Aequationes Math. 58 (1999) 46–59
0001-9054/99/020046-14 $ 1.50+0.20/0

Aequationes Mathematicae

Mean value property and associated functional equations

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Dedicated to Professor János Aczél on the occasion of his 75^{th} birthday

Summary. The main result says that, under some general conditions, if a function composed with a difference quotient is a difference quotient, then it must be continuous and affine. We show that it is an effective tool in solving some functional equations associated with a mean value property.

Mathematics Subject Classification (1991). 39B22.

Keywords. Functional equation involving difference quotient, mean value property, generalized logarithmic means, Stolarsky means.

Introduction

Let $I \subset \mathbb{R}$ be an open interval. A function $M : I \times I \to I$ is a *mean* on I if for all $x,\, y \in I$,

$$\min\{x,y\} \leq M(x,y) \leq \max\{x,y\}.$$

A mean M on I is *strict* if for all $x,\, y \in I$, $x \neq y$,

$$\min\{x,y\} < M(x,y) < \max\{x,y\}.$$

The classical mean value theorem associates to every differentiable function $g : I \to \mathbb{R}$ a strict mean M_g on I such that

$$\frac{g(x)-g(y)}{x-y} = g'\left(M_g(x,y)\right), \qquad x \neq y,\;\; x,\, y \in I. \tag{1}$$

If $h := g'$ is invertible, we can write the mean M_g in the form

$$M_g(x,y) = h^{-1}\Big(\frac{g(x)-g(y)}{x-y}\Big), \qquad x \neq y,\;\; x,\, y \in I.$$

In this context the problem of characterizations of means which can be written in this form arises. In a natural way it leads to the equation

$$h^{-1}\Big(\frac{g(x)-g(y)}{x-y}\Big) = H^{-1}\Big(\frac{f(x)-f(y)}{x-y}\Big), \qquad x \neq y, \ x, y \in I,$$

(the uniqueness of the representation) which reduces to the functional equation

$$\frac{g(x)-g(y)}{x-y} = \phi\Big(\frac{f(x)-f(y)}{x-y}\Big), \qquad x \neq y, \ x, y \in I,$$

with three unknown functions f, g and ϕ. Assuming that f is strictly convex or strictly concave, we prove that $\phi(u) = au + b$ and $g(x) = af(x) + bx + c$, for some a, b, $c \in \mathbb{R}$ (Theorem 1). This is the basic result of the paper. It turns out to be a convenient and effective tool in solving the functional equations of the form

$$\frac{g(x)-g(y)}{x-y} = h\left(M(x,y)\right), \qquad x \neq y, \qquad x, y \in I, \tag{2}$$

where M is a strict mean.

In Section 2 we apply Theorem 1 to the functional equation (1) where $M = E_p : (0,\infty)^2 \to (0,\infty)$, $p \in \mathbb{R}$, is the generalized logarithmic mean (cf. Bullen, Mitrinović, Vasić [4], p. 346):

$$E_p(x,y) = \begin{cases} \left(\frac{x^p - y^p}{p(x-y)}\right)^{1/(p-1)}, & 0 \neq p \neq 1 \\ E_0(x,y), & p = 0 \\ E_1(x,y), & p = 1 \end{cases}, \tag{3}$$

for all x, $y > 0$, $x \neq y$, and $E_p(x,x) := x$, for all $p \in \mathbb{R}$, and $x > 0$, where

$$E_0(x,y) := \frac{x-y}{\log x - \log y}, \qquad E_1(x,y) := e^{-1}\left(y^y x^{-x}\right)^{1/(y-x)},$$

to get some characterizations of power and logarithmic functions. From the main results of Section 2 we obtain the following corollary:

In all cases E_p, $p \neq q$, $pq \neq 0$, is the mean associated, by (1), with the function $g(x) = x^p$; and the main result of Section 2 shows that this association is, up to a constant factor and a linear function, unique; more generally if (2) is satisfied with $M = E_p$ then $g(x) = ax^p + bx + c$, $h(x) = g'(x)$. Analogous results are shown for the cases $p = 0, 1$.

