(x) + b_0 x = 0, \quad x \in \mathbf{R},$$

then f satisfies the functional equation

$$(2.4) \quad f^{n+1}(x) + a_n f^n(x) + \dots + a_1 f(x) + a_0 x = 0, \quad x \in \mathbf{R}.$$

Proof. Let r_1, \dots, r_{n+1} be the roots of the polynomial P . Since $Q|P$, we can assume that r_1, \dots, r_n are the roots of the polynomial Q . Making use of Vieta formulas, we can write the functional equations (2.3) and (2.4) in the forms:

$$(2.5) \quad \sum_{k=0}^n (-1)^k \tau_k^n(r_1, \dots, r_n) f^{n-k}(x) = 0, \quad x \in \mathbf{R},$$

and

$$(2.6) \quad \sum_{k=0}^{n+1} (-1)^k \tau_k^{n+1}(r_1, \dots, r_{n+1}) f^{n+1-k}(x) = 0, \quad x \in \mathbf{R}.$$

Suppose that a function $f : \mathbf{R} \rightarrow \mathbf{R}$ satisfies equation (2.3). Thus we have

$$\sum_{k=1}^{n+1} (-1)^{k-1} \tau_{k-1}^n(r_1, \dots, r_n) f^{n+1-k}(x) = 0, \quad x \in \mathbf{R},$$

or equivalently,

$$\sum_{k=1}^n (-1)^{k-1} \tau_{k-1}^n(r_1, \dots, r_n) f^{n+1-k}(x) + (-1)^n r_1 \cdots r_n x = 0, \quad x \in \mathbf{R}.$$

Multiplying this equation by $(-1)r_{n+1}$ gives

$$(2.7) \quad \sum_{k=1}^n (-1)^k r_{n+1} \tau_{k-1}^n(r_1, \dots, r_n) f^{n+1-k}(x) + (-1)^{n+1} r_1 \cdots r_{n+1} x = 0,$$

for all $x \in \mathbf{R}$. Replacing in (2.5) x by $f(x)$ we get

$$\sum_{k=0}^n (-1)^k \tau_k^n(r_1, \dots, r_n) f^{n+1-k}(x) = 0, \quad x \in \mathbf{R},$$

which can be written in the form

$$(2.8) \quad f^{n+1}(x) + \sum_{k=1}^n (-1)^k \tau_k^n(r_1, \dots, r_n) f^{n+1-k}(x) = 0, \quad x \in \mathbf{R}.$$

Adding the equations (2.7) and (2.8) by sides we get

$$\begin{aligned} f^{n+1}(x) + \sum_{k=1}^n (-1)^k (\tau_k^n(r_1, \dots, r_n) + r_{n+1} \tau_{k-1}^n(r_1, \dots, r_n)) f^{n+1-k}(x) \\ + (-1)^{n+1} r_1 \cdots r_{n+1} x = 0, \end{aligned}$$

for all $x \in \mathbf{R}$. Now applying Lemma 1 and the definitions of

$$\tau_0^{n+1}(r_1, \dots, r_{n+1}) \text{ and } \tau_{n+1}^{n+1}(r_1, \dots, r_{n+1})$$

we hence get

$$\sum_{k=0}^{n+1} (-1)^k \tau_k^{n+1}(r_1, \dots, r_{n+1}) f^{n+1-k}(x) = 0, \quad x \in \mathbf{R},$$

which shows that f satisfies equation (2.4). This completes the proof. \square

Proof of Theorem 1. Let r_1, \dots, r_m be the roots of the polynomial P . Since $Q|P$, we can assume that r_1, \dots, r_n are the roots of the polynomial Q . Put

$$p_k(x) := (x - r_1)(x - r_2) \cdots (x - r_n), \quad k = n, \dots, m.$$

Thus we have $Q = p_n$, and $P = p_m$. Since $p_k|p_{k+1}$ for $k = n, \dots, m-1$, the theorem is an obvious consequence of the Lemma 2. \square

Theorem 1 has the following practical meaning. Knowing the theory of equation (1.1) of the order $n \in \mathbb{N}$, and $m > n, m \in \mathbb{N}$, then one can establish a lot of solutions of equation (1.1) of the higher order m .

Therefore, in the sequel, we confine our investigations mainly to the case of the second order, i.e. to Eq.(1.2).

Lemma 3. *Suppose that $f : \mathbb{R} \rightarrow \mathbb{R}$ is a solution of Eq.(1.1). Then*

- (i) *f is one-to-one;*
- (ii) *f is strictly monotone and onto, provided f is continuous.*

Proof. (i) If $f(y_1) = f(y_2)$ then $f^k(y_1) = f^k(y_2)$ for all $k \in \mathbb{N}$. Hence, by Eq.(1.1)

$$\begin{aligned} a_0 y_1 &= f^{n+1}(y_1) - a_n f^n(y_1) - \dots - a_1 f(y_1) \\ &= f^{n+1}(y_2) - a_n f^n(y_2) - \dots - a_1 f(y_2) = a_0 y_2. \end{aligned}$$

Since $a_0 \neq 0$, it follows that $y_1 = y_2$, and, consequently, f is one-to-one. By the continuity, f must be monotone.

To prove (ii) write Eq.(1.1) in the form

$$f^{n+1}(x) - a_n f^n(x) - \dots - a_1 f(x) = a_0 x.$$

It follows that the left-hand side is unbounded on each of the intervals (a, ∞) and $(-\infty, a)$. Now the continuity of f on \mathbb{R} implies that f must be left and right unbounded. \square

A similar reasoning allows to prove the following

Remark 1. Let $k \in \mathbb{N}$, and a function $F : \mathbb{R}^k \rightarrow \mathbb{R}$ be fixed. Suppose that $f : \mathbb{R} \rightarrow \mathbb{R}$ satisfies the functional equation

$$F(f(x), f^2(x), \dots, f^k(x)) = x, \quad x \in \mathbb{R},$$

Then f is one-to-one; and f is strictly monotone, provided f has the Darboux property.

3 Characteristic Equation in the case $n = 2$

In this chapter we deal with equation (1.2). We shall see that the existence and uniqueness of continuous solution of this equation depends on the behavior of roots of the characteristic equation

$$r^2 = a_1 r + a_0, \quad a_0 \neq 0.$$

Now, using the Vieta formulas, we rewrite Eq.(1.2) into the form

$$(3.9) \quad f^2(x) = (r_1 + r_2)f(x) - r_1 r_2 x,$$

where r_1, r_2 are, in general, nonzero complex roots of the characteristic equation

$$r^2 = a_1 r + a_0, \quad a_0 \neq 0.$$

Obviously r_1, r_2 satisfy the relations

$$r_1 + r_2 = a_1, \quad r_1 r_2 = -a_0.$$

By Lemma 3 Eq.(1.2) is also equivalent to

$$(3.10) \quad f^{-2}(x) = \left(\frac{1}{r_1} + \frac{1}{r_2}\right)f^{-1}(x) - \frac{1}{r_1 r_2}x,$$

called the *dual equation* of Eq.(1.2), where $1/r_1, 1/r_2$ are the characteristic roots of (3.10).

Lemma 4 . Let $f: \mathbb{R} \rightarrow \mathbb{R}$ be an arbitrary solution of Eq.(1.2).

