PRACE NAUKOWE UNIWERSYTETU ŚLĄSKIEGO W KATOWICACH NR 2

PRACE MATEMATYCZNE 1, 1969

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On some properties of solutions of a functional equation

In the present paper we are concerned with the functional equation

(1) $$\varphi(x) = h(x, \varphi[f(x)]),$$

where $f(x)$ and $h(x, y)$ are known real-valued functions of real variables and $\varphi(x)$ is unknown.

We assume the following hypotheses:

(I) $f(x)$ is defined and continuous in an interval $\langle 0, a)$ and

(2) $$0 < f(x) < x \quad \text{for} \quad x \in (0, a);$$

(II) $h(x, y)$ is defined in a domain Ω such that $(0, 0) \in \Omega$; moreover, for every $x \in \langle 0, a)$, the set $\Omega_x \overset{\mathrm{df}}{=} \{y : (x, y) \in \Omega\}$ is a nonempty open interval and $A_x \subset \Omega_x$, where $A_x \overset{\mathrm{df}}{=} h(x, \Omega_{f(x)})$;

(III) $h(x, 0)$ is continuous at $x = 0$, $h(0, 0) = 0$ and

(3) $$|h(x, y_1) - h(x, y_2)| \leqslant s|y_1 - y_2|, \quad 0 < s < 1,$$

holds in a neighbourhood of the point $(0, 0)$.

We shall prove the following

THEOREM. *Let hypotheses (I)—(III) be fulfilled. Then equation (1) has exactly one solution φ defined in $\langle 0, a)$, continuous at $x = 0$ and such that $\varphi(0) = 0$.*

1°. *If, moreover, $f(x)$ is strictly increasing in $\langle 0, a)$ and $h(x, y)$ is increasing with respect to either variable in Ω, then φ is increasing in $\langle 0, a)$ (if, for every fixed y, $h(x, y)$ is a strictly increasing function of x, then φ is strictly increasing in $\langle 0, a)$).*

2°. *If, besides (I)—(III) and hypotheses of 1°, we assume that $f(x)$ is convex in $\langle 0, a)$, Ω is a convex domain and $h(x, y)$ is a convex function of two variables in Ω, then φ is convex in $\langle 0, a)$.*

Proof. We choose a $c > 0$ and a $d > 0$ such that (3) holds in the set

$$D = \{(x, y) : 0 \leqslant x \leqslant c, \ -d \leqslant y \leqslant d\}.$$

By (III) we may assume that c has been chosen in such a manner that

(4) $$|h(x, 0)| \leqslant (1 - s)d \quad \text{for} \quad x \in \langle 0, c\rangle.$$

Let F be the set of functions φ defined in $\langle 0, c)$, continuous at $x=0$ and such that

(5) $$\varphi(0)=0; \quad |\varphi(x)|\leq d \quad \text{for} \quad x\in\langle 0, c).$$

The set F with the metric

$$\varrho(\varphi_1, \varphi_2) = \sup_{\langle 0,c)} |\varphi_1(x)-\varphi_2(x)|$$

is a complete metric space. We define the transform

(6) $$\psi(x) = h(x, \varphi[f(x)])$$

for $\varphi\in F$. It follows from (2) that

(7) $$|\varphi[f(x)]|\leq d \quad \text{for} \quad \varphi\in F \quad \text{and} \quad x\in\langle 0, c).$$

Now, from (3) and (4) we get

(8) $$|\psi(x)| = |h(x, \varphi[f(x)])|\leq|h(x, \varphi[f(x)])-h(x, 0)| + |h(x, 0)|\leq\alpha|\varphi[f(x)]|+|h(x, 0)|.$$

Hence and from (2) we see that $\lim_{x\to+0}\psi(x)=0$. Since $\psi(0)=h(0, \varphi(0))=h(0, 0)=0$, $\psi(x)$ is continuous at $x=0$. From (8) and (7) we obtain

$$|\psi(x)|\leq\alpha d+(1-\alpha)d = d.$$

This proves that (6) transforms F into itself. Further, we have in virtue of (3) for

$$\psi_1(x) = h(x, \varphi_1[f(x)]), \quad \psi_2(x) = h(x, \varphi_2[f(x)]), \quad \varphi_1, \varphi_2\in F,$$

$$|\psi_1(x)-\psi_2(x)| = |h(x, \varphi_1[f(x)])-h(x, \varphi_2[f(x)])| \leq\alpha|\varphi_1[f(x)]-\varphi_2[f(x)]|,$$

whence

$$\varrho(\psi_1, \psi_2)\leq\alpha\varrho(\varphi_1, \varphi_2),$$

i.e., (6) is a contraction map. On account of BANACH's theorem there exists exactly one solution $\varphi\in F$ of equation (1). This solution has a unique extension onto the whole interval $\langle 0, a)$ (cf. [1], p. 70, Theorem 3.2).

REMARK 1. In the book [1] Theorem 3.2 has been proved under the assumption of the continuity of $h(x, y)$ in Ω but Theorem 3.2 will remain valid without this assumption (and the proof is the same), except that then the solution need not be continuous.