The case $p = 2k$, $k \in \mathbb{N}$, and g, $h : \mathbb{R} \to \mathbb{R}$, treated independently, gives, as a special case, the result of J. Aczél [1].

Note that the function h is the derivative of the corresponding function g. It is, under some regularity assumptions, an immediate consequence of the functional equation (2).

In Section 3 we apply Theorem 1 to describe the solutions g and h of the more general functional equation

$$\frac{g(x) - g(y)}{x^q - y^q} = h\big(E_{q,p}(x, y)\big), \qquad x \neq y, \tag{4}$$

where $E_{q,p}$ is the two parameter family of Stolarsky means defined below in (7), related to the Cauchy mean value theorem. As above $E_{q,p}$, $p \neq q$, $pq \neq 0$, is associated, by (4), with the functions $g(x) = x^p$, $h(x) = \frac{g'(x)}{(x^q)'}$; again, this association is shown to be unique in the sense described above. The other cases of p, q have analogous results.

In Section 4 we apply Theorem 1 to some special functional equations of the form (2).

Functional equation (2) in which M is the arithmetic mean was considered by J. Aczél in [1], by J. Aczél and M. Kuczma in [2], [3] for quasi-arithmetic, geometric and harmonic means, and for the quasi-arithmetic mean by M. Kuczma in [9], by different methods.

For a fairly broad view of the means considered here, together with many references and some indication of directions for their future study, see A. Horwitz [7].

1. When a composition of a function and a difference quotient is a difference quotient

Let $I \subset \mathbb{R}$ be an interval and $f : I \to \mathbb{R}$ a function. Put

$$\Delta := \{(x, x) : x \in I\},$$

and denote by J_f the range of the function of two variables which is the difference quotient of f:

$$(I \times I \setminus \Delta) \ni (x, y) \longrightarrow \frac{f(x) - f(y)}{x - y}.$$

A key role is played by the following

Lemma. *Let $I \subset \mathbb{R}$ be an open interval and $f : I \to \mathbb{R}$ strictly convex or strictly concave on I. Then $\phi : J_f \to \mathbb{R}$ satisfies the functional equation*

$$\phi\left(\frac{f(x) - f(y)}{x - y}\right) = \frac{x - z}{x - y}\phi\left(\frac{f(x) - f(z)}{x - z}\right) + \frac{z - y}{x - y}\phi\left(\frac{f(z) - f(y)}{z - y}\right), \tag{5}$$

for all x, y, $z \in I$, $x \neq y \neq z \neq x$, if, and only if, for some a, $b \in \mathbb{R}$,

$$\phi(u) = au + b, \qquad u \in J_f.$$

Proof. Suppose first that f is strictly convex. Then J_f is an open interval and for every $u_0 \in J_f$ there exist $\delta = \delta(u_0) > 0$, $\alpha = \alpha(u_0)$, $\beta = \beta(u_0)$, $\alpha, \beta \in [0,1]$, $\alpha < \beta$, such that for every $u, v \in (u_0 - \delta, u_0 + \delta) \cap J_f$ and $t \in (\alpha, \beta)$, the system of equations

$$\frac{f(x) - f(z)}{x - z} = u, \qquad \frac{f(z) - f(y)}{z - y} = v, \qquad \frac{x - z}{x - y} = t,$$

has exactly one solution $x = x(u,v,t),\ y = y(u,v,t),\ z = z(u,v,t) \in I$.