(i) if the characteristic roots r_1, r_2 of Eq.(1.2) are different, then

$$(3.11) \quad f^n(x) = A_2(n)(f(x) - r_1 x) - A_1(n)(f(x) - r_2 x), \quad n \in \mathbb{N}_0,$$

where

$$A_i(n) = \frac{r_i^n}{r_2 - r_1}, \quad i = 1, 2;$$

and

$$(3.12) \quad f^{-n}(x) = B_2(n)\left(f^{-1}(x) - \frac{x}{r_1}\right) - B_1(-n)\left(f^{-1}(x) - \frac{x}{r_2}\right), \quad n \in \mathbb{N}_0,$$

where

$$B_i(-n) = \frac{r_i^{-n}}{r_2^{-1} - r_1^{-1}}, \quad i = 1, 2;$$

(ii) if $r_1 = r_2 = r$ then

$$(3.13) \quad f^n(x) = nr^{n-1}f(x) - (n-1)r^n x, \quad n \in \mathbb{N}_0,$$

and

$$(3.14) \quad f^n(x) = nr^{1-n}f^{-1}(x) - (n-1)r^{-n}x, \quad n \in \mathbb{N}_0,$$

Proof. Using the form (3.9) of Eq.(1.2) we have

$$f(f(x)) - r_2f(x) = r_1(f(x) - r_2x).$$

Putting $g(x) := f(x) - r_2x$ we can write this equation in a shorter way

$$g(f(x)) = r_1g(x).$$

Hence by an easy induction we have $g(f^n(x)) = r_1^n g(x)$, i.e.,

$$f^{n+1}(x) - r_2f^n(x) = r_1^n(f(x) - r_2x).$$

Similarly we also get

$$f^{n+1}(x) - r_1f^n(x) = r_2^n(f(x) - r_1x).$$

Now the last two formulas imply (3.11). Repeating the same reasoning for (3.10) we get (3.12). Furthermore, if $r_1 = r_2 = r$, then Eq.(3.9) has the form

$$f^2(x) = 2rf(x) - r^2x,$$

and Eq.(3.10) has the form

$$f^{-2}(x) = 2r^{-1}f(x) - r^{-2}x.$$

Thus (3.13) and (3.14) follow by the induction. \square

Remark 2. All the formulas of Lemma 4 can be obtained from the general solution of the linear difference equation

$$x_{k+2} = a_1x_{k+1} + a_0x_k.$$

Conversely, from these formulas, one can easily get the general solution of the difference equation. For arbitrary $x_0, x_1 \in \mathbb{R}$ define the sequence $\{x_n\}, n \in \mathbb{Z}$, recursively:

$$(3.15) \quad x_{n+2} = (r_1 + r_2)x_{n+1} - r_1r_2x_n, \quad n \in \mathbb{N}_0;$$

$$(3.16) \quad x_{-n-2} = (r_1^{-1} + r_2^{-1})x_{-n-1} - r_1^{-1}r_2^{-1}x_{-n}, \quad n \in \mathbb{N}.$$

By Lemma 4 we have

$$(3.17) \quad x_n = A_2(n)(x_1 - r_1x_0) - A_1(n)(x_1 - r_2x_0), \quad n \in \mathbb{N}_0,$$

$$(3.18) \quad x_{-n+1} = B_2(-n)(x_0 - x_1/r_1) - B_1(-n)(x_0 - x_1/r_2), \quad n \in \mathbb{N}_0.$$

It is remarkable that functional equation (1.2) is not linear in the sense of the linearity of the adequate difference equation.

Remark 3. Note that, using Vieta formulas for arbitrary n , one can introduce the dual equation in general case, and it is not difficult to observe that a more general counterparts of Lemma 1 hold true.

Lemma 5 . Suppose that characteristic roots r_1, r_2 of Eq. (1.2) are real, $|r_1| < |r_2|$, and $f: \mathbf{R} \rightarrow \mathbf{R}$ is a continuous solution of Eq. (1.2).

(i) if $r_1 > 0$ and $r_2 > 0$, then

$$r_1 \leq \frac{f(x_2) - f(x_1)}{x_2 - x_1} \leq r_2, \quad x_1, x_2 \in \mathbf{R}, \quad x_1 \neq x_2.$$

(ii) if $r_1 < 0$ and $r_2 > 0$, then

$$r_2 \leq \frac{f(x_2) - f(x_1)}{x_2 - x_1}, \quad x_1, x_2 \in \mathbf{R}, \quad x_1 \neq x_2.$$

when f is increasing; and

$$f(x) = r_1 x + c, \quad x \in \mathbf{R}.$$

for some $c \in \mathbf{R}$, when f is decreasing.

(iii) if $r_1 > 0$ and $r_2 < 0$, then

$$0 \leq \frac{f(x_2) - f(x_1)}{x_2 - x_1} \leq r_1, \quad x_1, x_2 \in \mathbf{R}, \quad x_1 \neq x_2.$$

when f is increasing; and

$$f(x) = r_2 x + c, \quad x \in \mathbf{R},$$

for some $c \in \mathbf{R}$, when f is decreasing.

(iv) if $r_1 < 0$ and $r_2 < 0$, then

$$r_2 \leq \frac{f(x_2) - f(x_1)}{x_2 - x_1} \leq r_1, \quad x_1, x_2 \in \mathbf{R}, \quad x_1 \neq x_2.$$

Proof. Step 1. From (3.11) in Lemma 4 and the assumption that $|r_1| < |r_2|$, it follows that there exists

$$(3.19) \quad u(x) := \lim_{n \rightarrow \infty} r_2^{-n} f^n(x) = \frac{f(x) - r_1 x}{r_2 - r_1}, \quad x \in \mathbf{R}.$$

Since, according to Lemma 3, f is strictly monotone, the iterates f^n are increasing for even n . As the limit of a sequence of increasing functions, u has to be nondecreasing. Thus (3.19) implies the function $x \mapsto f(x) - r_1 x$ is nondecreasing in \mathbf{R} since $r_2 > r_1$, that is, for $x_1 < x_2$ we have

$$f(x_1) - r_1 x_1 \leq f(x_2) - r_1 x_2,$$

i.e.

$$r_1(x_2 - x_1) \leq f(x_2) - f(x_1),$$

and therefore

$$(3.20) \quad r_1 \leq \frac{f(x_2) - f(x_1)}{x_2 - x_1}, \quad x_1, x_2 \in \mathbf{R}, \quad x_1 \neq x_2.$$

Similarly, from (3.12) in Lemma 4 and the assumption $|r_1| < |r_2|$ it follows that there exists

$$(3.21) \quad v(x) := \lim_{n \rightarrow \infty} r_1^n f^{-n}(x) = \frac{r_2^{-1}x - f^{-1}(x)}{r_2^{-1} - r_1^{-1}}, \quad x \in \mathbf{R}.$$

By the same arguments as in the case of u , the function $v : \mathbf{R} \rightarrow \mathbf{R}$ must be nondecreasing, and the function $x \mapsto (r_2^{-1}x - f^{-1}(x))$ is nonincreasing in \mathbf{R} since $r_2^{-1} - r_1^{-1} < 0$. Take $x_1 < x_2$. In view of (3.20), f is strictly increasing, so $f(x_1) < f(x_2)$ and

$$r_2^{-1}f(x_1) - x_1 \geq r_2^{-1}f(x_2) - x_2,$$

i.e.

$$(3.22) \quad \frac{f(x_2) - f(x_1)}{x_2 - x_1} \leq r_2, \quad x_1, x_2 \in \mathbf{R}, \quad x_1 \neq x_2.$$

Relations (3.20) and (3.22) complete the proof of (i).

