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MATHEMATICS
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On the Existence of Differentiable Solutions of a Functional Equation

by

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Summary. The fundamental theorem on the existence of differentiable solutions of the functional equation

$$\varphi(x) = h(x, \varphi[f(x)])$$

is due to B. Choczewski [1], (see also M. Kuczma [3], p. 90). There it is assumed that $|f'(x)| < 1$ in a neighbourhood of the fixed point of f. In this paper it is proved that this hypothesis is superfluous.

In the present paper we are concerned with the C^r solutions of the functional equation

$$\varphi(x) = h(x, \varphi[f(x)]), \tag{1}$$

where φ is an unknown function.

Let I denote an interval. The problem of the existence of C^r solutions in I for Eq. (1) was investigated by B. Choczewski [1] (see also [3], chap. IV) under the following hypotheses (i), (ii), (2) and (3).

(i) $f \in C^r[I]$, $r \geq 1$, and for a certain $\xi \in I$ we have

$$0 < \frac{f(x) - \xi}{x - \xi} < 1 \quad \text{for } x \in I,\ x \neq \xi;$$

(ii) h is defined and of class C^r in a region Ω containing the point (ξ, η) such that $h(\xi, \eta) = \eta$; for every $x \in I$ the set $\Omega_x = \{y : (x, y) \in \Omega\}$ is a nonempty open interval and $h(f(x), \Omega_{f(x)}) \subset \Omega_x$;

$$\left| [f'(\xi)]^r \frac{\partial h}{\partial y}(\xi, \eta) \right| < 1; \tag{2}$$

$$|f'(x)| \leq 1 \quad \text{in a neighbourhood of } x = \xi. \tag{3}$$

Define the functions $h_k(x, y, y_1, \ldots, y_k)$ by the recurrent relations

$$h_1(x, y, y_1) = \frac{\partial h}{\partial x}(x, y) + f'(x) \frac{\partial h}{\partial y}(x, y)\, y_1,$$

$$h_{k+1}(x, y, y_1, \ldots, y_{k+1}) = \frac{\partial h_k}{\partial x} + f'(x)\left(\frac{\partial h_k}{\partial y} y_1 + \ldots + \frac{\partial h_k}{\partial y_k} y_{k+1}\right)$$

for $k = 1, \ldots, r-1$.

We denote by Φ the class of functions φ defined in I such that for every $x \in I$ we have $\varphi[f(x)] \in \Omega_x$ and $\varphi(\xi) = \eta$.

We have the following (cf. [1], also [3], p. 85)

LEMMA. *Suppose that hypotheses* (i) *and* (ii) *are fulfilled. If $\varphi \in \Phi$ is a C^r solution of Eq.* (1) *in I, then we have*

$$\eta_k = h_k(\xi, \eta, \eta_1, \ldots, \eta_k) \quad (k = 1, \ldots, r), \tag{4}$$

where

$$\eta_k = \varphi^{(k)}(\xi) \quad (k = 1, \ldots, r). \tag{5}$$

The following fundamental theorem is due to B. Choczewski [1] (see also [3], p. 90).

THEOREM 1. *If hypotheses* (i), (ii), (2) *and* (3) *are fulfilled, then for any system of values $\eta_1, \ldots, \eta_r$ satisfying Eqs.* (4) *there exists at least one C^r solution $\varphi \in \Phi$ of Eq.* (1) *in I fulfilling conditions* (5).

Recently the present author has proved [4] the following stronger result.

THEOREM 2. *If hypotheses* (i), (ii), (2) *and* (3) *are fulfilled, then for any system $\eta_1, \ldots, \eta_r$ satisfying Eqs.* (4) *there exists exactly one C^r solution $\varphi \in \Phi$ of Eq.* (1) *in I fulfilling conditions* (5).

It turns out, however, that in Theorems 1 and 2 hypothesis (3) is superfluous. Namely, in this paper we shall prove the following

THEOREM 3. *If hypotheses* (i), (ii) *and* (2) *are fulfilled, then for any system of values $\eta_1, \ldots, \eta_r$ satisfying Eqs.* (4) *there exists exactly one C^r solution $\varphi \in \Phi$ of Eq.* (1) *in I fulfilling conditions* (5).