Let us fix arbitrary $u_0 \in J_f$. Since

$$\frac{f(x) - f(y)}{x - y} = tu + (1-t)v,$$

equation (5) implies that

$$\phi(tu + (1-t)v) = t\phi(u) + (1-t)\phi(v), \qquad u, v \in (u_0 - \delta, u_0 + \delta) \cap J_f, \quad t \in (\alpha, \beta).$$

hence applying an argument of Daróczy and Páles [5], which is based on the identity

$$\frac{u+v}{2} = t\left(t\frac{u+v}{2} + (1-t)u\right) + (1-t)\left(tv + (1-t)\frac{u+v}{2}\right)$$

we infer that ϕ is Jensen affine on $(u_0 - \delta, u_0 + \delta) \cap J_f$, i.e.,

$$\phi\left(\frac{u+v}{2}\right) = \frac{\phi(u) + \phi(v)}{2}, \qquad u, v \in (u_0 - \delta, u_0 + \delta) \cap J_f.$$

Moreover, for all u, $v \in (u_0 - \delta, u_0 + \delta) \cap J_f$, $v < u$, we have

$$\phi(tu + (1-t)v) \le \max(\phi(u), \phi(v)), \qquad t \in (\alpha, \beta),$$

which proves that ϕ is bounded above on the interval $((u-v)\alpha + v, (u-v)\beta + v)$. It follows that ϕ is continuous and, consequently, (cf. M. Kuczma [8], p. 316, Theorem 3 and Theorem 2), there are $a = a(u_0)$, $b = b(u_0) \in \mathbb{R}$ such that

$$\phi(u) = au + b, \qquad u \in (u_0 - \delta, u_0 + \delta) \cap J_f.$$

Now it is obvious that a and b do not depend on u_0 and

$$\phi(u) = au + b, \qquad u \in J_f.$$

In the case when f is strictly concave the proof is analogous. □

Remark 1. Assume that the function f in this lemma is differentiable. Then for all x, $y \in I$, $y < x$, and for arbitrary partition $y = t_0 < t_1 < \ldots < t_n = x$, equation (5) implies that

$$(x-y)\phi\left(\frac{f(x)-f(y)}{x-y}\right) = \sum_{k=1}^{n} \phi(f'(s_k))(t_k - t_{k-1})$$

for some $s_k \in (t_{k-1}, t_k)$, $k = 1, \ldots, n$; and consequently,

$$\phi\left(\frac{f(x)-f(y)}{x-y}\right) = \frac{1}{x-y}\int_y^x \phi(f'(t))dt.$$

Remark 2. If the function $f : I \to \mathbb{R}$ has the following property:
"*for every* u, $v \in J_f$, *and every* $t \in (0,1)$ *there exist* x, y, $z \in J$ *such that*

$$\frac{f(x)-f(z)}{x-z} = u, \qquad \frac{f(z)-f(y)}{z-y} = v, \qquad \frac{x-z}{x-y} = t,\text{"}$$

then the proof of the lemma is immediate. In fact, equation (5) implies that for all u, $v \in J_f$, $t \in (0,1)$,

$$\phi(tu + (1-t)v) = t\phi(u) + (1-t)\phi(v)$$

and consequently, ϕ is affine and continuous. Simple geometrical considerations show that the functions $f : (0,\infty) \to \mathbb{R}, f = \log$; and $f : \mathbb{R} \to \mathbb{R}$, $f(x) := x^{2k}$, $k \in \mathbb{N}$, $f := \exp$, have this property.

The above lemma allows us to prove the main result of the paper which reads as follows

Theorem 1. *Let* $I \subset \mathbb{R}$ *be an open interval,* f, $g : I \to \mathbb{R}$, *and let* J_f *stand for the range of the two variable function*

$$(x,y) \to \frac{f(x)-f(y)}{x-y}, \qquad x, y \in I, \ x \neq y.$$

Suppose that f *is strictly convex or strictly concave on* I. *Then* $\phi : J_f \to \mathbb{R}$ *satisfies the functional equation*

$$\frac{g(x)-g(y)}{x-y} = \phi\left(\frac{f(x)-f(y)}{x-y}\right), \qquad x, y \in I, \quad x \neq y, \tag{6}$$

if, and only if, there exist a, b, $c \in \mathbb{R}$ *such that*

$$\phi(u) = au + b, \qquad u \in J_f,$$

and

$$g(x) = af(x) + bx + c, \qquad x \in I.$$

Proof. Suppose that ϕ satisfies equation (6). We can write this equation in the form