Step 2. If $r_1 < 0$, $r_2 > 0$, then $r_2^{-1} - r_1^{-1} > 0$. By (3.21) the function $x \mapsto (r_2^{-1}x - f^{-1}(x))$ is nondecreasing. When f is increasing we have $f(x_1) < f(x_2)$ for $x_1 < x_2$, and consequently,

$$r_2^{-1}f(x_1) - x_1 \leq r_2^{-1}f(x_2) - x_2,$$

which implies the inequality in (ii) of the lemma. On the other hand, when f is decreasing, f^n is also decreasing for each odd n , and the function u in (3.19) must be nonincreasing since $r_2 > 0$. In Step 1 we have proved that u is nondecreasing, so u must be a constant function. This together with (3.19) completes the proof of (ii).

The proofs of (iii) and (iv) are analogous. \square

Remark 4. Lemma 5(i) says that f and f^{-1} are strongly monotone, i.e.,

$$(f(x_1) - f(x_2))(x_1 - x_2) \geq r_1|x_1 - x_2|^2,$$

and

$$(f^{-1}(x_1) - f^{-1}(x_2))(x_1 - x_2) \geq r_2|x_1 - x_2|^2.$$

Lemma 6 . If the solution $f : \mathbf{R} \rightarrow \mathbf{R}$ of Eq. (1.2) has a nonzero fixed point, then one of its characteristic roots equals 1.

Proof. Assume $f(x_0) = x_0$, $x_0 \neq 0$. It follows from (1.2) that $x_0 = a_1 x_0 + a_0 x_0$, i.e., $a_1 + a_0 = 1$. Since the characteristic roots r_1 and r_2 satisfy $r_1 + r_2 = a_1$ and $r_1 r_2 = -a_0$, we have $r_1 + r_2 - r_1 r_2 = 1$, i.e., $(1 - r_1)(1 - r_2) = 0$. Hence either $r_1 = 1$ or $r_2 = 1$. \square

4 Second Order Equation-Nabeya's Theory and Some Complementary Results

4.1 Noncritical Cases as $r_1 r_2 > 0$

In what follows r_1 and r_2 denote real characteristic roots of Eq. (1.2).

Case 1: $1 < r_1 < r_2$.

Theorem 2 . Suppose that $1 < r_1 < r_2$.

(i) if $f : \mathbf{R} \rightarrow \mathbf{R}$ is a continuous solution of Eq. (1.2), then $f(0) = 0$ and f , strictly increasing, satisfies the "two-side" Lipschitzian condition

$$r_1 \leq \frac{f(x) - f(x')}{x - x'} \leq r_2, \quad x, x' \in \mathbf{R}, x \neq x'.$$

(ii) Moreover, Eq. (1.2) has a continuous solution depending on an arbitrary function. More exactly, for every $x_0 > 0$, $x_1 > x_0$, and $f_0 : [x_0, x_1] \rightarrow \mathbf{R}$ such that

$$(4.23) \quad r_1 x_0 \leq x_1 \leq r_2 x_0,$$

$$(4.24) \quad f_0(x_0) = x_1, \quad f_0(x_1) = (r_1 + r_2)x_1 - r_1 r_2 x_0,$$

and

$$(4.25) \quad r_1 \leq \frac{f_0(x) - f_0(x')}{x - x'} \leq r_2, \quad x, x' \in [x_0, x_1], x \neq x',$$

there exists a unique continuous function $p : (0, \infty) \rightarrow (0, \infty)$ satisfying Eq. (1.2) on $(0, \infty)$, and $p(x) = f_0(x)$ for $x \in [x_0, x_1]$. For arbitrary two initial functions f_{01} and f_{02} like f_0 , the function

$$(4.26) \quad f(x) := \begin{cases} p_1(x) & x > 0 \\ 0 & x = 0, \\ -p_2(x) & x < 0 \end{cases}$$

is a continuous solution of Eq. (1.2) in \mathbf{R} , where p_1 and p_2 are functions like p determined as above by f_{01} and f_{02} . The formula (4.26) gives all continuous solutions of Eq. (1.2) in \mathbf{R} .

Proof. For given $x_0 > 0$ and $x_1 > 0$ in (4.23), the sequences $\{x_n\}_{n \in \mathbf{N}}$ and $\{x_{-n}\}_{n \in \mathbf{N}}$, defined by (3.15) and (3.16), are strictly monotone and

$$\lim_{n \rightarrow \infty} x_n = \infty, \quad \lim_{n \rightarrow \infty} x_{-n} = 0.$$

Since (4.24) and (4.25) imply that the given initial function f_0 satisfies

$$f_0(x_0) = x_1, \quad f_0(x_1) = x_2, \quad \text{and} \quad f_0 : [x_0, x_1] \rightarrow [x_1, x_2]$$

is a "two-side" Lipschitzian homeomorphism, we can define recursively homeomorphisms $f_n : [x_n, x_{n+1}] \rightarrow [x_{n+1}, x_{n+2}]$, $n \in \mathbf{N}_0$, such that

$$(4.27) \quad f_n(x_n) = x_{n+1}, \quad f_n(x_{n+1}) = x_{n+2},$$

and

$$(4.28) \quad r_1 \leq \frac{f_n(x) - f_n(x')}{x - x'} \leq r_2, \quad x, x' \in [x_n, x_{n+1}], \quad x \neq x'.$$

In fact, for a defined f_n in (4.27) and (4.28), we let

$$(4.29) \quad f_{n+1}(x) = (r_1 + r_2)x - r_1 r_2 f_n^{-1}(x), \quad x \in [x_{n+1}, x_{n+2}].$$

Obviously (4.27) implies $f_{n+1}(x_{n+1}) = x_{n+2}$, and $f_{n+1}(x_{n+2}) = x_{n+3}$. Moreover, by (4.28) we have

$$r_2^{-1} \leq \frac{f_n^{-1}(x) - f_n^{-1}(x')}{x - x'} \leq r_1^{-1}, \quad x, x' \in [x_{n+1}, x_{n+2}], \quad x \neq x',$$

so it is not difficult to deduce

$$r_1 \leq \frac{f_{n+1}(x) - f_{n+1}(x')}{x - x'} \leq r_2, \quad x, x' \in [x_{n+1}, x_{n+2}], \quad x \neq x'.$$

By induction the definition of f in (4.27) and (4.28) is correct.

Similarly, we can also define recursively homeomorphisms $f_{-n} : [x_{-n+1}, x_{-n+2}] \rightarrow [x_{-n}, x_{-n+1}]$, $n \in \mathbf{N}_0$, such that

$$(4.30) \quad f_{-n}(x_{-n+1}) = x_{-n}, \quad f_{-n}(x_{-n+2}) = x_{-n+1},$$

and

$$r_2^{-1} \leq \frac{f_{-n}(x) - f_{-n}(x')}{x - x'} \leq r_1^{-1}, \quad x, x' \in [x_{-n+1}, x_{-n+2}], \quad x \neq x'.$$

In fact we let

$$(4.31) \quad f_{-1}(x) = (r_1^{-1} + r_2^{-1})x - (r_1 r_2)^{-1} f_0(x), \quad x \in [x_0, x_1],$$

and

$$(4.32) \quad f_{-n-1}(x) = (r_1^{-1} + r_2^{-1})x - (r_1 r_2)^{-1} f_{-n}^{-1}(x), \quad x \in [x_{-n}, x_{-n+1}].$$

We omit the same induction procedure here.