Proof. Without loss of generality we may assume that $\xi = 0$. From (i) we have $0 \leqslant f'(0) \leqslant 1$. Hence in view of Theorem 2 it is sufficient to consider the case when $f'(0) = 1$. In this case we have by (2) the inequality $\left|\frac{\partial h}{\partial y}(0, \eta)\right| < 1$. Consequently (see [3], p. 73), Eq. (1) has a unique solution $\varphi \in \Phi$ continuous in I. We shall prove that this solution is of class C^r in I. For this purpose we introduce the sequence of equations

$$\varphi(x) = h(x, \varphi[f_n(x)]), \tag{6}$$

where

$$f_n(x) = t_n f(x), \quad 0 < t_n < 1, \quad \lim_{n\to\infty} t_n = 1. \tag{7}$$

In view of Theorem 2, for every n, $n = 1, 2, \ldots$, Eq. (6) has exactly one solution $\varphi_n \in \Phi$ of class C^r in I, since $|f_n'(0)| = t_n < 1$. Thus we have

$$\varphi_n(x) = h(x, \varphi_n[f_n(x)]), \quad x \in I, \ (n = 1, 2, \ldots).$$

Differentiating both sides of this equality we have

$$\varphi_n'(x)=\frac{\partial h}{\partial x}\left(x,\varphi_n[f_n(x)]\right)+f_n'(x)\frac{\partial h}{\partial y}\left(x,\varphi_n[f_n(x)]\right)\varphi_n'[f_n(x)].$$

This means that $\Phi_n(x)=\varphi_n'(x)$ satisfies the equation

$$\Phi(x)=H_n\left(x,\Phi[f_n(x)]\right) \tag{8}$$

where

$$H_n(x,v)=\frac{\partial h}{\partial x}\left(x,\varphi_n[f_n(x)]\right)+f_n'(x)\frac{\partial h}{\partial y}\left(x,\varphi_n[f_n(x)]\right)v.$$

Since f_n tends to $f_0=f$ uniformly in every compact subset of I, the sequence φ_n tends to φ uniformly in every compact subset of I (see th. 1 in [2]). Hence it follows that H_n uniformly converges to the function

$$H_0(x,v)=\frac{\partial h}{\partial x}\left(x,\varphi[f(x)]\right)+f'(x)\frac{\partial h}{\partial y}\left(x,\varphi[f(x)]\right)v,$$

on every compact set contained in $I\times R$ (where R denotes the set of real numbers). We have

$$\left|\frac{\partial H_n}{\partial v}(0,v)\right|=\left|f_n'(0)\frac{\partial h}{\partial y}(0,\eta)\right|=t_n\left|\frac{\partial h}{\partial y}(0,\eta)\right|<1,\quad (n=0,1,2,\ldots).$$

Therefore (see [3], p. 73) $\Phi_n(x)=\varphi_n'(x)$ is the unique solution of Eq. (8) which is continuous in I $(n=1,2,\ldots)$. Moreover, this equation has for $n=0$ a unique solution Φ_0 continuous in I. Applying again the theorem on the continuous dependence of continuous solutions (cf. [2], th. 1) we obtain $\lim\limits_{n\to\infty}\varphi_n'(x)=\Phi_0(x)$, uniformly on every compact subset of I. Since $\lim\limits_{n\to\infty}\varphi_n(x)=\varphi(x)$, this implies that $\varphi\in C^1[I]$ and $\varphi'(x)=\Phi_0(x)$ for $x\in I$.

In the same manner (differentiating equation (8)) we can prove that $\varphi'\in C^1[I]$, i.e. $\varphi\in C^2[I]$. Repeating this procedure r-times we obtain $\varphi\in C^r[I]$. This completes the proof.

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[4] J. Matkowski, *On the uniqueness of differentiable solutions of a functional equation*, Bull. Acad. Polon. Sci., Sér. Sci. Math. Astronom. Phys., 18 (1970), 253—255.

И. Матковски, О существовании дифференцируемых решений функционального уравнения

Содержание. Основная теорема, касающаяся существования дифференциальных решений уравнения

$$\varphi(x) = h(x, \varphi[f(x)])$$

происходит от Б. Хочевского. В вышеприведенной теореме принимается, что $|f'(x)| \leq 1$ в некоторой окрестности стабильной точки функции f. В настоящей работе доказывается, что это предположение является излишним.