$$g(x) = (x - y)\phi\left(\frac{f(x) - f(y)}{x - y}\right) + g(y).$$

Since the left hand side does not depend on y, it follows that

$$g(x) = (x - z)\phi\left(\frac{f(x) - f(z)}{x - z}\right) + g(z), \qquad x,\, z \in I,\ x \neq z,$$

and

$$g(y) = (y - z)\phi\left(\frac{f(y) - f(z)}{y - z}\right) + g(z), \qquad y,\, z \in I,\ y \neq z.$$

Hence, making use of the identity $g(x) - g(z) = [g(x) - g(y)] + [g(y) - g(z)]$, we infer that

$$\begin{aligned}&(x - z)\phi\left(\frac{f(x) - f(z)}{x - z}\right)\\ &= (x - y)\phi\left(\frac{f(x) - f(y)}{x - y}\right) + (y - z)\phi\left(\frac{f(y) - f(z)}{y - z}\right),\end{aligned}$$

for all $x,\, y,\, z \in I$, $x \neq z \neq y \neq x$, i.e. that ϕ satisfies the functional equation (5). Now the Lemma gives $\phi(u) = au + b$ for some $a,\, b \in \mathbb{R}$, and all $u \in J_f$. Setting $\phi(u) = au + b$, $u \in J_f$, in (6) we conclude that $g(x) = af(x) + bx + c$ for all $x \in I$.

Since the converse implication is obvious, the proof is completed. □

2. The mean value property and related functional equations for generalized logarithmic means

Let $I_0 \subset \mathbb{R}$ be an arbitrary interval. If $M : I_0 \times I_0 \to \mathbb{R}$ is a mean on I_0 then, obviously, for every subinterval $I \subset I_0$ we have $M(I \times I) = I$.

We shall consider functional equation (2) in which M is a given mean on I_0 and the functions $g,\, h : I \to \mathbb{R}$, $I \subset I_0$, are unknown. It should be emphasized that we do not assume any regularity conditions of the functions g and h. However let us note the following

Remark 3. Suppose that $M : I_0 \times I_0 \to I_0$ is a mean and $g,\, h : I \to \mathbb{R}$, with $I \subset I_0$, satisfy equation (2). If M is continuous on the diagonal $\Delta := \{(x, x) : x \in I_0\}$, and h is continuous, then g is differentiable and $g' = h$.

In this section we apply Theorem 1 to determine the functions g and h satisfying the functional equation (2) where $I_0 = (0, \infty)$ and $M = E_p$ is a fixed generalized logarithmic mean defined by (3).

The first result of this section reads as follows.

Theorem 2. *Let $I \subset (0, \infty)$ be an open interval, and $p \in \mathbb{R}$, $0 \neq p \neq 1$. Then g, $h : I \to \mathbb{R}$ satisfy the functional equation*

$$\frac{g(x) - g(y)}{x - y} = h\left[\left(\frac{x^p - y^p}{p(x - y)}\right)^{1/(p-1)}\right], \qquad x, y \in I, \ x \neq y,$$

if, and only if, there are a, b, $c \in \mathbb{R}$ such that

$$g(x) = ax^p + bx + c, \qquad h(x) = g'(x), \quad x \in I.$$

Proof. Let I_p be the pre-image of the interval I for the power function $u \to (u/p)^{1/(p-1)}$. Define $\phi : I_p \to \mathbb{R}$ by

$$\phi(u) := h\left[\left(\frac{u}{p}\right)^{1/(p-1)}\right], \qquad u \in I_p,$$

and $f : I \to \mathbb{R}$ by $f(x) = x^p$, $x \in I$. Then the considered functional equation takes the form (6) (here $J_f = I_p$). Since the function f is of the class C^1 and f' is strictly monotonic, in view of Theorem 1 there are a, b, $c \in \mathbb{R}$, such that

$$\phi(u) = au + b, u \in J_p,$$

and

$$g(x) = af(x) + bx + c = ax^p + bx + c, \qquad x \in I.$$

The definition of ϕ gives

$$h(x) = apx^{p-1} + b, \qquad x \in I.$$

As the converse implication is easy to verify, the proof is completed. $\square$

Since E_2 is the arithmetic mean restricted to the set $(0, \infty)^2$, taking $p = 2$, in the above result we obtain the following