Since

$$\begin{aligned} f_n(x_{n+1}) &= f_{n+1}(x_{n+1}), \quad n \in \mathbf{N}_0, \\ f_{-1}^{-1}(x_0) &= f_0(x_0), \\ f_{-n}^{-1}(x_{-n+1}) &= f_{-n+1}^{-1}(x_{-n+1}), \quad n \in \mathbf{N}_0, \quad n \geq 2, \end{aligned}$$

the function

$$p(x) := \begin{cases} f_n(x) & x \in [x_n, x_{n+1}], \quad n \in \mathbf{N}_0 \\ f_{-n}^{-1}(x) & x \in [x_{-n}, x_{-n+1}], \quad n \in \mathbf{N} \end{cases}$$

is a continuous function from $(0, \infty)$ into itself and satisfies Eq.(1.2) by (4.29), (4.31) and (4.32). Moreover we can extend the function p continuously at the end point 0 such that $p(0) = 0$, since (4.30) implies $p(x_{-n}) = x_{-n+1}$ and x_{-n} tends to 0 as $n \rightarrow \infty$. Furthermore, by Lemma 5(i), the above construction allows to obtain all continuous solutions of Eq.(1.2) in $(0, \infty)$.

Observe now that a function $q : (-\infty, 0) \rightarrow (-\infty, 0)$ is a continuous solution of Eq.(1.2) iff the function $p : (0, \infty) \rightarrow (0, \infty)$ defined by $p(x) = -q(-x)$ is a continuous solution of Eq.(1.2) iff $p = f|_{(0, \infty)}$ and $q = f|_{(-\infty, 0)}$ satisfy Eq.(1.2) on $(0, \infty)$ and $(-\infty, 0)$, respectively, and $f(0) = 0$. Therefore, every continuous solution $f : \mathbf{R} \rightarrow \mathbf{R}$ of Eq.(1.2) must be of the form (4.26). This completes the proof. \square

Remark 5. Take $x_1 = r_1 x_0$ (resp. $x_1 = r_2 x_0$) in Theorem 2 we get, as the only possible solution, $f(x) = r_1 x$ (resp. $f(x) = r_2 x$) for $x \in (0, \infty)$. In fact, in this case there is only one initial function f_0 satisfying (4.23)-(4.25), namely $f_0(x) = r_1 x$ (resp. $f_0(x) = r_2 x$).

Case 2: $0 < r_1 < r_2 < 1$.

This case can be reduced to the previous one by considering the equivalent equation (3.10).

Case 3: $0 < r_1 < 1 < r_2$.

Theorem 3 . Suppose that $0 < r_1 < 1 < r_2$.

- (i) If $f: \mathbf{R} \rightarrow \mathbf{R}$ is a continuous solution of Eq. (1.2), then f is strictly increasing.
- (ii) If, additionally, f has a fixed point, then

$$f(x) = \begin{cases} r_1 x & x \geq 0 \\ r_2 x & x < 0 \end{cases}, \quad i, j = 1, 2.$$

(iii) Moreover, every continuous solution $f: \mathbf{R} \rightarrow \mathbf{R}$ of Eq. (1.2) without fixed points depends on arbitrary initial function. More exactly, for $x_0 = 0$ and every $x_1 > 0$ (resp. $x_1 < 0$) and for every function $f_0: [x_0, x_1] \rightarrow \mathbf{R}$ (resp. $f_0: [x_1, x_0] \rightarrow \mathbf{R}$) such that

$$\begin{aligned} f_0(x_0) = f_0(0) = x_1, \quad f_0(x_1) = (r_1 + r_2)x_1, \\ r_1 \leq \frac{f_0(x) - f_0(x')}{x - x'} \leq r_2, \quad x, x' \neq 0, \quad x \neq x'. \end{aligned}$$

there exists a unique continuous function $f: \mathbf{R} \rightarrow \mathbf{R}$ satisfying Eq. (1.2) and $f(x) = f_0(x)$ on $[x_0, x_1]$ (resp. on $[x_1, x_0]$).

Proof. By Lemma 5(i) f is strictly increasing. By Lemma 6 we see that the only available fixed point of f is 0. Suppose $f(0) = 0$. Obviously either $0 < f(x) < x$ or $f(x) > x$ for $x > 0$. In the first case we have that f^n approaches 0 as $n \rightarrow \infty$, so (3.11) in Lemma 2 implies that $f(x) = r_1 x$. In the second case, $0 < f^{-1}(x) < x$ for $x > 0$, so it follows from (3.12) in Lemma 4, with the same arguments as in the first case, that $f^{-1}(x) = r_2^{-1}x$, i.e., $f(x) = r_2 x$. The discussion for $x < 0$ is analogous. Thus the result (ii) is proved as well.

As follows we use the method of construction in the proof of Theorem 2 to prove the result (iii). For the given x_0 and $x_1 > 0$ in the hypotheses of (iii), the sequence $\{x_n\}$ defined by (3.15) (resp. $\{x_{-n}\}$ defined by (3.16)) is strictly increasing (resp. decreasing) and tends to ∞ (resp. $-\infty$) as $n \rightarrow \infty$. Now in quite a similar way to the proof of Theorem 2 we can define the sequences $\{f_n\}$ and $\{f_{-n}\}$ of functions by (4.29) and (4.32). Then the function

$$f(x) := \begin{cases} f_n(x) & x \in [x_n, x_{n+1}], \quad n \in \mathbf{N}_0 \\ f_{-n}^{-1}(x) & x \in [x_{-n}, x_{-n+1}], \quad n \in \mathbf{N}_0 \end{cases}$$

is continuous and satisfies Eq. (1.2). The case $x_1 < 0$ can be discussed analogously. \square

Case 4: $r_1 < r_2 < -1$.

Theorem 4. Suppose that $r_1 < r_2 < -1$.

(i) If $f: \mathbf{R} \rightarrow \mathbf{R}$ is a continuous solution of Eq. (1.2), then f has a unique fixed point 0 and f , strictly increasing, satisfy the "two-side" Lipschitzian condition

$$r_1 \leq \frac{f(x) - f(x')}{x - x'} \leq r_2, \quad x, x' \in \mathbf{R}, \quad x \neq x'.$$

(ii) Moreover, Eq. (1.2) has a continuous solution depending on an arbitrary function, that is, it can be given by the formula

$$(4.33) \quad f(x) = \begin{cases} -p(x) & x \geq 0 \\ p(-x) & x < 0 \end{cases}$$

where $p: [0, \infty) \rightarrow [0, \infty)$ is an arbitrary solution of the functional equation

$$p^2(x) = ((-r_1) + (-r_2))p(x) - (-r_1)(-r_2)x, \quad x \in [0, \infty).$$

Here p has been constructed in Theorem 1 for Case 1.