For the proof of 1° we define the space F_1

$$F_1 = \{\varphi\in F : \varphi \text{ is increasing in } \langle 0, c)\}.$$

Let us take $0\leq x_1<x_2\leq d$. From the monotonicity of $f(x)$ and from (2) we get

$$0\leq f(x_1)<f(x_2)\leq d.$$

For $\varphi\in F_1$ we have

$$0\leqslant\varphi[f(x_1)]\leqslant\varphi[f(x_2)]\leqslant a.$$

Now from (6) and the monotonicity of $h(x, y)$ we obtain

$$\psi(x_1)=h(x_1,\varphi[f(x_1)])\leqslant h(x_2,\varphi[f(x_1)])\leqslant h(x_2,\varphi[f(x_2)])=\psi(x_2),$$

i.e., ψ is increasing in $\langle 0, c\rangle$. Hence and from the first part of the proof it follows that $\psi\in F_1$. Since F_1 is a complete metric space, the unique solution φ continuous at $x=0$ and such that $\varphi(0)=0$ must belong to F_1. Thus φ is increasing in $\langle 0, c\rangle$. Now we shall prove that φ is increasing in $\langle 0, a\rangle$. For this purpose we denote by x_0 the supremum of all b such that φ is monotonic in $\langle 0, b\rangle$ and suppose that $x_0<a$. From (2) we get $f(x_0)<x_0$. It follows from the continuity of $f(x)$ that there exists a number $\alpha>x_0$ such that for $x\in\langle x_0, \alpha)$ we have $f(x)<x_0$. Hence for $0\leqslant x_1<x_2\leqslant\alpha$ we obtain

$$\varphi(x_1)=h(x_1,\varphi[f(x_1)])\leqslant h(x_2,\varphi[f(x_2)])=\varphi(x_2),$$

i.e., φ is monotonic in $\langle 0, \alpha\rangle$, $\alpha>x_0$. This contradiction completes the proof of the monotonicity of φ in $\langle 0, a\rangle$.

If, for every fixed y, $h(x, y)$ is a strictly increasing function of the variable x, then for $0\leqslant x_1<x_2<a$ we have

$$\varphi(x_1)=h(x_1,\varphi[f(x_1)])<h(x_2,\varphi[f(x_1)])\leqslant h(x_2,\varphi[f(x_2)])=\varphi(x_2),$$

which proves that φ is strictly increasing.

REMARK 2. It is not sufficient to suppose that $h(x, y)$ is strictly increasing in the variable y. For instance, the unique solution continuous at $x=0$ of the equation

$$\varphi(x)=\tfrac{1}{2}\varphi[f(x)]$$

is $\varphi(x)\equiv 0$, and $h(x, y)=\frac{1}{2}y$ is strictly monotonic with respect to y.

For the proof of 2° we define the space F_2:

$$F_2=\{\varphi\in F_1:\varphi \text{ is convex in } \langle 0, c\rangle\}.$$

From the convexity and monotonicity of f, h, and φ we get for $0\leqslant x_1\leqslant c$, $0\leqslant x_2\leqslant c$ and ψ given by formula (6)

$$\psi\left(\frac{x_1+x_2}{2}\right)=h\left(\frac{x_1+x_2}{2},\varphi\left[f\left(\frac{x_1+x_2}{2}\right)\right]\right)\leqslant h\left(\frac{x_1+x_2}{2},\varphi\left[\frac{f(x_1)+f(x_2)}{2}\right]\right)$$

$$\leqslant h\left(\frac{x_1+x_2}{2},\frac{\varphi[f(x_1)]+\varphi[f(x_2)]}{2}\right)\leqslant\frac{h(x_1,\varphi[f(x_1)])+h(x_2,\varphi[f(x_2)])}{2}=$$

$$=\frac{\psi(x_1)+\psi(x_2)}{2}.$$

Hence $\psi(x)$ is convex in $\langle 0, c\rangle$. In view of the inclusion $F_2\subset F_1$ and of the preceding part of the proof, (6) transforms F_2 into itself. Evidently, F_2 is a complete metric space, and thus the unique solution φ continuous at $x=0$ and such that $\varphi(0)=0$ must be convex. Similary as in preceding case we can prove that φ is convex in $\langle 0, a\rangle$.

REMARK 3. The first part of the theorem is implicitly contained in the book [1] (comp. the remark at the end of § 4, Chapter III, p. 54).

REFERENCE

[1] M. Kuczma: *Functional Equations in a Single Variable*, Monografie Matematyczne 46 (PWN, Warszawa 1968).

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O PEWNYCH WŁASNOŚCIACH ROZWIĄZAŃ RÓWNANIA FUNKCYJNEGO

Streszczenie

W pracy dowodzi się twierdzenia które orzeka, że jeżeli dana funkcja f i h są monotoniczne (względnie monotoniczne i wypukłe) to rozwiązanie równania (1) jest monotoniczne (wypukłe).

Odebrano do Redakcji 1 sierpnia 1969 r.