Corollary 1. *Let $I \subset (0, \infty)$ be an open interval. Then g, $h : I \to \mathbb{R}$ satisfy the functional equation*

$$\frac{g(x) - g(y)}{x - y} = h\left(\frac{x + y}{2}\right), \qquad x, y \in I, \ x \neq y,$$

if, and only if, there are a, b, c $\in \mathbb{R}$ *such that*

$$g(x) = ax^2 + bx + c, \quad h(x) = g'(x), \qquad x \in I.$$

Let us note that in the papers by J. Aczél [1], and Sh. Haruki [6] the case $I = \mathbb{R}$ was considered. In this connection we refer the reader to Theorem 5 below.

Since E_{-1} coincides with the geometric mean, taking $p = -1$ in Theorem 2 gives

Corollary 2. *Let* $I \subset (0,\infty)$ *be an open interval. Then* $g,\ h : I \to \mathbb{R}$ *satisfy the functional equation*

$$\frac{g(x) - g(y)}{x - y} = h\left(\sqrt{xy}\right), \qquad x, y \in I, \ x \neq y,$$

if, and only if, there are a, b, c $\in \mathbb{R}$ *such that*

$$g(x) = \frac{a}{x} + bx + c, \quad h(x) = g'(x), \qquad x \in I.$$

In Theorem 2 we assume that $0 \neq p \neq 1$, i.e., in the considered functional equation, neither the logarithmic mean E_0 nor the identic mean E_1 is admitted. Now we shall treat these two cases seperately.

Theorem 3. *Let* $I \subset (0,\infty)$ *be an open interval. Then* $g,\ h : I \to \mathbb{R}$ *satisfy the functional equation*

$$\frac{g(x) - g(y)}{x - y} = h\left(\frac{x - y}{\log x - \log y}\right), \qquad x, y \in I, \ x \neq y,$$

if, and only if, there are a, b, c $\in \mathbb{R}$ *such that*

$$g(x) = a \log x + bx + c, \quad h(x) = g'(x), \qquad x \in I.$$

Proof. Let $f := \log$. Then $J_f = \{x^{-1} : x \in I\}$. Define $\phi : J_f \to \mathbb{R}$ by $\phi(u) := h(u^{-1})$, $u \in J_f$. Then the considered functional equation can be written in the form (6). Since log is strictly concave, the result follows from Theorem 1.

Remark 4. Note that Remark 2 supplies us with a shorter proof of Theorem 3.

Theorem 4. *Let* $I \subset (0,\infty)$ *be an open interval. Then* $g, h : I \to \mathbb{R}$ *satisfy the functional equation*

$$\frac{g(x) - g(y)}{x - y} = h\left(e^{-1}\left(y^y x^{-x}\right)^{1/(y-x)}\right), \qquad x, y \in I, \ x \neq y,$$

if, and only if, there are a, b, $c \in \mathbb{R}$ such that

$$g(x) = ax(\log x - 1) + bx + c, \qquad h(x) = g'(x), \qquad x \in I.$$

Proof. We have

$$E_1(x,y) := e^{-1}\left(y^y x^{-x}\right)^{1/(y-x)} = \exp\circ\log\left(e^{-1}\left(y^y x^{-x}\right)^{1/(y-x)}\right)$$

$$= \exp\left(\frac{x(\log x - 1) - y(\log y - 1)}{x-y}\right)$$

for all x, $y > 0$, $x \neq y$. Hence, setting

$$f(x) := x(\log x - 1), \qquad x > 0,$$

$\phi := h \circ \exp_{|\log(I)}$, we can write the considered functional equation in the form (6). Since f is strictly convex in $(0,\infty)$, the result follows from Theorem 1.