Proof. By Lemma 5(iv) the solution f is strictly increasing and satisfies the "two-side" Lipschitzian condition. Thus

$$\frac{f(x) - f(x_0)}{x - x_0} \leq r_2, \quad x, x_0 \in \mathbf{R}, \quad x \neq x_0.$$

If $f(x_0) > x_0$ (resp. $f(x_0) < x_0$) for some $x_0 \in \mathbf{R}$, then

$$f(x) \leq f(x_0) + r_2(x - x_0) \rightarrow -\infty \quad \text{as } x \rightarrow +\infty,$$

(resp.

$$f(x) \geq f(x_0) + r_2(x - x_0) \rightarrow +\infty \quad \text{as } x \rightarrow -\infty),$$

that is, there must be $x_1 > x_0$ (resp. $x_1 < x_0$), such that

$$f(x_1) < x_0 < x_1,$$

(resp.

$$f(x_1) > x_0 > x_1, \quad),$$

By the continuity f must have a fixed point. by Lemma 6, $f(0) = 0$ and $f(x) \neq x$ for $x \neq 0$.

In order to prove (ii) it suffices to check that f defined by (4.33) satisfies Eq. (1.2). For $x \geq 0$ we have

$$\begin{aligned} f(f(x)) &= f(-p(x)) = p(-(-p(x))) = p^2(x) \\ &= ((-r_1) + (-r_2))p(x) - (-r_1)(-r_2)x \\ &= (r_1 + r_2)(-p(x)) - r_1 r_2 x = (r_1 + r_2)f(x) - r_1 r_2 x. \end{aligned}$$

Similarly for $x < 0$. \square

Case 5: $-1 < r_1 < r_2 < 0$.

This case can be reduced to the Case 4 by considering the equivalent equation (3.10).

Case 6: $r_1 < -1 < r_2 < 0$.

Theorem 5 . Suppose that $r_1 < -1 < r_2 < 0$. Then every continuous solution of Eq.(1.2) is strictly decreasing and 0 is its unique fixed point. Moreover,

$$r_1 \leq \frac{f(x) - f(x')}{x - x'} \leq r_2, \quad x, x' \in \mathbf{R}, \quad x \neq x'.$$

The proof is given by Lemma 5(iv) and Lemma 4 as for Theorem 3(i).

4.2 Noncritical Cases as $r_1 r_2 < 0$

Case 7: $r_1 < 0, r_1 \neq -1, r_2 > 0, r_2 \neq 1, r_2 \neq -r_1$.

In this case the possible continuous solutions of Eq.(1.2) are its characteristic solutions.

Theorem 6 . Suppose that $r_1 < 0, r_1 \neq -1, r_2 > 0, r_2 \neq 1$ and $r_2 \neq -r_1$. If $f : \mathbf{R} \rightarrow \mathbf{R}$ is a continuous solution of Eq.(1.2), then $f(x) = r_1 x$ or $f(x) = r_2 x$ for $x \in \mathbf{R}$.

Proof. It is discussed in the following different cases.

(i) Case $-1 < r_1 < 0, 0 < r_2 < 1$ and $|r_1| < r_2$.

In view of Lemma 5(ii) every decreasing solution is of the form $f(x) = r_1 x + c$. Substituting this function in Eq.(1.2) we can check easily that $c = 0$. Thus $f(x) = r_1 x$ is the unique decreasing solution. On the other hand, we consider its increasing solutions. For indirect proof we assume that there exists a continuous increasing solution f of Eq.(1.2) different from the characteristic solution $x \mapsto r_2 x$. By Lemma 6 the function f has no any other fixed points than 0. Since $|r_1| < 1$ and $|r_2| < 1$, in this case by Lemma 4, $f^n(x)$ approaches 0 for $x \in \mathbf{R}$ as $n \rightarrow \infty$. The monotonicity implies $f(0) = 0$. Hence by Lemma 3(ii) we obtain

$$(4.34) \quad r_2 x < f(x) < x, \quad x > 0$$

and

$$(4.35) \quad x < f(x) < r_2 x, \quad x < 0$$

Note the reason why the inequalities (4.34) and (4.35) are strict is that $f(x) \neq r_2 x$ for every $x \neq 0$. In fact, if $f(x_0) = r_2 x_0$ for some $x_0 > 0$, the inequality of Lemma 5(ii) implies $f(x) \leq r_2 x$ for all $x \in [0, x_0]$, and then we have from (4.34)-(4.35) that $f(x) = r_2 x$ for $x \in [0, x_0]$, so $f(x) = r_2 x$ for $x \geq 0$ by the continuously extension and increasing iteration of Eq.(1.2). Similarly for $x_0 < 0$. Therefore, for every $x > 0$, the sequence $\{f^n(x)\}$ should be strictly decreasing and

$$r_2^n x < f^n(x) < x,$$

and $\{f^{-n}(x)\}$ should be strictly increasing and

$$x < f^{-n}(x) < r_2^{-n} x.$$

Take $x_0 > 0$ and put $x_n = f^n(x_0)$, $n \in \mathbb{Z}$. Since $\{x_{-n+1}\}_{n \in \mathbb{N}_0}$ satisfies (3.16), by (3.18) and the monotonicity of $\{f^{-n}(x)\}$ we have $x_{-n+1} < x_{-n}$, i.e.,

$$\begin{aligned} B_2(-n)(x_0 - r_1^{-1}x_1) &= B_1(-n)(x_0 - r_2^{-1}x_1) \\ &< B_2(-n-1)(x_0 - r_1^{-1}x_1) = B_1(-n-1)(x_0 - r_2^{-1}x_1). \end{aligned}$$

Multiplying both sides by the negative constant $(r_2^{-1} - r_1^{-1})r_1 r_2$, we get $r_1^{-n}G > r_2^{-n}H$, i.e.,

$$(r_1^{-1}r_2)^n G > H, \quad n \in \mathbb{N}_0,$$

where

$$G = r_2 x_0 - x_1 - r_1 r_2 x_0 + r_1 x_1, \quad H = r_1 x_0 - x_1 - r_1 r_2 x_0 + r_2 x_1.$$

If $G > 0$ then

$$-\infty = \lim_{k \rightarrow \infty} (r_1^{-1}r_2)^{2k+1} G \geq H,$$

which implies a contradiction. If $G < 0$ then

$$G \geq \lim_{k \rightarrow \infty} (r_1 r_2^{-1})^{2k} H = 0,$$

which is also a contradiction. Consequently, $G = 0$, i.e., $f(x_0) = x_1 = r_2 x_0$. This conflicts with (4.34). The proofs are analogous for $x_0 < 0$ and $x < 0$. This completes the proof in the case (i).

(ii) Case $r_1 < -1, r_2 > 1$ and $|r_1| > r_2$.

This case can be reduced to the previous one considering the dual Eq.(3.10) of Eq.(1.2) for f^{-1} .

(iii) Case $-1 < r_1 < 0, 0 < r_2 < 1$ and $|r_1| > r_2$.

In a similar way to the proof of (i), we see by Lemma 3(iii), where r_1 and r_2 are interchanged, that $f(x) = r_1 x$ is the unique decreasing solution (the constant c in the formula of Lemma 3(iii) must equal 0). On the other hand, we consider its increasing solutions. For indirect proof we assume that there is a continuous increasing solution f of Eq. (1.2) different from the characteristic solution $x \mapsto r_2 x$. By Lemma 6 the function f has no any other fixed points than 0. Since $|r_1| < 1$ and $|r_2| < 1$, in this case by Lemma 4,

$f^n(x)$ approaches 0 for $x \in \mathbf{R}$ as $n \rightarrow \infty$. The monotonicity implies $f(0) = 0$. Hence by Lemma 5(iii) and the same arguments as in the case (i), we obtain the strict inequalities

$$(4.36) \quad 0 < f(x) < r_2 x < x, \quad x > 0$$

and

$$x < r_2 x < f(x) < 0, \quad x < 0.$$

Therefore, for every $x > 0$, the sequence $\{f^n(x)\}$ should be strictly decreasing, that is, for arbitrary $x_0 > 0$, the sequence $\{x_n : x_n = f^n(x_0)\}$, $n \in \mathbf{N}_0$, satisfies $x_{n+1} < x_n$. By (3.17) we have $r_1^n G < r_2^n H$, i.e.,

$$(r_1^{-1} r_2)^n H > G, \quad n \in \mathbf{N}_0,$$

where

$$G = x_1 - r_2 x_0 - r_1 x_1 + r_1 r_2 x_0, \quad H = x_1 - r_1 x_0 - r_2 x_1 + r_1 r_2 x_0.$$

If $G > 0$ then

$$0 = \lim_{k \rightarrow \infty} (r_1^{-1} r_2)^{2k} H \geq G,$$

which implies a contradiction. If $G < 0$ then

$$0 = \lim_{k \rightarrow \infty} (r_1^{-1} r_2)^{2k+1} H \leq G,$$

which is also a contradiction. Consequently, $G = 0$, i.e., $f(x_0) = x_1 = r_2 x_0$. This conflicts with (4.36). The proofs are analogous for $x_0 < 0$ and $x < 0$. This completes the proof in the case (iii).

(iv) Case $r_1 < -1$, $r_2 > 1$ and $|r_1| < r_2$.

This case can be reduced to the case (iii) by considering the dual Eq. (3.10) of Eq. (1.2) for f^{-1} .

(v) Case $r_1 < -1$ and $0 < r_2 < 1$.

Similarly to the cases (i) and (ii) we apply Lemma 3(iii) interchanging r_1 and r_2 . Obviously, $f(x) = r_1 x$ is the unique decreasing solution (the constant c in the

formula of Lemma 5(iii) must be equal 0). On the other hand, suppose there is a continuous increasing solution f . By Lemma 5(iii)

$$0 \leq \frac{f(x_2) - f(x_1)}{x_2 - x_1} \leq r_2, \quad x_1, x_2 \in \mathbf{R}, \quad x_1 \neq x_2.$$

Since $0 < r_2 < 1$ we see by the contraction principle that f has a unique fixed point. By lemma 6 the function f has no any other fixed points than 0. Thus $f(0) = 0$, $f(x) \leq r_2 x < x$ for $x > 0$ and $f(x) \geq r_2 x$ for $x < 0$. The monotonicity implies that $\{f^n(x)\}$ tends to 0 as $n \rightarrow \infty$. By (3.11) in Lemma 4 we see that the inequalities $r_1 < -1$ and $0 < r_2 < 1$ imply that $f(x) = r_2 x$ for all $x \in \mathbf{R}$. This completes the proof in the case (v).

(vi) Case $-1 < r_1 < 0$ and $r_2 > 1$.

This case follows immediately from the case (iv) and the dual equation (3.10) of Eq.(1.2) for f^{-1} .

This completes the proof of Theorem 6. \square

4.3 Special Cases: $|r_1| = |r_2|$

Case 8: $r_1 = r_2 = r \neq 0$.

Theorem 7 . Suppose that $r_1 = r_2 = r$, $r \neq 0$ and that $f : \mathbf{R} \rightarrow \mathbf{R}$ is a continuous solution of Eq.(1.2).

(i) If $r \neq 1$ then $f(x) = rx$, $x \in \mathbf{R}$.

(ii) If $r = 1$ then $f(x) = x + c$, $x \in \mathbf{R}$ for some $c \in \mathbf{R}$.

Proof. By Lemma 4(ii) we have

$$(4.37) \quad u(x) := \lim_{n \rightarrow \infty} \frac{f^n(x)}{nr^{n-1}} = f(x) - rx, \quad x \in \mathbf{R},$$

$$(4.38) \quad v(x) := \lim_{n \rightarrow \infty} \frac{r^{n-1}f^{-n}(x)}{n} = f^{-1}(x) - r^{-1}x, \quad x \in \mathbf{R}.$$

For $r > 1$ the function f must be increasing. In fact, if f is decreasing, then, by Lemma 3, the monotonicity is strict and f^2 is increasing. Putting n even in (4.37) and (4.38) we see that both u and v are nondecreasing, but the function $x \mapsto u(x) = f(x) - rx$ is clearly decreasing. Since f is increasing, (4.37) and (4.38) imply that u , v , and $v \circ f$ are nondecreasing, that is, for $x_1 < x_2$,

$$f(x_1) - rx_1 \leq f(x_2) - rx_2, \quad \text{and} \quad r^{-1}(x_1 - f(x_1)) \leq r^{-1}(x_2 - f(x_2)).$$

Hence $f(x_2) - f(x_1) = r(x_2 - x_1)$ for all $x_1, x_2 \in \mathbf{R}$. Consequently, taking arbitrary $x_1 = x$, and a fixed x_2 gives

$$(4.39) \quad f(x) = rx + c, \quad x \in \mathbf{R},$$

where $c := f(x_2) - rx_2$ is a constant. On the other hand, for $r < 0$, the function f must be decreasing. In fact, if f is increasing, the function $x \mapsto u(x) = f(x) - rx$ is clearly increasing, but putting n even in (4.37) and (4.38) we see that both u and v are nonincreasing. Since f is decreasing, putting n odd in (4.37) and (4.38) we see that both u and v are nonincreasing and $v \circ f$ is nondecreasing, i.e., for $x_1 < x_2$,

$$f(x_1) - rx_1 \geq f(x_2) - rx_2, \quad \text{and} \quad r^{-1}(x_1 - f(x_1)) \leq r^{-1}(x_2 - f(x_2)).$$

Because $r < 0$, we have $f(x_2) - f(x_1) = r(x_2 - x_1)$ for all $x_1, x_2 \in \mathbf{R}$, and consequently, f is of the form (4.39). Substituting $f(x) = rx + c$ in (3.9), the equivalent form of (1.2), one gets $c(r - 1) = 0$. Thus $c = 0$, and the proof is completed. \square

Remark 6. Consider the equation

$$g\left(\frac{2x - g(x)}{m}\right) = mx, \quad g \in C^0(\mathbf{R}, \mathbf{R}),$$

proposed by I.C. Bivens [1]. Setting $h(x) = g(x)/m$ we see that $h(2x - h(x)) = x$, and evidently, $h: \mathbf{R} \rightarrow \mathbf{R}$ is one-to-one and onto. Hence, for $f = h^{-1}$ we get

$$f^2(x) = 2f(x) - x, \quad x \in \mathbf{R},$$

i.e., a special case of Eq.(1.2). Since the characteristic roots are $r_1 = r_2 = 1$, all continuous solutions of this equation are of the form $f(x) = x + c$, $x \in \mathbf{R}$, for some $c \in \mathbf{R}$.

Case 9: $r_1 = -r$, $r_2 = r > 0$.