Obviously, the natural domain of the arithmetic mean is $\mathbb{R}^2$. On the other side the common domain of all members of E_p is $(0,\infty)^2$. This explains why in Corollary 1, where the case E_2 is considered, we assume that $I \subset (0,\infty)$. To avoid this inconvenience let us note the following result which is a complement of Theorem 2.

Theorem 5. *Let $I \subset \mathbb{R}$ be an open interval and k a fixed positive integer. Then g, $h : I \to \mathbb{R}$ satisfy the functional equation*

$$\frac{g(x) - g(y)}{x - y} = h\left(\frac{x^{2k} - y^{2k}}{2(x-y)}\right), \qquad x, y \in I, \ x \neq y,$$

if, and only if, there are a, b, $c \in \mathbb{R}$ such that

$$g(x) = ax^{2k} + bx + c, \qquad h(x) = g'(x), \qquad x \in I.$$

For the proof it is sufficient to assume $p = 2k$ in the argument used in the proof of Theorem 2. Note also that an application of Remark 2 one gives a shorter direct proof.

Taking $k = 1$ and $I = \mathbb{R}$ we obtain the result of J. Aczél [1].

3. The mean value property and related functional equations for Stolarsky means

Now we consider an important class of means that is related to the Cauchy mean value theorem, the two parameter family of Stolarsky means $E_{q,p} : (0,\infty)^2 \to (0,\infty)$ $(p, q \in \mathbb{R})$ defined by (cf. [10], also [4], p. 345)

$$E_{q,p}(x,y) := \begin{cases} \left(\frac{q}{p} \cdot \frac{x^p - y^p}{x^q - y^q}\right)^{1/(p-q)}, & p \neq q;\ pq \neq 0;\ x \neq y \\ \left(\frac{1}{q} \cdot \frac{x^q - y^q}{\log x - \log y}\right)^{1/q}, & p = 0;\ q \neq 0;\ x \neq y \\ \left(\frac{1}{p} \cdot \frac{x^p - y^p}{\log x - \log y}\right)^{1/p}, & p \neq 0;\ q = 0;\ x \neq y \\ \left(e^{\frac{-1}{q}}\right) \cdot \left(\frac{y^{y^q}}{x^{x^q}}\right)^{\frac{1}{y^q - x^q}}, & p = q \neq 0;\ x \neq y \\ \sqrt{xy}, & p = q = 0;\ x \neq y \\ x, & p,\ q \in \mathbb{R};\ x = y \end{cases} . \tag{7}$$

This family contains the generalized logarithmic means, as we have

$$E_p = E_{1,p}, \qquad p \in \mathbb{R}.$$

Note that the harmonic mean $E_{-2,-1}$ is not an element of the family of the generalized logarithmic means.

We shall apply Theorem 1 and the results of the previous section to determine the functions $g,\, h : I \to \mathbb{R}$, $I \subset (0,\infty)$, satisfying the functional equation

$$\frac{g(x) - g(y)}{x^q - y^q} = h(E_{q,p}(x,y)), \qquad x,\, y \in I,\ \ x \neq y.$$

Also here we do not assume any regularity conditions of g and h. Note however the following obvious

Remark 5. Let $M : I_0 \times I_0 \to I_0$ be a mean, $I \subset I_0$ an interval, and γ, g, $h : I \to \mathbb{R}$ arbitrary functions. Suppose that γ is bijective and

$$\frac{g(x) - g(y)}{\gamma(x) - \gamma(y)} = h(M(x,y)), \qquad x,\, y \in I,\ \ x \neq y.$$

If M is continuous on the diagonal $\Delta := (x,x) : x \in I_0$, the function γ is differentiable, and h is continuous, then g is differentiable and

$$h = \frac{g'}{\gamma'}.$$

We start with the following

Theorem 6. *Let $I \subset (0,\infty)$ be an open interval, and $p,\ q \in \mathbb{R}$, $p \neq q$, $pq \neq 0$. Then $g,\ h : I \to \mathbb{R}$ satisfy the functional equation*

$$\frac{g(x)-g(y)}{x^q-y^q} = h\left[\left(\frac{q}{p}\cdot\frac{x^p-y^p}{x^q-y^q}\right)^{1/(p-q)}\right], \qquad x,\ y \in I,\ x \neq y,$$

if, and only if, there are $a,\ b,\ c \in \mathbb{R}$ such that

$$g(x) = ax^p + bx^q + c, \qquad h(x) = \frac{g'(x)}{(x^q)'}, \qquad x \in I.$$

Proof. Replacing x and y by $x^{1/q}$ and $y^{1/q}$ in the considered functional equation, we get

$$\frac{g\left(x^{1/q}\right)-g\left(y^{1/q}\right)}{x-y} = h\left[\left(\frac{q}{p}\cdot\frac{x^{p/q}-y^{p/q}}{x-y}\right)^{1/(p-q)}\right], \qquad x,\ y \in I^q,\ x \neq y, \quad (8)$$

where $I^q := x^q : x \in I$. Let $I_{p,q}$ be the pre-image of the interval I^q for the power function $u \to \left(p^{-1}qu\right)^{1/(p-q)}$. Introducing the functions $\phi : I_{p,q} \to \mathbb{R}$, $f : I^q \to \mathbb{R}$ and $G : I^q \to \mathbb{R}$ defined by

$$\phi(u) := h\left[\left(\frac{q}{p}\cdot u\right)^{1/(p-q)}\right], \qquad u \in I_{p,q},$$

$$f(x) := x^{p/q}, \qquad G(x) := g\left(x^{1/q}\right), \qquad x \in I^q,$$

we can write the functional equation (8) in the form

$$\frac{G(x)-G(y)}{x-y} = \phi\left(\frac{f(x)-f(y)}{x-y}\right), \qquad x,\ y \in I^q,\ x \neq y.$$

Since the function f is continuously differentiable and f' is strictly monotonic, we can apply Theorem 1 (with $J_f = I_{p,q}$). Thus there exist $a,\ b,\ c \in \mathbb{R}$, such that

$$\phi(u) = au + b, \qquad u \in J_p,$$

and

$$G(x) = af(x) + bx + c = ax^{p/q} + bx + c, \qquad x \in I.$$

Now the definitions of ϕ, G and f give

$$h(x) = apq^{-1}x^{p-q} + b, \qquad g(x) = ax^p + bx^q + c, \qquad x \in I.$$

Since these functions satisfy the considered functional equation, the proof is completed. $\square$

Taking $p = -1$ and $q = -2$ in Theorem 6 gives the following result for the harmonic mean.

Corollary 3. *Let $I \subset (0, \infty)$ be an open interval. The functions g, $h : I \to \mathbb{R}$ satisfy the functional equation*

$$\frac{g(x) - g(y)}{x^{-2} - y^{-2}} = h\left(\frac{2xy}{x+y}\right), \qquad x,\, y \in I,\ x \neq y,$$

if, and only if, there are a, b, $c \in \mathbb{R}$ such that

$$g(x) = \frac{a}{x} + \frac{b}{x^2} + c, \qquad h(x) = \frac{g'(x)}{(x^{-2})'}, \qquad x \in I.$$

The functional equation of Theorem 6 makes no sense for $q = 0$. However for $E_{0,p}, p \neq 0$, (i.e. for the mean $E_{q,p}$ with $q = 0$ and $p \neq 0$), as a simple consequence of Theorem 3, we obtain the following

Corollary 4. *Let $I \subset (0, \infty)$ be an open interval and $p \in \mathbb{R}$, $p \neq 0$. Then g, $h : I \to \mathbb{R}$ satisfy the functional equation*

$$\frac{g(x) - g(y)}{x^p - y^p} = h\left(\left(\frac{1}{p} \cdot \frac{x^p - y^p}{\log x - \log y}\right)^{1/p}\right), \qquad x,\, y \in I,\ x \neq y,$$

if, and only if, there are a, b, $c \in \mathbb{R}$ such that

$$g(x) = ap \cdot \log x + bx^p + c, \qquad h(x) = \frac{g'(x)}{(x^p)'}, \qquad x \in I.$$

Applying again Theorem 1 (or Theorem 4 with g replaced by $g(x^{1/q})$ we obtain the following result for the mean $E_{0,q}$ with $q \neq 0$.