Now Eq.(1.2) is equivalent to

$$(4.40) \quad f^2(x) = r^2x, \quad x \in \mathbf{R},$$

i.e., Eq.(3.9) reduces to a problem on iterative roots which has been considered by M.Kuczma[4]. His Theorem 15.7 and 15.9 in Chapter XV of [5] show that Eq.(4.40) not only has continuous increasing solutions but also has continuous decreasing solutions, all of which depend on arbitrarily given function. In particular, when $r = 1$, his Theorem 15.2 in Chapter XV of [5] indicates that Eq.(4.40) has a decreasing solution, a so-called *involutory function* depending on an arbitrary function, but $f(x) = x$, $x \in \mathbf{R}$, is its unique increasing solution.

4.4 Critical Cases

Case 10: $r_2 = 1, r_1 > 0, r_1 \neq 1$.

Theorem 8 . Suppose that $r_2 = 1, r_1 > 0, r_1 \neq 1$, and that $f : \mathbf{R} \rightarrow \mathbf{R}$ is a continuous solution of Eq. (1.2). Then f has one of the following forms

$$\begin{aligned} f(x) &= x, & x \in \mathbf{R}, \\ f(x) &= \begin{cases} x, & x \leq a, \\ r_1 x + (1 - r_1)a, & x > a, \end{cases} \\ f(x) &= \begin{cases} r_1 x + (1 - r_1)a, & x \leq a, \\ x, & x > a, \end{cases} \\ f(x) &= \begin{cases} r_1 x + (1 - r_1)a, & x \leq a, \\ x, & a \leq x \leq b, \\ r_1 x + (1 - r_1)b, & x \geq b \end{cases} \end{aligned}$$

where $a, b \in \mathbf{R}$, $a < b$. Moreover, all these functions are continuous solutions of Eq. (1.2).

Proof. Consider first the case $0 < r_1 < 1$. Let $F := \{x \in \mathbf{R} : f(x) = x\}$, the set of all fixed points of f . The set F is a closed interval (or consists of only one point). In fact, F is clearly closed. If there are two points $a, b \in F, a < b$, such that $f(x) \neq x$ for all $x \in [a, b]$, then $f(x) > x$ for all $x \in (a, b)$ or $f(x) < x$ for all $x \in (a, b)$, i.e.,

$$\frac{f(x) - f(a)}{x - a} > \frac{x - a}{x - a} = 1, \quad x \in (a, b),$$

or

$$\frac{f(b) - f(x)}{b - x} < \frac{b - x}{b - x} = 1, \quad x \in (a, b).$$

This contradicts to the result of Lemma 5(i). Thus F must be a close interval. If $F = \mathbf{R}$ then $f(x) = x$ for all $x \in \mathbf{R}$. If $F = (-\infty, a]$, then Lemma 1, Lemma 5(i), and the fact that $f(F) = F$ imply that f is strictly increasing from (a, ∞) onto itself. By the inequality in Lemma 5(i) we have $a < f(x) < x$ for $x \in (a, \infty)$. Hence $f^n(x) \rightarrow a$ as $n \rightarrow \infty$. It follows from (3.11) in Lemma 4 that $f(x) = r_1 x + (1 - r_1)a$ for $x > a$. Similar discussions for $F = [a, \infty)$ and $F = [a, b]$ give the desired solutions. The case $r_1 > 1$ can be reduced to the previous one by considering the equivalent equation (3.10). \square

Case 11: $r_2 = 1, r_1 < 0, r_1 \neq -1$.

Theorem 9 . Suppose that $r_2 = 1, r_1 < 0, r_1 \neq -1$, and that $f : \mathbf{R} \rightarrow \mathbf{R}$ is a continuous solution of Eq. (1.2). Then $f(x) = x$, for all $x \in \mathbf{R}$, or $f(x) = r_1 x + c$, for all $x \in \mathbf{R}$, where c is a constant in \mathbf{R} .

Proof. Consider the case $-1 < r_1 < 0$. By (3.11) in Lemma 4,

$$g(x) := \lim_{n \rightarrow \infty} f^n(x) = (r_2 - r_1)^{-1}(f(x) - r_1 x).$$

If f is increasing then g must be strictly increasing continuous, and from \mathbf{R} onto \mathbf{R} . Thus

$$f(g(x)) = f(\lim_{n \rightarrow \infty} f^n(x)) = \lim_{n \rightarrow \infty} f^{n+1}(x) = g(x), \quad x \in \mathbf{R}$$

This means that $f(x) = x$ for all $x \in \mathbf{R}$. On the other hand, if f is decreasing then Lemma 5(ii) implies that $f(x) = r_1 x + c$, $x \in \mathbf{R}$, for some $c \in \mathbf{R}$.

The case $r_1 < -1$ can be reduced to the previous one by considering the equivalent Eq.(3.10). \square

Case 12: $r_1 = -1, r_2 > 0, r_2 \neq 1$.

Theorem 10 . Suppose that $r_1 = -1, r_2 > 0, r_2 \neq 1$, and that $f : \mathbf{R} \rightarrow \mathbf{R}$ is continuous solution of Eq.(1.2). Then $f(x) = -x$, for all $x \in \mathbf{R}$, or $f(x) = r_2$ for all $x \in \mathbf{R}$.

Proof. Consider the case $0 < r_2 < 1$. By Lemma 3(iii), replacing the role of r_1 and r_2 , we can get the unique decreasing solution $f(x) = -x$, $x \in \mathbf{R}$. If f is increasing by Lemma 5(iii)

$$0 \leq \frac{f(x_2) - f(x_1)}{x_2 - x_1} \leq r_2 = 1, \quad x_1, x_2 \in \mathbf{R}, \quad x_1 \neq x_2.$$

Thus $0 < r_2 < 1$ implies that f , as a contraction, has a unique fixed point, which in view of Lemma 6, must be 0. Naturally, we also have $f(x) < x$ (resp. $f(x) > x$) for $x > 0$ (resp. $x < 0$), so $f^n(x)$ tends to 0 for $x \in \mathbf{R}$ as $n \rightarrow \infty$. On the other hand, by Lemma 4 we have

$$f^n(x) = \frac{r_2^n}{r_2 + 1}(f(x) + x) - \frac{(-1)^n}{r_2 + 1}(f(x) - r_2 x), \quad x \in \mathbf{R}.$$

Therefore, $f(x) = r_2 x$ for all $x \in \mathbf{R}$. Furthermore, the case $r_2 > 1$ can be reduced to the previous one by considering the equivalent Eq.(3.10). \square

Case 13: $r_1 = -1, r_2 < 0, r_2 \neq -1$.

Theorem 11 . Suppose that $r_1 = -1, r_2 < 0, r_2 \neq -1$, and that $f : \mathbf{R} \rightarrow \mathbf{R}$ is a continuous solution of Eq.(1.2). Then $f(x) = -x$, for all $x \in \mathbf{R}$, or $f(x) = r_2 x$ for all $x \notin [a, b]$, for some constants a and b such that $-\infty \leq a \leq 0$ and $0 \leq b \leq +\infty$.

Remark 7. From the proof of Theorem 4 we see that if $r_1 < 0, r_2 < 0, r_1 \neq r_2$, then $f(0) = 0$, and $f(x) \neq x$ for all $x \neq 0$.