Theorem 7. *Let $I \subset (0, \infty)$ be an open interval, and $q \in \mathbb{R}$, $q \neq 0$. Then g, $h : I \to \mathbb{R}$ satisfy the functional equation*

$$\frac{g(x) - g(y)}{x^q - y^q} = h\left(e^{-1/q}\left(x^{-x^q} y^{y^q}\right)^{1/(y^q - x^q)}\right), \qquad x,\, y \in I,\ x \neq y,$$

if, and only if, there are a, b, $c \in \mathbb{R}$ such that

$$g(x) = ax^q\left(\log x^q - 1\right) + bx^q + c, \qquad h(x) = aq \cdot \log x + b, \qquad x \in I.$$

4. Some other means and applications of Theorem 1

In this section we show that Theorem 1 allows to characterize all elementary functions by their mean value property.

Theorem 8. *Let $I \subset \mathbb{R}$ be an open interval. Then g, $h : I \to \mathbb{R}$ satisfy the functional equation*

$$\frac{g(x)-g(y)}{x-y} = h\left(\log \frac{e^x - e^y}{x-y}\right), \qquad x,\, y \in I,\ x \neq y,$$

if, and only if, there are a, b, $c \in \mathbb{R}$ such that

$$g(x) = a \cdot e^x + bx + c, \qquad h(x) = g'(x), \qquad x \in I.$$

Proof. Put $\phi := h \circ \log$ and apply Theorem 1. □

Theorem 9. *Let $I \subset (0,\pi)$ be an open interval. Then g, $h : I \to \mathbb{R}$ satisfy the functional equation*

$$\frac{g(x)-g(y)}{x-y} = h\left(\arccos \frac{\sin x - \sin y}{x-y}\right), \qquad x,\, y \in I,\ x \neq y,$$

if, and only if, there are a, b, $c \in \mathbb{R}$ such that

$$g(x) = a \cdot \sin x + bx + c, \qquad h(x) = g'(x), \qquad x \in I.$$

Proof. The function $f := \sin_{|(0,\pi)}$ satisfies the assumption of Theorem 1. Putting $\phi := h \circ \arccos$ we can apply Theorem 1. □

In a similar way one can characterize the remaining trigonometric functions.

Theorem 10. *Let $I \subset (0,\infty)$ be an open interval. Then g, $h : I \to \mathbb{R}$ satisfy the functional equation*

$$\frac{g(x)-g(y)}{x-y} = h\left(\left(\frac{x-y}{\arctan x - \arctan y} - 1\right)^{1/2}\right), \qquad x,\, y \in I,\ x \neq y,$$

if, and only if, there are a, b, $c \in \mathbb{R}$ such that

$$g(x) = a \cdot \arctan x + bx + c, \qquad h(x) = g'(x), \qquad x \in I.$$

Proof. The function $f := \arctan_{|(0,\infty)}$ satisfies the assumption of Theorem 1. Putting $\phi(u) := h\left[\left(u^{-1} - 1\right)^{1/2}\right]$ we can apply Theorem 1. □

5. Final remark

Under some regularity assumptions on functions g and h, a more general functional equation

$$\frac{g(x)-g(y)}{h(x)-h(y)} = \phi(M(x,y)), \qquad x, y \in I,\ x \neq y,$$

related to Cauchy mean value theorem, will be considered in our next paper.

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Manuscript received: April 20, 1998 and, in final form, September 12, 1998.