Proof. Consider first the case $-1 < r_1 < 0$. By Lemma 5(iv)

$$(4.41) \quad -1 \leq \frac{f(x_2) - f(x_1)}{x_2 - x_1 - 1} \leq r_2 < 0, \quad x_1, x_2 \in \mathbf{R}, \quad x_1 \neq x_2.$$

Clearly f is strictly decreasing. Now we claim that $F^* := \{x \in \mathbf{R} : f(x) = -x\}$ is a connected closed interval containing 0. In fact, F^* is closed and, from Remark 5, $0 \in F^*$. Assume that there are two points a and b in F^* , $a < b$, such that $f(x) > -x$ (resp. $f(x) < -x$) for all x in (a, b) . Then

$$(4.42) \quad \frac{f(b) - f(x)}{b - x} < \frac{-b - (-x)}{b - x} = 1, \quad x \in (a, b),$$

resp.

$$(4.43) \quad \frac{f(x) - f(a)}{x - a} > \frac{-x - (-a)}{x - a} = 1, \quad x \in (a, b).$$

Obviously both (4.42) and (4.43) contradict (4.41), i.e., the claim is proved. Without loss of generality, we let $F^* = [a, b]$ for some constants $-\infty \leq a \leq 0$ and $0 \leq b \leq +\infty$. For $x \in [a, b]$, $f(x) = -x$; For $x \notin [a, b]$, $f(x) = r_2 x$. In fact, if $f(x) \neq r_2 x$ for a certain $x \notin [a, b]$, then from (4.41)

$$-x < f(x) < r_2 x < 0, \quad x > 0,$$

or

$$0 < r_2 x < f(x) < -x, \quad x > 0,$$

Since f is strictly decreasing,

$$0 < r_2^2 x < f(r_2 x) < f^2(x) < f(-x) < x, \quad x > 0,$$

or

$$0 > r_2^2 x > f(r_2 x) > f^2(x) > f(-x) > x, \quad x < 0.$$

The increasing monotonicity of f^2 implies that $f^{2k}(x)$ tends monotonically to 0. By (3.11) in Lemma 4 we see that $f(x) = r_2 x$. This contradicts to the choice of x . Furthermore, the case $r_2 < -1$ can be reduced to the previous one by considering the equivalent equation (3.10). \square .

4.5 No Real Roots

To make this paper selfcontained we shall prove the following.

Theorem 12 (cf. Nabeya [8]). Eq.(1.2) has no continuous solutions on \mathbf{R} if it has no real characteristic roots.

Proof. For reduction to absurdity we assume that Eq.(1.2) has a continuous solution $f : \mathbf{R} \rightarrow \mathbf{R}$. By Lemma 3 the function f is monotone, onto, and consequently, f^2 is strictly increasing. Let the complex characteristic roots of Eq.(1.2) be denoted by

$$r_1 = a - ib = S \exp(-i\theta), \quad r_2 = a + ib = S \exp(i\theta),$$

where $a, b \in \mathbf{R}$, $b > 0$, $S > 0$, and $\theta \in (0, \pi)$. By Lemma 6, $f(x) \neq x$ for every $x \neq 0$. Obviously the sign of the sequence $\{f^{n+1}(x) - f^n(x)\}$ is the same (resp. alternate between -1 and 1) for arbitrary fixed $x \neq 0$ when f is strictly increasing (resp. decreasing). However, from (3.11) in Lemma 4 we have

$$\begin{aligned} f^n(x) &= \frac{r_2^n}{r_2 - r_1}(f(x) - r_1 x) + \frac{r_1^n}{r_2 - r_1}(r_2 x - f(x)) \\ &= b^{-1} S^n (\sin(n\theta)) f(x) - b^{-1} S^{n+1} (\sin(n-1)\theta) x. \end{aligned}$$

Then

$$f^{n+1}(x) - f^n(x) = r_2^n U(x) + r_1^n V(x),$$

where

$$U(x) := \frac{r_2 - 1}{r_2 - r_1}(f(x) - r_1 x), \quad V(x) := \frac{r_2 - 1}{r_2 - r_1}(r_2 x - f(x)).$$

It is not difficult to check that $\overline{U(x)} = V(x)$, so for a fixed $x \neq 0$ we can let

$$U(x) = T \exp(it), \quad V(x) = T \exp(-it),$$

where $T \geq 0$ and $t \in [0, 2\pi)$. Thus

$$f^{n+1}(x) - f^n(x) = S^n T (\exp(i(n\theta + t)) + \exp(-i(n\theta + t))) = 2S^n T \cos(n\theta + t).$$

When $T > 0$, this formula gives a contradiction with the property of sign of the sequence $\{f^{n+1}(x) - f^n(x)\}$ stated above; when $T = 0$ we see that $U(x) = V(x) = 0$, that is, $f(x) = r_1 x = r_2 x$ for all $x \neq 0$ i.e., $r_1 = r_2$, which is a contradiction. This completes the proof. \square

4.6 Final Remarks

The results of S Nabeya [8] concerning Eq.(1.2) are similar (or even the same) but our paper has its own distinguishing feature in the following:

1. Our paper deals with all cases of r_1 and r_2 , the characteristic values, especially with the cases where $r_1 = -1$ and $r_2 > 0$ in subsection 4.4 case 12, and $r_1 = -r, r_2 = r$ in subsection 4.3, case 9.

2. In methodology, Nabeya [8] often discusses the sign of $f^{n+1}(x) - f^n(x)$ as a sequence of n , but we use the sequence $\{x_n\}$ defined in (3.15) and (3.16) by a difference equation to construct inductively the solutions, which can be seen explicitly in the the proofs of the theorems.

3. In Nabeya's paper [8] the characteristic values r_1 and r_2 are supposed to be that of difference equation

$$\begin{cases} a_{n+1} = a_n & +ac_n \\ b_{n+1} = a_n & +bc_n \\ c_{n+1} = b_n & +cc_n \end{cases}$$

which is set up by the relation that $f^{n+1}(x) = f(f^n(x))$, but in our paper, in the light of Euler's idea to consider formally the solution of exponential function for ordinary differential equations, we deduce the characteristic equation by assuming formally that $f(x) = rx$, $x \in \mathbf{R}$, is a solution of the iterative equation.

4. Some statements of our results are different and more concrete, e.g. in Theorem 3, Theorem 8 and Theorem 11.

5 Some Consequences for General Equation of Order n

As an obvious consequence of the previous section we obtain the following

Corollary. *Let $a_k \in \mathbf{R}$, $k = 1, \dots, n$, $a_0 \neq 0$. Suppose the polynomial*

$$r^{n+1} - a_n r^n - a_{n-1} r^{n-1} - \dots - a_0$$

has two roots $r_1, r_2 \in \mathbf{R}$ such either $1 < r_1 < r_2$, or $0 < r_1 < r_2 < 1$, or $0 < r_1 < 1 < r_2$, or $r_1 < r_2 < -1$, or $-1 < r_1 < 1 < r_2 < 0$, then the continuous solution of Eq.(1.1) depends on arbitrary function. Moreover every continuous solution is a homeomorphism of \mathbf{R} .

It is not difficult to prove

Theorem 13 . *Let $a_k \geq 0$, $k = 1, \dots, n$, $a_0 \neq 0$, be such that $a_0 + a_1 + \dots + a_n = 1$. If $f : \mathbf{R} \rightarrow \mathbf{R}$ is a continuous solution of equation (1.1) then $f(x) = x$ for all $x \geq 0$.*

An interesting result has been recently proved by W.Jarczyk[3].

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